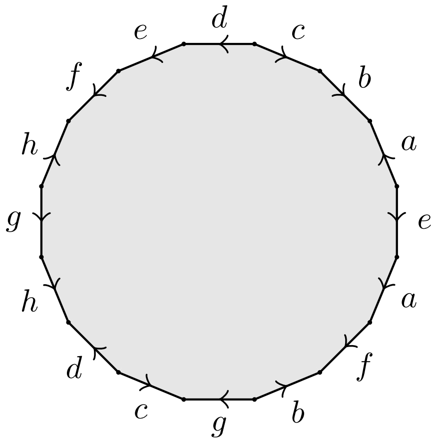
Old exam and practice exam questions *Topologie en Meetkunde* (WISB341).

A. Henriques

Problem 1 State the definition of a manifold.
Prove that if M and N are manifolds, then their product $M \times N$ is also a manifold.

Solution: A manifold a space X such that every point $x \in X$ has a neighborhood that is homeomorphic to $\mathbb{R}^n$. Given a point $(x, y) \in M \times N$, pick neighborhoods $U \subset M$ of x and $V \subset N$ of y such that $U \cong \mathbb{R}^m$ and $V \cong \mathbb{R}^n$. Then $U \times V \cong \mathbb{R}^{m+n}$ is a neighborhood of (x, y).

Problem 2 State the classification theorem for compact surfaces.
Let Σ be the surface obtained by glueing the sides of a regular 16-gon according to the following pattern:



To which surface in the classification is Σ homeomorphic?

Solution: A compact surface is homeomorphic to either S^2, a connected sum of copies of T^2, or a connected sum of copies of P^2, and those are pairwise non-homeomorphic. The surface Σ is orientable and has Euler characteristic given by $3 - 8 + 1 = -4$. It is therefore homeomorphic to $T^2 \# T^2 \# T^2$.

Problem 3 Given two natural numbers $m < n$, the product $S^m \times S^n$ of the m-dimensional sphere with the n-dimensional sphere is a CW-complex with four cells.
What are the dimensions of those cells?
Describe the m-skeleton of that CW complex.
Describe the n-skeleton of that CW complex.

Solution: The cells have dimenions 0, m, n, and $m + n$. The m-skeleton is S^m, and the n-skeleton is $S^n \vee S^m$.

Problem 4 State the definition of homotopy equivalence.
Prove that if X and Y are two spaces that are homotopy equivalent, then the products $X \times S^1$ and $Y \times S^1$ are also homotopy equivalent.

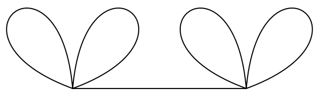
Solution: $X \approx Y$ if there are maps $f : X \to Y$, $g : Y \to X$, such that $f \circ g \sim 1_Y$ and $g \circ f \sim 1_X$. The maps $f \times 1_{S^1} : X \times S^1 \to Y \times S^1$, and $g \times 1_{S^1} : Y \times S^1 \to X \times S^1$ satisfy $(f \times 1_{S^1}) \circ (g \times 1_{S^1}) \sim 1_{Y \times S^1}$ and $(g \times 1_{S^1}) \circ (f \times 1_{S^1}) \sim 1_{X \times S^1}$, and thus form a homotopy equivalence between $X \times S^1$ and $Y \times S^1$.

Problem 5 Consider a triangulation of $T^2 \# T^2$ such that at every vertex, exactly seven triangles meet. How many triangles are there in total in that triangulation?

Solution: The Euler characteristic $F - E + V$ is equal to $\chi(T^2 \# T^2) = -2$. We have $E = \frac{3}{2}F$ and $V = \frac{3}{7}F$, therefore $\chi(T^2 \# T^2) = F(1 - \frac{3}{2} + \frac{3}{7}) \Rightarrow F = (-2)/(1 - \frac{3}{2} + \frac{3}{7}) = 28$.

Problem 6 The surface $T^2 \# T^2$ admits a CW complex structure whose 1-skeleton is the following graph:

Γ :

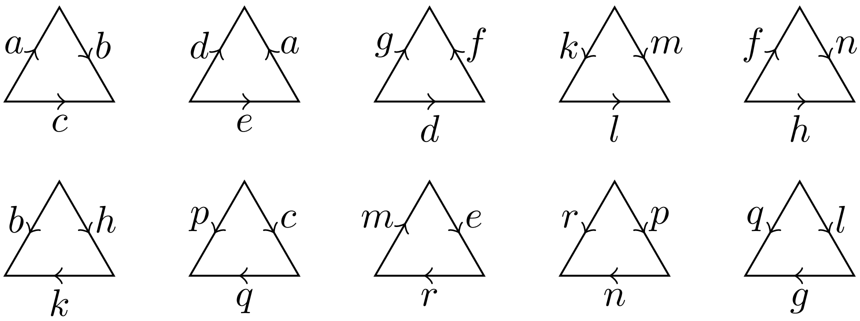


Describe an attaching map $f : S^1 \to \Gamma$ such that $\Gamma \cup_f e_2 = T^2 \# T^2$.

Solution: If the 1-skeleton was $S^1 \vee S^1 \vee S^1 \vee S^1$, then the attaching map would be given by the word $aba^{-1}b^{-1}cdc^{-1}d^{-1}$. Calling the extra horizontal edge in Γ by the letter e, the attaching map is then given by $aba^{-1}b^{-1}ecdc^{-1}d^{-1}e^{-1}$.

Problem 7 State the classification theorem for compact 2-dimensional manifolds (no boundary).

Consider the 2-dimensional manifold built by glueing together the following ten triangles:



What is the Euler characteristic of that manifold?
Is that manifold orientable?
To which manifold in the classification is the above manifold homeomorphic?

Solution: Every compact connected surface is homeomorphic to S^2, a connected sum of copies of P^2, or a connected sum of copies of T^2. Moreover, the surfaces in the above list are pairwise non-homeomorphic. The manifold above has 10 triangles, 15 edges, and 6 vertices. Its Euler characteristc is therefore equal to $10 - 15 + 6 = 1$. The only manifold in the classification that has Euler characteristic 1 is P^2, which is not orientable. That manifold is therefore not orientable.

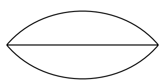
Problem 8 Define what it means for two spaces to be homotopy equivalent.
Show that the following two spaces are homotopy equivalent:

– The sphere S^2 minus n points.
– The wedge of $n-1$ copies of S^1.

Solution: Two spaces X and Y are homotopy equivalent is there exist maps $X \to Y$ and $Y \to X$ such that the composites $X \to Y \to X$ and $Y \to X \to Y$ are homotopic to Id_X and Id_Y, respectively. S^2 minus n points is homeomorphic to $\mathbb{R}^2$ minus $n-1$ points, which deformation retracts onto a wedge of $n-1$ copies of S^1.

Problem 9 What is the definition of an n-dimensional manifold?
Prove that the space $X := [0,1] \times \{0,1,2\}/(0,0) \sim (0,1) \sim (0,2); (1,0) \sim (1,1) \sim (1,2)$

X :



is not a 1-dimensional manifold.

Solution: An n-dimensional manifold is a space X such that every point $p \in X$ has a neighborhood that is homeomorphic to $\mathbb{R}^n$. If X is a 1-dimensional manifold, then for every point $p \in X$ and every neighborhood $\tilde{U} \subset X$ of p, there exists a smaller neighborhood $U \subset \tilde{U}$ that is homeomorphic to $\mathbb{R}$. Pick $p := [(0,0)]$ and $\tilde{U} = X \setminus [(1,0)]$. If the point p had a neighborhood $U \subset \tilde{U}$ that is homeomorphic to $\mathbb{R}$, then $U \setminus p$ would have two connected components. But any such neighborhood has at least three connected components.

Problem 10 State the classification theorem for compact surfaces.
Prove (without using the above classification theorem) that the Klein bottle is homeomorphic to

the connected sum of two projective planes.

Solution: The classification theorem says that every compact surface is homeomorphic to either S^2, a connected sum of copies of T^2, or a connected sum of copies of P^2, and that those are pairwise non-homeomorphic. The Klein bottle is given by  while $P^2 \# P^2$ is given . One can see the homeomorphism by subdividing each one of those squares into two triangles  and , and then rearranging the triangles.

Problem 11 Exhibit an equivalence relation on the two-dimensional torus T^2, whose quotient space is the two-dimensional sphere S^2.

Solution: Take the whole 1-skeleton $S^1 \vee S^1$ to be an equivalence class, and all the other equivalence classes to be single points.

Problem 12 The complex projective plane $X := \mathbb{C}P^2$ is a CW complex with three cells.
What is the 2-skeleton $X^{(2)}$ of that space?
Write down a formula for the attaching map $S^3 \to X^{(2)}$ of the 4-cell.

Solution: The 2-skeleton is S^2, and the attaching map $S^3 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\} \to S^2 = \mathbb{C} \cup \{\infty\}$ sends (z, w) to z/w.

Problem 13 State the definition of a deformation retract.
Show with explicit formulas that S^1 is a deformation retract of $\mathbb{R}^2 \setminus \{0\}$.

Solution: A subspace $A \subset X$ is a deformation retract if there is a map $r : X \to A$ such that $r|_A = 1_A$ and $r \sim 1_X$. The map $r : \mathbb{R}^2 \setminus \{0\} \to S^1$ is $x \mapsto x/\|x\|$, and the homotopy $h : r \sim 1_X$ sends $(x, t) \in (\mathbb{R}^2 \setminus \{0\}) \times [0, 1]$ to $(1 - t)x + t(x/\|x\|) \in \mathbb{R}^2 \setminus \{0\}$.

Problem 14 Fix a number n. Show that if there exists a triangulation of the two-dimensional torus T^2 such that at every vertex exactly n triangles meet, then $n = 6$.

Solution: The Euler characteristic $F - E + V$ is equal to $\chi(T^2) = 0$. We have $E = \frac{3}{2}F$ and $V = \frac{3}{n}F$, therefore $0 = F(1 - \frac{3}{2} + \frac{3}{n}) \Rightarrow 1 - \frac{3}{2} + \frac{3}{n} = 0 \Rightarrow n = 6$.

Extra exercises:

Problem 15 Recall the definition of D^n and S^n.
Prove that S^n is homeomorphic to D^n modulo its boundary.

Problem 16 Prove that a CW complex is compact iff it has finitely many cells.

Problem 17 Give the full definition of the term "manifold"
(along with a full description of the terms "Hausdorff" and "paracompact").

Problem 18 Prove that for n even, the manifold $\mathbb{R}P^n$ is not orientable.