Practice exam for *Foundations of Mathematics* A. Henriques, October 2013.

Problem 1 Determine which ones of the following statements are true, and which ones are false:

- $\mathcal{P}(\mathbb{N}) \subset \mathcal{P}(\mathbb{Z})$
- $\mathbb{N} \subset \mathcal{P}(\mathbb{Z})$
- $\{\emptyset\} \subset \mathcal{P}(\mathbb{N})$
- $\mathcal{P}(\mathbb{N}) \in \mathcal{P}(\mathbb{Z})$
- $\mathbb{N} \in \mathcal{P}(\mathbb{Z})$
- $\{1, 2, 3\} \in \mathcal{P}(\mathbb{N})$

Problem 2 Give an example of three sets A, B, C, such that $A \in B$, $B \in C$, $A \in C$, but $B \not\subseteq C$.

Problem 3 Describe the following sets by means of a picture:

- $\{x \in \mathbb{R} : x^2 > 2\}$
- $\{x \in \mathbb{Q} : x^2 = 2\}$
- $\{x \in \mathbb{R} : |x - 2| \leq 1\}$
- $\{x \in \mathbb{R} : \exists n \in \mathbb{Z}, x > n^2\}$
- $\{x \in \mathbb{R} : \forall n \in \mathbb{Z}, x > n^2\}$
- $\{(x, y) \in \mathbb{R}^2 : x + y \leq 1\}$

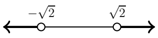
Problem 4 Show by means of a truth table that if $P \Rightarrow Q$ and $Q \Rightarrow R$, then $P \Rightarrow R$:

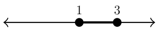
Problem 5 Prove that for every $n \in \mathbb{Z}$, the number $\frac{1}{2}(n^2 + n)$ is an integer.

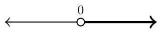
Problem 6 Prove that if $a \equiv b \pmod{n}$, then $4a \equiv 4b \pmod{2n}$.

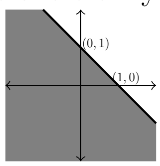
Solution: 1: Yes, every subset of $\mathbb{N}$ is also a subset of $\mathbb{Z}$. 2: No, the elements of $\mathbb{N}$ are numbers, and no number is an element of $\mathcal{P}(\mathbb{Z})$. 3: Yes, the empty set is an element of $\mathcal{P}(\mathbb{N})$. 4: No, the elements of $\mathcal{P}(\mathbb{Z})$ are sets of numbers, and $\mathcal{P}(\mathbb{N})$ is not a set of numbers. 5: Yes, $\mathbb{N}$ is a subset of $\mathbb{Z}$. 6: Yes, $\{1, 2, 3\}$ is a subset of $\mathbb{N}$.

Solution: This problem has many possible solutions. Here is an example that work: $A = \{1\}$, $B = \{\{1\}, 2\}$, $C = \{\{1\}, \{\{1\}, 2\}\}$.

Solution: 1: 

2: The empty set because $\pm\sqrt{2}$ are irrational numbers. 3: 

4: 

5: The empty set because squares can get arbitrarily large, and it is impossible for a number to be bigger than every square. 6: 

Solution: We have to show that $((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$ always holds.

P	Q	R	$P \Rightarrow Q$	$Q \Rightarrow R$	$(P \Rightarrow Q) \wedge (Q \Rightarrow R)$	$P \Rightarrow R$	our expression
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Solution: We split the problem in two cases. Case 1: n is even. Then $n = 2k$ for some $k \in \mathbb{Z}$, and $n^2 + n = 4k^2 + 2k = 2(2k^2 + k)$. It follow that $\frac{1}{2}(n^2 + n) = 2k^2 + k$, which is clearly an integer. Case 2: n is odd. Then $n = 2k + 1$ for some $k \in \mathbb{Z}$, and $n^2+n = (4k^2+4k+1)+(2k+1) = 2(2k^2+3k+1)$. It follow that $\frac{1}{2}(n^2+n) = 2k^2+3k+1$, which is also clearly an integer.

Solution: By definition, $a \equiv b \pmod{n}$ means that there exists an integer number k such that $a - b = kn$. We want to show that $4a \equiv 4b \pmod{2n}$, that is, there exists an integer ℓ such that $4a - 4b = \ell 2n$. For that, we may take $\ell = 2k$: $4a - 4b = 4(a - b) = 4kn = 2\ell n$.

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No solutions will be made available.

Problem 1 Among the following statements, determine which ones are tautologies, and which ones are contradictions:

- $P \Rightarrow (P \Rightarrow Q)$
- $(P \wedge (P \Leftrightarrow Q)) \Rightarrow Q$
- $(P \Rightarrow Q) \wedge P \wedge (\sim Q)$
- $(P \wedge (Q \vee R)) \Leftrightarrow ((P \wedge Q) \vee R)$.

Problem 2 Prove the following statement:
Given an integer $x \in \mathbb{Z}$, x is even if and only if $5 - x$ is odd.

Problem 3 Let $a, b, n \in \mathbb{Z}$ be integer numbers. Define, using mathematical symbols only, what it means that $a \equiv b \pmod{n}$.

Problem 4 Let $A := \{1, 2, 3, 4, 5\}$, and $B := \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}$. Which of the following statements are true?

- $A \subseteq B$
- $A \in B$
- $B \subset \mathcal{P}(A)$
- $\forall x \in A, \{x\} \in B$
- $\forall x \in B, x \subset A$
- $\bigcup_{x \in B} x = A$
- $\exists x, x \in A \cap B$
- The cardinality of $A \cup B$ is 5.
- B is a partition of A.
- $B \subset \mathbb{N}$.

Problem 5 For $x \in \{-10, -9, -8, \ldots, 7, 8, 9, 10\}$, consider the following sentences:

$P(x) : x$ is odd. $\quad Q(x) : x \geq 1. \quad R(x) : x \in \{-1, 0, 1\}$.

For which $x \in \{-10, \ldots, 9, 10\}$ are the following statements true?

- $(\sim P(x)) \wedge Q(x)$
- $Q(x) \vee R(x)$
- $R(x) \Rightarrow P(x)$
- $\sim (Q(x) \wedge R(x))$.
- $P(x) \Leftrightarrow Q(x)$