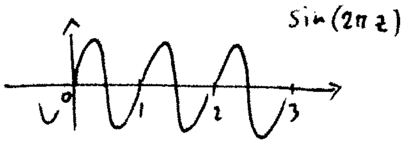
- $\dfrac{1}{e^z-1}$ has simple poles where $e^z=1$: $z = 2\pi i k$, $k \in \mathbb{Z}$

- $\dfrac{\sin(2\pi z)}{z^3(2z-1)}$



at 0: pole of order $3-1=2$

at ½: pole of order $1-1=0$ (removable singularity)

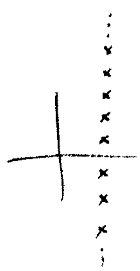
- $\sin(1/z) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{-n}$ essential singularity at 0: infinitely many terms in the Laurent series

- $\overline{z}$ not a holomorphic function (otherwise no singularities)

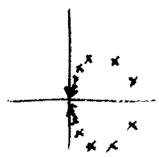
- $\dfrac{1}{\exp(\frac{1}{z})+2}$ poles whenever $\exp(\frac{1}{z}) = -2$, that is

$$\frac{1}{z} = \log(-2) + 2\pi i k \qquad k \in \mathbb{Z}$$

$$= \log(2) + 2\pi i\,(k+\tfrac{1}{2}) \quad k \in \mathbb{Z}$$

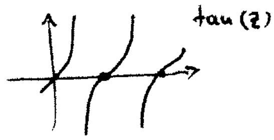


$$z = \left(\log(2) + 2\pi i\,(k+\tfrac{1}{2})\right)^{-1}, \quad k \in \mathbb{Z}$$



this set has an accumulation point at $z=0$: the latter is not an isolated singularity.

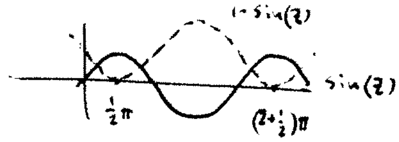
Recall : $\operatorname{res}\left(\frac{f}{g};a\right) = \frac{f(a)}{g'(a)}$ when $f(a) \neq 0$ and $g'(a) \neq 0$.



(i) $\tan(\pi z)$ has simple zeros at all integer points

these become the poles of $\frac{\pi}{\tan(\pi z)}$

$$\operatorname{res}\left(\frac{\pi}{\tan(\pi z)}; k\right) = \frac{\pi}{\pi} = 1 \qquad \forall k \in \mathbb{Z}.$$



(ii) $1 - \sin(z) = 0$ for $z = (2k + \frac{1}{2})\pi$, $k \in \mathbb{Z}$

(how does one check that these are all ?

$1 - \frac{e^{iz} - (e^{iz})^{-1}}{2i} = 0$ is a quadratic equation in e^{iz}

$\Rightarrow$ two solutions mod 2π.)

Fix $a := (2k + \frac{1}{2})\pi$.

Around $z = a$, $1 - \sin(z) = \frac{1}{2}(z-a)^2 + \mathcal{O}(z-a)^4$ $\leftarrow$ quadratic term in the expansion of cos.

$$\operatorname{res}\left(\frac{z^2 - z}{1 - \sin(z)}; a\right) = \operatorname{res}\left(\frac{z^2 - z}{\frac{1}{2}(z-a)^2}; a\right)$$

We have poles of order two, so we only care about the first two terms of any Taylor / Laurent expansion.

$$= 2 \cdot (z^2 - z)'\Big|_{z=a} = 4z - 2\Big|_{z=a} = 4a - 2 = 4(2k + \tfrac{1}{2})\pi - 2$$

$$= 8k\pi + 2\pi - 2$$

(iii) $\frac{\cos(z) - 1}{(e^z - 1)^2}$ has double poles at $z = 2\pi i k$ $k \in \mathbb{Z}$

(except at $k = 0$, where $\cos(z) - 1$ also vanishes to second order $\Rightarrow$ removable singularity)

These are covert double poles, and we have no alternative but to calculate the principal part of the Laurent series : first <u>two</u> terms.

$W = z - 2\pi i k$

$$(e^z - 1)^2 = (e^W - 1)^2 = (W + \tfrac{1}{2}W^2 + \dots)^2 = W^2 + W^3 + \dots$$

$$\cos(z) - 1 = \cos(W + 2\pi i k) - 1 = (\cos(2\pi i k) - 1) - (\sin(2\pi i k))\, W + \dots$$

$$= (\cosh(2\pi k) - 1) - i\sinh(2\pi k)\, W + \dots$$

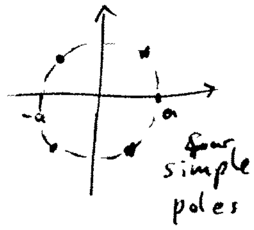
$$= A + BW + \dots$$

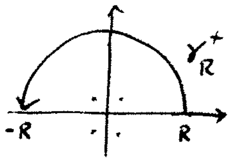
$$\frac{A + BW}{W^2 + W^3} = (A + BW)(W^{-2} - W^{-1} + \dots) = AW^{-2} + (B - A)W^{-1} + \dots$$

$$\operatorname{res}\left(\frac{\cos(z) - 1}{(e^z - 1)^2}; 2\pi i k\right) = \operatorname{res}\left(\frac{A + BW}{W^2 + W^3}; 0\right) = B - A$$

$$= -i\sinh(2\pi k) - \cosh(2\pi k) + 1$$

$$\int_{-\infty}^{\infty} \frac{1}{x^4+a^4}\,dx$$





$$\left| \int_{\gamma_R^+} \frac{1}{z^4+a^4}\,dx \right| < \pi R \cdot \frac{1}{R^4-a^4} \xrightarrow{R\to\infty} 0 .$$

$$\int_{-\infty}^{\infty} \frac{1}{x^4+a^4}\,dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{x^4+a^4}\,dx + \lim_{R\to\infty} \int_{\gamma_R^+} \frac{1}{z^4+a^4}\,dz$$

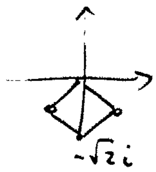
$$= \lim_{R\to\infty} \oint \frac{1}{z^4+a^4}\,dz = 2\pi i \left[\operatorname{res}\left(\frac{1}{z^4+a^4}; a e^{\frac{\pi i}{4}}\right) + \operatorname{res}\left(\frac{1}{z^4+a^4}; a e^{\frac{3\pi i}{4}}\right) \right]$$

have you seen this symbol?

$$= 2\pi i \left[\frac{1}{4z^3}\Big|_{z=ae^{\frac{\pi i}{4}}} + \frac{1}{4z^3}\Big|_{z=ae^{\frac{3\pi i}{4}}} \right]$$

$$= 2\pi i \left[\frac{1}{4a^3 e^{\frac{3\pi i}{4}}} + \frac{1}{4a^3 e^{\frac{9\pi i}{4}}} \right]$$

$$= \frac{\pi i}{2a} \left[e^{-\frac{3\pi i}{4}} + e^{-\frac{\pi i}{4}} \right]$$



$$= \frac{\pi i}{2a} \cdot (-\sqrt{2} i) = \frac{\pi}{\sqrt{2}\, a^3}$$

$\boxtimes$

$-1 < a < 1$

$$I = \int_0^{2\pi} \frac{d\theta}{1 - a\cos(\theta)} = \oint_C \frac{dz}{iz\left(1 - a\frac{z+z^{-1}}{2}\right)} = -2i \oint_C \frac{1}{az^2+2z+a}\, dz$$

$$e^{i\theta} = z$$

$$\theta \in [0, 2\pi] \iff z \in C$$

$$\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+z^{-1}}{2}$$

$$ie^{i\theta}\, d\theta = dz$$

$$d\theta = \frac{dz}{iz}$$

Residues at $z = \frac{-2 \pm \sqrt{4-4a^2}}{2a} = \frac{-1 \pm \sqrt{1-a^2}}{a}$

Which ones are inside C ?

$$\left|\frac{-1-\sqrt{1-a^2}}{a}\right| = \frac{1+\sqrt{1-a^2}}{|a|} > \frac{1}{|a|} > 1 \Rightarrow \underline{\text{outside}}.$$

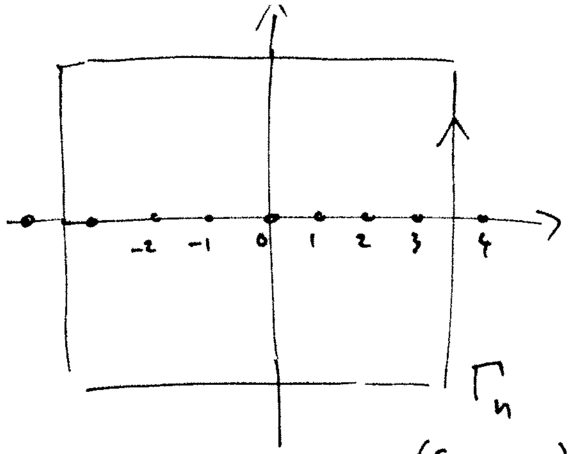
$\Rightarrow \frac{-1+\sqrt{1-a^2}}{a}$ is <u>inside</u> C since otherwise we would have $I = 0$, which is clearly absurd as $\frac{1}{1-a\cos(\theta)} > 0$.

$$I = 2\pi i \operatorname{res}\left(\frac{-2i}{az^2+2z+a}\, ; \frac{-1+\sqrt{1-a^2}}{a}\right)$$

$$= 4\pi \operatorname{res}\left(\frac{1}{az^2+2z+a}\, ; \frac{-1+\sqrt{1-a^2}}{a}\right)$$

$$= 4\pi \left.\frac{1}{2az+2}\right|_{z=\frac{-1+\sqrt{1-a^2}}{a}} = \frac{4\pi}{2\cancel{a}\,\frac{\cancel{-1}+\sqrt{1-a^2}}{\cancel{a}} \cancel{+2}} = \frac{2\pi}{\sqrt{1-a^2}}$$

$\boxtimes$



(for $n=3$)

$\frac{\pi}{w^2 \sin(\pi w)}$ has sigles poles for $w \in \mathbb{Z} \setminus \{0\}$ and a triple pole at $w=0$.

$$\boxed{\operatorname{res}\left(\frac{\pi}{w^2 \sin(\pi w)} ; n\right) = \frac{(-1)^n}{n^2} \quad \text{for } n \neq 0}$$

$$\operatorname{res}\left(\frac{\pi/w^2}{\sin(\pi w)} ; n\right) = \left.\frac{\pi/w^2}{\pi \cos(\pi w)}\right|_{w=n} = \frac{\pi/n^2}{\pi \cdot (-1)^n} = \frac{(-1)^n}{n^2}$$

In order to compute $\operatorname{res}\left(\frac{\pi}{w^2 \sin(\pi w)} ; 0\right)$, let us first compute the Laurent series of $(\sin(\pi w))^{-1}$:

$$\frac{1}{\sin(\pi w)} = \frac{1}{\pi w - \frac{1}{6}\pi^3 w^3 + \dots} = \frac{1}{\pi w} \cdot \frac{1}{1 - \frac{1}{6}\pi^2 w^2 + \dots}$$

$$= \frac{1}{\pi w}\left(1 + \frac{1}{6}\pi^2 w^2 + \dots\right) = \frac{1}{\pi w} + \frac{1}{6}\pi w + \dots$$

$$\Rightarrow \frac{\pi}{w^2 \sin(\pi w)} = \frac{1}{w^3} + \frac{\pi^2}{6} w^{-1} + \dots \quad \Rightarrow \boxed{\operatorname{res}\left(\frac{\pi}{w^2 \sin(\pi w)} ; 0\right) = \frac{\pi^2}{6}.}$$

---

have you seen this symbol?

$$\oint_{\Gamma_n} \frac{\pi}{w^2 \sin(\pi w)}\, dw \xrightarrow{n \to \infty} 0$$

By the Estimation Theorem, because the length of Γ_n is $4(2n+1)$ whereas the size of the integrand is $\leq c \cdot \frac{1}{n^2}$

using that $\csc(\pi w) = \frac{1}{\sin(\pi w)}$ is uniformly bounded on the Γ_n

$$\Rightarrow \sum \text{all residues} = 0$$

$$\left(\sum_{n \in \mathbb{Z} \setminus \{0\}} (-1)^n \frac{1}{n^2}\right) + \frac{\pi^2}{6} = 0 \quad \Rightarrow \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2} = -\frac{\pi^2}{12} \quad \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

$\boxtimes$