Homological algebra (Oxford, fall 2015)

André Henriques

(week 1) October 13th:

Given a ring R, the tensor product over R of a right module M with a left module N is denoted $M \otimes_R N$. It is the abelian group generated by symbols $m_1 \otimes n_1 + \ldots + m_k \otimes n_k$, under the equivalence relation generated by

$$\begin{aligned} (m+m') \otimes n &= m \otimes n + m' \otimes n, \\ m \otimes (n+n') &= m \otimes n + m \otimes n', \\ \text{and} \qquad mr \otimes n &= m \otimes rn. \end{aligned}$$

A chain complex of R-modules $C_\bullet = (C_n, d_n)_{n\in\mathbb{Z}}$ is a collection of R-modules C_n and R-module maps $d_n : C_n \to C_{n-1}$, called 'differentials', subject to the unique axiom $d_n \circ d_{n+1} = 0$. This axiom is sometimes abusively abbreviated $d^2 = 0$.

Given a chain complex $C_\bullet$, we define

$$\begin{aligned} Z_n &:= \ker(d_n) \\ B_n &:= \mathrm{im}(d_{n+1}) \\ H_n &:= Z_n/B_n \end{aligned}$$

Z_n is called the 'module of n-cycles', B_n is called the 'module of n-boundaries', and H_n is called the 'n^{th} homology module' of $C_\bullet$. Note that the axiom $d^2 = 0$ is equivalent to the statement that $B_n \subset Z_n$. So the quotient $H_n = Z_n/B_n$ makes sense.

A chain complex that satisfies $Z_n = B_n$ is called *exact*, and an exact chain complex is also called an 'exact sequence'. It is interesting to note that the operation $- \otimes_{\mathbb{Z}} \mathbb{Z}/2$ sends the exact sequence

$$\ldots 0 \to 0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2 \to 0 \to 0 \ldots$$

to the sequence

$$\ldots 0 \to 0 \to \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{\cong} \mathbb{Z}/2 \to 0 \ldots,$$

which is not exact.

Exercise 1. Show that if p and q are distinct prime numbers, then $\mathbb{Z}/p \otimes_{\mathbb{Z}} \mathbb{Z}/q = 0$.

Exercise 2. Prove that for any abelian group A, there is a canonical isomorphism $A \otimes_{\mathbb{Z}} \mathbb{Z}/2 \cong A/2A$.

Exercise 3. Let k be a field, and let V and W be k-vector spaces. Prove that if $\{v_1, \ldots, v_n\}$ is a basis of V and $\{w_1, \ldots, w_m\}$ is a basis of W, then $\{v_i \otimes w_j \,|\, i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}\}$ is a basis of $V \otimes_k W$.

Exercise 4. Prove that for any ring R and any left R-module M, there is a canonical isomorphism $R^n \otimes_R M \cong M^n$.

Exercise 5. Show that if k is a field and if V is a k-vector space, then the operation $- \otimes_k V$, sends exact sequences to exact sequences.

Exercises 1–5 should be handed in on Tuesday Oct 20th at 5pm in the hand-in area in the mezzanine.

October 15th:

So far, we've seen chain complexes (as well as exact sequences, and short exact sequences, cycles, boundaries, homology, ...) of abelian groups and, more generally, of R-modules. Does it make sense to talk about chain complexes of sets? How about chain complexes of groups? We'll see that the answer is *no* (or maybe one could, but they would behave very differently from chain complexes of R-modules).

It only really makes sense to talk about chain complexes of *blah*'s if *blah*'s form an <u>abelian category</u> (and the category of sets or that of groups are not abelian categories).

A category is a thing that has a collection of *objects* and that has, for any two objects x and y, a set of *morphisms* from x to y. The collection of all morphisms from x to y is denoted $\mathrm{Hom}(x,y)$ and one writes $f : x \to y$ to indicate that f is an element of $\mathrm{Hom}(x,y)$ (even though f might not be a map in the set-theoretic sense). Every object x should have an identity morphism $1_x : x \to x$, and there should be a composition operation

$$(f,g) \mapsto g \circ f : \mathrm{Hom}(x,y) \times \mathrm{Hom}(y,z) \to \mathrm{Hom}(x,z).$$

This is all subject to the axioms $f \circ 1 = f$, $1 \circ f = f$, and $(f \circ g) \circ h = f \circ (g \circ h)$.

Given two categories $\mathcal{C}$ and $\mathcal{D}$, a functor $F : \mathcal{C} \to \mathcal{D}$ is a thing that sends objects of $\mathcal{C}$ to objects of $\mathcal{D}$, and that sends morphisms $x \to y$ to morphisms $F(x) \to F(y)$. It should satisfy $F(1_x) = 1_{F(x)}$ and $F(f \circ g) = F(f) \circ F(g)$. *Example:* taking the n^{th} homology module defines a functor

$$\begin{aligned} H_n : \{\text{Chain complexes of } R\text{-modules}\} &\to \{R\text{-modules}\}. \\ C_\bullet &\mapsto H_n(C_\bullet) \\ (f_\bullet : C_\bullet \to D_\bullet) &\mapsto (H_n(f_\bullet) : H_n(C_\bullet) \to H_n(D_\bullet)) \end{aligned}$$

A *terminal object* is an object that admits exactly one morphism to it from any other object. An *initial object* is an object that admits exactly one morphism from it to any other object. A *zero object* is an object that is both initial and terminal.

A *product* of two objects $x, y \in \mathcal{C}$ is a diagram

$$\begin{tikzcd} z \arrow[r] \arrow[d] & x \\ y & \end{tikzcd}$$

that satisfies the universal property

$$\begin{tikzcd} \forall z' \arrow[dr, "\exists!"] \arrow[drr, "\forall"] \arrow[ddr, "\forall"'] & & \\ & z \arrow[r] \arrow[d] & x \\ & y & \end{tikzcd}$$

Exercise 6. Check that, in the category of sets, the Cartesian product $X \times Y := \{(x,y) \,|\, x \in X, y \in Y\}$, along with it two projections $(x,y) \mapsto x$ and $(x,y) \mapsto y$, satisfies the universal property of a product.

(week 2) October 20th:

Category:	Sets	Groups	Ab. groups	com. rings
terminal object	$\{*\}$	$\{e\}$	$\{0\}$	$\{*\}$
initial object	$\emptyset$	$\{e\}$	$\{0\}$	$\mathbb{Z}$
zero object	$\nexists$	✓	✓	$\nexists$
product	$X \times Y$	$G \times H$	$A \times B$	$R \times S$
coproduct	$X \sqcup Y$	$G * H$	$A \times B$	$R \otimes_{\mathbb{Z}} S$

In a category with zero object, the *zero map* from X to Y is the composite of the unique map $X \to 0$ with the unique map $0 \to Y$.

The tensor product $M \otimes_R N$, along with the map $M \times N \to M \otimes_R N : (m, n) \mapsto m \otimes n$, has the following universal property. For every abelian group A and every bilinear map $f : M \times N \to A$ that satisfies $f(mr, n) = f(m, rn)$, there is a unique linear map $M \otimes_R N \to A$ that makes the diagram

$$\begin{tikzcd} & & A \\ M \times N \arrow[r] \arrow[urr] & M \otimes_R N \arrow[ur] & \end{tikzcd}$$

commute.

Note that this cannot be expressed purely in the language of category theory, because there is no such thing as a "bilinear morphism".

A category *enriched over* $(Ab, \otimes_{\mathbb{Z}})$ is a category whose hom-sets are equipped with the structure of abelian groups, and whose composition is a morphism

$$\begin{aligned} \mathrm{Hom}(A, B) \otimes_{\mathbb{Z}} \mathrm{Hom}(B, C) &\to \mathrm{Hom}(A, C) \\ f \otimes g &\mapsto g \circ f \end{aligned}$$

A category is called *additive* if it is enriched over $(Ab, \otimes_{\mathbb{Z}})$, it has a zero object, and it admits finite products.

Lemma: In an additive category, product and coproduct agree. I.e., given two objects A and B, there are canonical maps $\iota_A : A \to A \times B$ and $\iota_B : A \to A \times B$ that exhibit $A \times B$ as the coproduct of A and B. (See Kobi's notes for a proof)

In any additive category, it is customary to write $A \oplus B$ for the product (equivalently coproduct) of A and B.

Exercise 7. Let R and S be two commutative rings, and let $R \sqcup S$ denote their coproduct in the category of commutative rings. Show that the underlying abelian group of $R \sqcup S$ is canonically isomorphic to $R \otimes_{\mathbb{Z}} S$. Equivalently, equip $R \otimes_{\mathbb{Z}} S$ with a ring structure, and construct two ring homomorphisms $R \to R \otimes_{\mathbb{Z}} S$ and $S \to R \otimes_{\mathbb{Z}} S$ which exhibit this ring as the coproduct of R and S.

Exercise*. Guess the definition of "C is a category enriched over $(D, \otimes)$". Write it down on paper. Once you've written it down to the best of your ability, check your answer against some online source.

October 22nd:

The *kernel* of a map $f : X \to Y$ is a morhpism $i : K \to X$ which is universal w.r.t the property that $f \circ i = 0$. Dually, the *cokernel* of a map $f : X \to Y$ is a morhpism $q : Y \to C$ which is universal w.r.t the property that $q \circ f = 0$.

In the category of Banach spaces, taking the cokernel means modding out by the *closure* of the image.

A *monomorphism* is a map f that satisfies $f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2$. Dually, an *epimorphism* is a map f that satisfies $g_1 \circ f = g_2 \circ f \Rightarrow g_1 = g_2$.

Definition of *abelian category* (see Kobi's notes): abelian categories are the general setup in which one can do homological algebra.

Lemma: kernels are monomorphisms; cokernels are epimorphisms.

Lemma: In an abelian category, a morphism which is simultaneously mono and epi is in fact an isomorhpism. (See Kobi's notes for a proof)

Exercise 8. Show that, in he category of R-modules, epimorphisms are the same thing as surjective maps, and monomorphisms are the same thing as injective maps.

An abelian group A is called *torsion-free* if $\forall n \in \mathbb{Z} \setminus \{0\}. \forall a \in A \setminus \{0\}. na \neq 0$.

Exercise 9. Show that, in he category of torsion-free abelian groups, the morphism $\cdot 2 : \mathbb{Z} \to \mathbb{Z}$ is an epimorphism (even though it's not surjective).

An abelian group A is called *divisible* if $\forall n \in \mathbb{Z} \setminus \{0\}. \forall a \in A. \exists b \in A. nb = a$.

Exercise 10. Show that, in he category of divisible abelian groups, the projection $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is a monomorphism (even though it's not injective).

Exercise 11. Show that, in an additive category, a morphism $f : X \to Y$ is a monomorphism if and only if its kernel is zero.

Exercises 6–11 should be handed in on Tuesday Oct 27th at 5pm in the hand-in area in the mezzanine.

(week 3) October 27th:

A sort exact sequence $0 \to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0$ is *split* if there exists a splitting $s : C \to B$ of the map $\pi : B \to C$, equivalently, if the map $\iota : A \to B$ admits a retraction $r : B \to A$. As opposed to the property of being a short exact sequence, the property of being a split short exact sequence is *equationally defined* (see Exercise 12). It follows that additive functors always send split short exact sequences to split short exact sequences.

An *exact* functor is one that sends short exact sequences to short exact sequences. A *right exact functor* is an additive functor such that for every short exact sequence $0 \to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0$ the sequence $FA \xrightarrow{F\iota} FB \xrightarrow{F\pi} FC \to 0$ is exact. A *left exact functor* is one such that $0 \to FA \xrightarrow{F\iota} FB \xrightarrow{F\pi} FC$ is exact.

Lemma 1. *The functor $- \otimes_R N$ is right exact.*

Proof. Given a short exact sequence of right R-modules $0 \to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0$, we need to show that $A \otimes_R N \to B \otimes_R N \to C \otimes_R N \to 0$ is exact. The surjectivity of $B \otimes_R N \to C \otimes_R N$ is easy, so let us focus on the harder argument: given an element $\sum b_i \otimes n_i \in B \otimes_R N$ that goes to zero in $C \otimes_R N$, we need to show that it comes from $A \otimes_R N$.

Since $\sum \pi(b_i) \otimes n_i = 0$ in $A \otimes_R N$, there exist elements c'_α, c''_α, n_α, c_β, n'_β, n''_β, c_γ, r_γ, n_γ such that

$$\begin{aligned}\sum_i \pi(b_i) \otimes n_i + \sum_\alpha (c'_\alpha + c''_\alpha) \otimes n_\alpha - c'_\alpha \otimes n_\alpha - c''_\alpha \otimes n_\alpha \\ + \sum_\beta c_\beta \otimes (n'_\beta + n''_\beta) - c_\beta \otimes n'_\beta - c_\beta \otimes n''_\beta \\ + \sum_\gamma c_\gamma r_\gamma \otimes n_\gamma - c_\gamma \otimes r_\gamma n_\gamma\end{aligned}$$

is zero in the free abelian group on the set of symbols "$c \otimes n$". If we mod out that free abelian group by the first set of relations $(c' + c'') \otimes n = c' \otimes n + c'' \otimes n$, then we get the abelian group $\bigoplus_{n \in N} C$. So, another way of saying that $\sum \pi(b_i) \otimes n_i$ is zero in $A \otimes_R N$ is to say that there exist elements c_β, n'_β, n''_β, c_γ, r_γ, n_γ such that

$$\sum_i \pi(b_i) \otimes n_i + \sum_\beta c_\beta \otimes (n'_\beta + n''_\beta) - c_\beta \otimes n'_\beta - c_\beta \otimes n''_\beta + \sum_\gamma c_\gamma r_\gamma \otimes n_\gamma - c_\gamma \otimes r_\gamma n_\gamma = 0 \text{ in } \bigoplus_{n \in N} C,$$

where "$c \otimes n$" now stands for the element c put in the n-th copy of C.

Pick preimages b_β, $b_\gamma \in B$ of $c_\beta, c_\gamma \in C$, and consider the element

$$y := \sum_i b_i \otimes n_i + \sum_\beta b_\beta \otimes (n'_\beta + n''_\beta) - b_\beta \otimes n'_\beta - b_\beta \otimes n''_\beta + \sum_\gamma b_\gamma r_\gamma \otimes n_\gamma - b_\gamma \otimes r_\gamma n_\gamma \quad \in \quad \bigoplus_{n \in N} B.$$

This element goes to 0 in $\bigoplus_{n \in N} C$ and therefore comes from some $x \in \bigoplus_{n \in N} A$.

Let $[x]$ denote the image of x in $A \otimes_R N$ and let $[y]$ denote the image of y in $B \otimes_R N$. Since $x \mapsto y$, it follows that $[x] \mapsto [y]$. We are done since $[y] = \sum_i b_i \otimes n_i$ in $B \otimes_R N$. ∎

Exercise 12. Let A, B, C be objects of an abelian category, and let $\iota : A \to B$, $\pi : B \to C$, $s : C \to B$, $r : B \to A$ be morphisms such that $\pi \circ \iota = 0$, $\pi \circ s = 1_C$, $r \circ \iota = 1_A$, $s \circ \pi + \iota \circ r = 1_B$. Show that (1) ι and s exhibit B as the sum of A and C;[1] (2) π and r exhibit B as the product of A and C; (3) ι and π form a short exact sequence; (4) s and r form a short exact sequence.

Exercise 13. Let $\mathcal{C}$ be an abelian cateogry. Show that $\mathrm{Hom}(M, -) : \mathcal{C} \to Ab$ and $\mathrm{Hom}(-, N) : \mathcal{C}^{op} \to Ab$ are left exact. Namely, given a short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{C}$ show that the sequences

$$\begin{aligned} &0 \to \mathrm{Hom}(M, A) \to \mathrm{Hom}(M, B) \to \mathrm{Hom}(M, C) \\ \text{and} \qquad &0 \to \mathrm{Hom}(C, N) \to \mathrm{Hom}(B, N) \to \mathrm{Hom}(A, N) \end{aligned}$$

are exact. Illustrate with examples that the above two functors are typically not exact.

October 29th: A module P is *projective* if $\mathrm{Hom}_R(P, -)$ is exact. A module I is *injective* if $\mathrm{Hom}_R(-, I)$ is exact. A module F is *flat* if $- \otimes_R F$ is exact. Equivalently, P is projective if for every solid arrow diagram there exists a dotted arrow making the diagram commute:

$$\begin{tikzcd} & P \arrow[dl, dotted, "\exists"'] \arrow[dr] & \\ B \arrow[rr, two heads] & & C \end{tikzcd}$$

In the same vein, I is injective if for every solid arrow diagram there exists a dotted arrow making the diagram commute:

$$\begin{tikzcd} & I & \\ A \arrow[ur] \arrow[rr, tail] & & B \arrow[ul, dotted, "\exists"'] \end{tikzcd}$$

Finally, a module is flat iff for every $A \rightarrowtail B$, the corresponding map $A \otimes_R F \to B \otimes_R F$ is injective.

Lemma 2. *An R-module is projective iff it's a direct summand of a free module.*

Proof. Let P be projective. Pick a surjective map from a free module $\pi : F \twoheadrightarrow P$. Since P is projective, π admits a splitting $s : P \to F$, the short exact sequence $0 \to \ker(\pi) \to F \to P \to 0$ splits, and $F = P \oplus \ker(\pi)$. □

If $R =$ (functions on some space X), and $V \to X$ is a vector bundle over X, then the set of sections of V is a projective module over R.

Lemma 3. *An abelian group is injective iff it is divisible.*

Proof. • Injective ⇒ Divisible: Let I be injective, and let $x \in I$ be an element. Given $m \in \mathbb{N}$, we need to show that $\exists y \in I$ such that $my = x$. Let $\langle x \rangle \subset I$ be the subgroup generated by x. Consider the diagram

$$\begin{tikzcd} I & \\ I \arrow[u, "\mathrm{id}"] \arrow[r, tail] & (I \oplus \mathbb{Q})/\{(bx, b) : b \in \mathbb{Z}\} \arrow[ul, dotted, "\exists"'] \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} I & \\ I \arrow[u, "\mathrm{id}"] \arrow[r, tail] & (I \oplus \mathbb{Q}/n\mathbb{Z})/\{(bx, b) : b \in \mathbb{Z}/n\mathbb{Z}\} \arrow[ul, dotted, "\exists"'] \end{tikzcd}$$

depending on whether $\langle x \rangle \cong \mathbb{Z}$ or $\langle x \rangle \cong \mathbb{Z}/n\mathbb{Z}$. Let $y \in I$ be the image of the element $[(0, -1/m)]$ under the dotted map. Then my is the image of $[(0, -1)] = [(x, 0)]$. But $[(x, 0)]$ is the image of x under the horizontal map, so $my = x$.

[1] The terms "sum" and "coproduct" are synonyms.

- Divisible $\Rightarrow$ Injective: Let D be divisible. We consider an extension problem

$$\begin{tikzcd} & D & \\ A \arrow[ur] \arrow[rr, tail] & & B \arrow[ul, dotted, "\exists ?"'] \end{tikzcd}$$

Let $A = A_0 \subset A_1 \subset A_1 \subset \ldots \subset A_\alpha \subset \ldots \subset B$ be a sequence of subgroups indexed by ordinals, such that $A_{\alpha+1} = \langle A_\alpha, b_\alpha \rangle$ for some element $b_\alpha \in B$, and $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ when α is a limit ordinal. By transfinite induction, it's enough to solve the extension problem

$$\begin{tikzcd} & D & \\ A_\alpha \arrow[ur] \arrow[rr, tail] & & A_{\alpha+1} \arrow[ul, dotted, "\exists ?"'] \end{tikzcd}$$

If the map $A_\alpha \oplus \mathbb{Z} \to A_{\alpha+1} : (a, n) \mapsto a + nb_\alpha$ is injective, then $A_{\alpha+1} = A_\alpha \oplus \mathbb{Z}$ and we may define the dotted arrow $A_{\alpha+1} \to D$ to send $b_\alpha \mapsto 0$.

If the map $A_\alpha \oplus \mathbb{Z} \to A_{\alpha+1} : (a, n) \mapsto a + nb_\alpha$ has a kernel, call it K, then that kernel is cyclic, generated by some element: $K = \langle (a_0, n_0) \rangle$. Let d be the image of a_0 in D, and let $e \in D$ be an element such that $n_0 e = d$. Then $b_\alpha \mapsto -e$ defines a map $A_{\alpha+1} = (A_\alpha \oplus \mathbb{Z})/K \to D$. $\square$

An abelian group is injective iff it's a $\bigoplus$ of (possibly infinitely many) copies $\mathbb{Q}$ and $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$. Given a field k, a $k[x]$-module is injective iff it's a $\bigoplus$ of (possibly infinitely many) copies $k(x)$ and $k[y, y^{-1}]/k[y]$ for $y = x - a$, $a \in k$.

Exercise 14. Prove that, in the category of abelian groups, every projective module is free. More generally, if R is a principal ideal domain, prove that in the category of R-modules every projective module is free.

Exercise 15. [2] Show that an abelian group is injective iff it's a $\bigoplus$ of (possibly infinitely many) copies $\mathbb{Q}$ and $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$.

Hint: Work by transfinite induction. Pick a non-zero element $x \in I$. If it's not torsion, construct a map $\mathbb{Q} \to I$, and use it to write I as a direct sum $I = I_0 \oplus \mathbb{Q}$. If it's torsion, assume wlog that its order is of the form p^n, construct a map $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z} \to I$, and use it to write I as a direct sum $I = I_0 \oplus \mathbb{Z}[\frac{1}{p}]/\mathbb{Z} \to I$. Don't forget to include a discussion of what happens at limit ordinals. Alternatively, you may use Zorn's lemma.

Exercises 12–15 should be handed in on Tuesday Nov 3rd at 5pm.

(week 4) November 3rd:

Stated without proof (or any kind of further explanation): If R is a commutative Noetherian ring, then there is a classification theorem for injective R-modules. For every prime ideal $\mathfrak{p} \subset R$, there is a certain injective module $I_\mathfrak{p}$, and any injective module is a direct sum of (possibly infinitely many) such.

Lemma 4. *A $\mathbb{Z}$-module is flat iff it's torsion free.*

Proof. $\Rightarrow$: If A is not torsion free, then

$$\exists\, n \in \mathbb{N} \text{ s.t. } \qquad A \otimes_\mathbb{Z} (\mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z}) = (A \otimes_\mathbb{Z} \mathbb{Z} \xrightarrow{\cdot n} A \otimes_\mathbb{Z} \mathbb{Z}) = (A \xrightarrow{\cdot n} A)$$

[2] Exercise 15 will be marked a first time, and handed back to you. If you wish to improve your answer, you may hand it in a second time the next week.

is not injective. Hence A is not flat.
$\Leftarrow$: Assume A is torsion free. Write A as

$$A = \operatorname{colim}_{\alpha \in P} A_\alpha$$

where P is the poset of finitely generated subgroups of A, and $A_\alpha \subset A$ is the subgroup that corresponds to α (which is to say $A_\alpha := \alpha$). By the classification theorem of finitely generated abelian groups, since each A_α is a finitely generated torsion-free abelian group, it's a free $\mathbb{Z}$-module. In particular, $A_\alpha \otimes_{\mathbb{Z}} -$ preserves monomorphisms. Given any $\mathbb{Z}$-module M, we have

$$A \otimes_{\mathbb{Z}} M = (\operatorname{colim}_{\alpha \in P} A_\alpha) \otimes_{\mathbb{Z}} M = \operatorname{colim}_{\alpha \in P}(A_\alpha \otimes_{\mathbb{Z}} M), \tag{1}$$

where the last equality is justified in Exercise 17. Therefore

$$A \otimes_{\mathbb{Z}} (M_1 \rightarrowtail M_2) = \underbrace{\operatorname{colim}_{\alpha \in P} \Big(\underbrace{A_\alpha \otimes_{\mathbb{Z}} (M_1 \rightarrowtail M_2)}_{\text{injective because } A_\alpha \text{ is free}} \Big)}_{\text{injective because } \operatorname{colim}_{\alpha \in P} \text{ preserves monomorphisms}}$$

and so A is flat. The very last step is justified in Exercise 18. It depends crucially on the fact that that P is a *directed* poset, i.e., that $\forall x, y \in P$, $\exists z \in P$ s.t. $z \geq x$ and $z \geq y$. $\square$

A *colimit* (also called *direct limit*) of a diagram is an object to which all the things in the diagram map, and which is universal w.r.t. that property:

$$\operatorname{colim}\left(\begin{tikzcd} A \arrow[r] \arrow[drr] & C \arrow[r] \arrow[dr] \arrow[drr] & E \arrow[r] \arrow[dr] & G & \cdots \\ B \arrow[r] \arrow[ur] & D \arrow[r] \arrow[urr] & F \arrow[r] & H & \cdots \end{tikzcd}\right) =$$

$$\begin{tikzcd} A \arrow[r] \arrow[drr] \arrow[ddrrrrr, dotted] & C \arrow[r] \arrow[dr] \arrow[drr] \arrow[ddrrrr, dotted] & E \arrow[r] \arrow[dr] \arrow[ddrrr, dotted] & G \arrow[ddrr, dotted] & \cdots & \\ B \arrow[r] \arrow[ur] \arrow[drrrrr, dotted] & D \arrow[r] \arrow[urr] \arrow[drrrr, dotted] & F \arrow[r] \arrow[drrr, dotted] & H \arrow[drr, dotted] & \cdots & \\ & & & & & Z \end{tikzcd}$$

(all little triangles commute)

s.t. for any other Z' with maps $A \to Z'$, $B \to Z'$, etc making the little triangles commute there exists a unique map $Z \to Z'$ s.t. yet more triangles commute. If the diagram is indexed by a *directed poset*, and if all the maps involved are inclusions, then 'colimit' is just the same thing as 'union'. The direct limit is denoted $\varinjlim$.

The dual notion to a colimt is is called a *limit* equivalently an *inverse limit*. The inverse limit is denoted $\varprojlim$.

A *projective resolution* of an object M of some abelian category is a chain complex $P_\bullet = (P_n, d_n : P_n \to P_{n-1})$ s.t. each P_n is projective, $P_n = 0$ for $n < 0$, and $H_i(P_\bullet)$ is concentrated in degree zero, where it's M. An *injective resolution* of an object M of some abelian category is a cochain complex $I^\bullet = (I^n, d^n : I^n \to I^{n+1})$ s.t. each I^n is injective, $I^n = 0$ for $n < 0$, and $H^i(I^\bullet)$ is concentrated in degree zero, where it's M. A *free resolution* is a projective resolution s.t. all the P_n's are free. Here's how you build a free resolution:

$$\begin{tikzcd} & \ker & \ker & \ker & \\ \cdots \arrow[ur, two heads] & \cap & \cap & \cap & \\ F_3 \arrow[uur, two heads] & F_2 \arrow[uur, two heads] & F_1 \arrow[uur, two heads] & F_0 \arrow[r, two heads] & M. \end{tikzcd}$$

The free resolution is then given by $F_\bullet = (\ldots \to F_3 \to F_2 \to F_1 \to F_0 \to 0 \to 0 \ldots)$. Dually, injective resolutions are built in the following way:

$$\begin{tikzcd}
 & \mathrm{coker} \arrow[dr] & \mathrm{coker} \arrow[dr] & \cdots \\
M \arrow[r, tail] & I_0 \arrow[u, two heads] & I_1 \arrow[u, two heads] & I_2
\end{tikzcd}$$

For such constructions to be possible, the abelian category needs to have *enough projectives* (respectively *enough injectives*), meaning that each object admits an epimorphism from a projective (a monomorphism to an injective).

The *Tor* and *Ext* groups are defined as follows:

Let M be a right R-module and N a left R-module. Then:

$$\mathrm{Tor}_i^R(M,N) = H_i(P_\bullet \otimes_R N) = H_i(M \otimes_R Q_\bullet)$$

where $P_\bullet$ is a projective resolution of M or $Q_\bullet$ is a projective resolution of N. Implicit in the above definition is the fact that $\mathrm{Tor}_i^R(M,N)$ doesn't depend on the choice of projective resolution, and doesn't depend on whether one resolves M or N (or both).

Let M and N be R-modules (either both right modules or both left modules). Then:

$$\mathrm{Ext}_R^i(M,N) = H^i(\mathrm{Hom}_R(P_\bullet, N)) = H^i(\mathrm{Hom}_R(M, I^\bullet)).$$

Here, $P_\bullet$ is a projective resolution of M and $I^\bullet$ is an injective resolution of N. Once again, the choice of resolution doesn't matter, neither does the choice of which of the two modules one decides to resolve.

Exercise 16. Disprove the following statement: "every torsion-free abelian group without p-divisible elements is free".

Exercise 17. Let P be a directed poset,[3] and let $\{N_\alpha\}_{\alpha\in P}$ be a diagram of right R-modules (and let's not assume that the maps in the diagram are injective). Show that for any left R-module M, there is a canonical isomorhpism

$$(\mathrm{colim}_{\alpha\in P} N_\alpha) \otimes_R M \;\cong\; \mathrm{colim}_{\alpha\in P}(N_\alpha \otimes_R M).$$

Exercise 18. Let $\{N_\alpha\}_{\alpha\in P}$ and $\{M_\alpha\}_{\alpha\in P}$ be two diagams of R-modules indexed by the same poset P. Let us also assume that P is a directed poset. Let $\{f_\alpha : N_\alpha \to M_\alpha\}_{\alpha\in P}$ be a natural transformation between the above two diagrams (see Lecture 3 of Kobi's notes for a definition). Show that if each f_α is a monomorphism, then

$$\mathrm{colim} f_\alpha : \mathrm{colim} N_\alpha \to \mathrm{colim} M_\alpha$$

is also a monomorphism

This exercise can be interpreted in the following way: let $(R\text{-Mod})^P$ denote the abelian category whose objects are P-indexed diagrams of R-modules, and whose morphisms are natural transformations between such diagrams. Then the functor

$$\mathrm{colim} \,:\, (R\text{-Mod})^P \longrightarrow R\text{-Mod}$$

is exact.

[3]The result is also true for arbitrary posets. But then, you just can't use the interpretation of 'colim' as 'union modulo relations'. For example, the direct sum is an instance of a colimit.

Exercise 19. Compute all the entires of the following tables:

$\mathrm{Tor}_1^{\mathbb{Z}}(A,B)$	$\mathbb{Z}$	$\mathbb{Z}/m\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{Q}/\mathbb{Z}$
$\mathbb{Z}$				
$\mathbb{Z}/n\mathbb{Z}$				
$\mathbb{Q}$				
$\mathbb{Q}/\mathbb{Z}$				

and

$\mathrm{Ext}^1_{\mathbb{Z}}(A,B)$	$\mathbb{Z}$	$\mathbb{Z}/m\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{Q}/\mathbb{Z}$
$\mathbb{Z}$				
$\mathbb{Z}/n\mathbb{Z}$				
$\mathbb{Q}$				
$\mathbb{Q}/\mathbb{Z}$				

Hint: some of the above Ext groups are best expressible in terms of the *profinite completion* $\widehat{\mathbb{Z}}$ of $\mathbb{Z}$. The profinite completion of $\mathbb{Z}$ is the inverse limit of all the $\mathbb{Z}/n\mathbb{Z}$, where the indexing poset is $\mathbb{N}$ ordered by 'a divides b'. Equivalently, $\widehat{\mathbb{Z}} = \varprojlim_{n\in\mathbb{N}} \mathbb{Z}/n!\mathbb{Z}$ with the usual order on $\mathbb{N}$.

Exercises 16–20 should be handed in on Tuesday Nov 10th at 5pm.

November 5th:

I started by computing $\mathrm{Tor}_*^{k[x,y]}\big(k[x,y]/(x,y), k[x,y]/(x-a,y-b)\big)$. It's identically zero unless $(0,0)=(a,b)$.

The *homological dimension* of a (let's say commutative) ring is the max over all R-modules of the shortest possible length of a projective resolution. This number turns out to also be the max over all modules of the length of a shortest injective resolution. If the ring is not commutative, then you can do this with left modules, and you can do this with right modules, and there's no guarantee that those two notions of homological dimension should agree: there's the homological dimension of the category of left R-modules (which one can compute using either injective or projective resolutions), and there's the homological dimension of the category of right R-modules.

Example: Every subgroup of a free abelian group is free, and so $\mathrm{hdim}(\mathbb{Z}) = 1$. The same can be concluded from the observation that every quotient of a divisible group is divisible. As a consequence, in the category of $\mathbb{Z}$-modules, $\mathrm{Tor}_n(A,B)=0$ and $\mathrm{Ext}^n(A,B)=0$ for every $n\geq 2$.

Example: If $R=\mathcal{O}_X$ is the ring of functions on some smooth affine variety X (defined over some field), then $\mathrm{hdim}(\mathcal{O}_X)=\dim(X)$.

Example: $\mathrm{hdim}(\mathbb{Z}/p^2)=\infty$, as can be seen from computing $\mathrm{Tor}_*^{\mathbb{Z}/p^2}(\mathbb{Z}/p,\mathbb{Z}/p)$.

Given a right exact functor $F:\mathcal{C}\to\mathcal{D}$ between abelian categories, the n-th *left derived functor* is given by

$$L_nF : M \mapsto H_n(F(P_\bullet))$$

where $P_\bullet$ is a projective resolution of M. The *right derived functor* of a left exact functor is defined similarly:

$$R^nF : M \mapsto H^n(F(I^\bullet))$$

where $I^\bullet$ is now an injective resolution.

Lemma 5. *Let $F:\mathcal{C}\to\mathcal{D}$ be a right exact functor. Then $L_0F=F$ (and similarly $R^0F=F$ is F is left exact).*

Proof. We must show that $L_0F(M)=F(M)$. The former is computed as

$$L_0F(M) = H_0\Big(\dots \overset{F(d_2)}{\to} F(P_1) \overset{F(d_1)}{\to} F(P_0)\to 0\Big) = F(P_0)/\mathrm{im}(F(d_1)),$$

where $P_\bullet$ is a projective resolution of M. Now

$$0\to P_1/\ker(d_1)\to P_0\to M\to 0$$

is exact, and so

$$F(P_1/\ker(d_1))\to F(P_0)\to F(M)\to 0$$

is also exact. Since $P_1 \to P_1/\ker(d_1)$ is epi and F preserves epis, the map $F(P_1) \to F(P_1/\ker(d_1))$ is epi, from which we deduce that the sequence

$$F(P_1) \overset{F(d_1)}{\to} F(P_0) \to F(M) \to 0$$

is also exact. We conclude that $F(M) = F(P_0)/\mathrm{im}(F(d_1))$, as desired. $\square$

A useful fact for computing Tor is that if P is a filtered poset, we have

$$\mathrm{Tor}(\mathrm{colim}\, M_\alpha, N) = \mathrm{colim}\, \mathrm{Tor}(M_\alpha, N)$$

Warning! The corresponding reasonable statement that one might conjecture for Ext fails miserably:

$$\mathrm{Ext}(\mathrm{colim}\, M_\alpha, N) \neq \lim \mathrm{Ext}(M_\alpha, N) \qquad \mathrm{Ext}(M, \lim N_\alpha) \neq \lim \mathrm{Ext}(M, N_\alpha)$$

(week 5) November 10th:

There are two ways of making the operation "take a projective resolution" into a functor:

(1) Take P_0 to be the free R-module on the underlying set of M. Take P_1 to be the free R-module on the underlying set of $\ker(P_0 \to M)$. Take P_2 to be the free R-module on the underlying set of $\ker(P_1 \to P_0)$. Etc. This construction has the advantage that if say a group G acts on M, then that group will also automatically act on the resolution $P_\bullet$ of M.

(2) View the operation "take a projective resolution" as a functor from our abelian category $\mathcal{C}$ to its derived category $D(\mathcal{C})$.

Definition: Let $\mathcal{A}$ be an abelian category. It's *derived category* $D(\mathcal{A})$ has:
- Object = chain complexes of projectives of $\mathcal{A}$
- Morphisms = chain maps modulo chain homotopy. (see Lecture 7 of Kobi's notes for a definition)

The notion of chain homotopy is made so that whenever $f_\bullet : C_\bullet \to D_\bullet$ and $g_\bullet : C_\bullet \to D_\bullet$ are chain homotopic maps, then $H_*(f_\bullet) = H_*(g_\bullet) : H_*(C_\bullet) \to H_*(D_\bullet)$. The fact that

$$\text{"take any projective resolution"} \;:\; \mathcal{A} \to D(\mathcal{A})$$

is a functor is proved in Prop 7.2 of Lecture 7 of Kobi's notes.

Here's a way of defining the nth derived functor of some right exact functor $F : \mathcal{A} \to \mathcal{B}$:

$$L_nF \;:\; \mathcal{A} \xrightarrow[\text{resolution}]{\text{take projective}} D(\mathcal{A}) \xrightarrow{\text{apply } F} \left(\mathrm{Ch}(\mathcal{B}); \tfrac{\text{chain maps modulo}}{\text{chain homotopy}}\right) \xrightarrow{H_n} \mathcal{B}$$

The *total derived functor* of F, or simply "the derived functor of F" is the functor

$$LF \;:\; \mathcal{A} \xrightarrow[\text{resolution}]{\text{take projective}} D(\mathcal{A}) \xrightarrow{\text{apply } F} \left(\mathrm{Ch}(\mathcal{B}); \tfrac{\text{chain maps modulo}}{\text{chain homotopy}}\right) \xrightarrow[\text{resolution}]{\text{take projective}} D(\mathcal{B}).$$

Exercise 20. Let R be a ring and let $r \in R$ be a non-zero-divisor. Show that $\mathrm{Tor}_1^R(R/rR, -)$ is the functor which sends a module to its submodule of r-torsion elements.[4]

Exercise 21. Consider the following two proofs:

[4] This is where the notation "Tor" comes from.

Lemma A: Let $\{M_\alpha\}_{\alpha\in P}$ be a diagram of R-modules indexed by some filtered poset P. Then we have a canonical isomorphism $$\mathrm{Tor}_n(\operatorname{colim} M_\alpha, N) = \operatorname{colim} \mathrm{Tor}_n(M_\alpha, N).$$ *Proof.* Let $F^\alpha_\bullet$ be the canonical free resolution of M_α. Then $F_\bullet := \operatorname{colim}_{\alpha\in P} F^\alpha_\bullet$ is the canonical free resolution of $M := \operatorname{colim}_{\alpha\in P} M_\alpha$. We then have $$\begin{aligned} &\operatorname{colim}_{\alpha\in P} \mathrm{Tor}^R_n(M_\alpha, N)\\ =&\operatorname{colim}_{\alpha\in P} H_n(F^\alpha_\bullet \otimes_R N)\\ =&H_n(\operatorname{colim}_{\alpha\in P}(F^\alpha_\bullet \otimes_R N))\\ =&H_n((\operatorname{colim}_{\alpha\in P} F^\alpha_\bullet) \otimes_R N)\\ =&H_n(F_\bullet \otimes_R N) = \mathrm{Tor}^R_n(M, N). \end{aligned}$$ □	**Lemma B:** Let $\{M_\alpha\}_{\alpha\in P}$ be a diagram of R-modules indexed by some filtered poset P. Then we have a canonical isomorphism $$\mathrm{Ext}^n(\operatorname{colim} M_\alpha, N) = \lim \mathrm{Ext}^n(M_\alpha, N).$$ *Proof.* Let $F^\alpha_\bullet$ be the canonical free resolution of M_α. Then $F_\bullet := \operatorname{colim}_{\alpha\in P} F^\alpha_\bullet$ is the canonical free resolution of $M := \operatorname{colim}_{\alpha\in P} M_\alpha$. We then have $$\begin{aligned} &\lim_{\alpha\in P} \mathrm{Ext}^R_n(M_\alpha, N)\\ =&\lim_{\alpha\in P} H^n(\mathrm{Hom}_R(F^\alpha_\bullet, N))\\ =&H^n(\lim_{\alpha\in P}(\mathrm{Hom}_R(F^\alpha_\bullet, N)))\\ =&H^n(\mathrm{Hom}_R(\operatorname{colim}_{\alpha\in P} F^\alpha_\bullet, N))\\ =&H^n(\mathrm{Hom}_R(F_\bullet, N)) = \mathrm{Ext}^n_R(M, N) \end{aligned}$$ □

Lemma A is correct and its proof is correct, but Lemma B is wrong and its proof is flawed. Find the error in the proof of Lemma B, and illustrate the mistake by providing a counterexample. Explain (by filling the details of the proof) why the corresponding step in the proof of Lemma A is ok.

Exercise 22. [5] Let A and C be two objects of some abelian category. Show that there is a canonical bijection between the abelian group $\mathrm{Ext}^1(C, A)$ and the set of isomorphism classes of extension $0 \to A \to B \to C \to 0$.[6] Here, two extensions (= short exact sequences) are called isomorphic if they fit into a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\mathrm{id}_A"] & B \arrow[r] \arrow[d, "\simeq"] & C \arrow[r] \arrow[d, "\mathrm{id}_C"] & 0 \\ 0 \arrow[r] & A \arrow[r] & B' \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

The main steps are as follows:

▶ Given an extension $0 \to A \to B \to C \to 0$ and a resolution $P_\bullet \to C$, construct a map of chain complexes from $(P_\bullet \to C \to 0)$ to $(0 \to A \to B \to C \to 0)$.
▶ Use this to define an element of $\mathrm{Ext}^1(C, A)$.
▶ Show that the resulting element of $\mathrm{Ext}^1(C, A)$ does not depend on the choice of map from $(P_\bullet \to C \to 0)$ to $(0 \to A \to B \to C \to 0)$.
▶ Given an element of $\mathrm{Ext}^1(C, A)$ represented by a map $P_1 \to A$ for some resolution $P_\bullet \to C$, define the group $B := (P_0 \oplus A)/P'_1$, where P'_1 is the quotient of P_1 by the image of $d_2 : P_2 \to P_1$.
▶ Show that B fits into a short exact sequence $0 \to A \to B \to C \to 0$.
▶ Show that this short exact sequence does not depend on the choice of resolution $P_\bullet \to C$.
▶ Finally, show that the above constructions are each other's inverses.

Exercises 20–22 should be handed in on Tuesday Nov 17th at 5pm.

November 12th:

A *quasi-isomorhpism* is a map $C_\bullet \to D_\bullet$ between chain complexes, which induces an isomorphism in homology $H_*(C_\bullet) \xrightarrow{\simeq} H_*(D_\bullet)$. A *projective resolution* of a chain complex $C_\bullet$ is a quasi-isomorhpism from a complex of projective modules. If the ambient abelian category has enough projectives, and $C_\bullet$ is bounded below, then projective resolutions always exist.

[5] Exercise 22 will be marked a first time, and handed back to you. If you wish to improve your answer, you may hand it in a second time the next week.

[6] This is where the notation "Ext" comes from.

If $0 \to A \to B \to C \to 0$ is a short exact sequence and F is a right exact functor, then there's a long exact sequence

$$\begin{tikzcd}
\cdots & \cdots & \cdots \arrow[dll] \\
L_2F(A) \arrow[r] & L_2F(B) \arrow[r] & L_2F(C) \arrow[dll] \\
L_1F(A) \arrow[r] & L_1F(B) \arrow[r] & L_1F(C) \arrow[dll] \\
F(A) \arrow[r] & F(B) \arrow[r] & F(C) \arrow[r] & 0
\end{tikzcd} \tag{2}$$

A similar story applies to left exact functors. The main examples which are relevant to us are:

$$\begin{tikzcd}
\cdots & \cdots & \cdots \arrow[dll] \\
\mathrm{Tor}_2^R(A, M) \arrow[r] & \mathrm{Tor}_2^R(B, M) \arrow[r] & \mathrm{Tor}_2^R(C, M) \arrow[dll] \\
\mathrm{Tor}_1^R(A, M) \arrow[r] & \mathrm{Tor}_1^R(B, M) \arrow[r] & \mathrm{Tor}_1^R(C, M) \arrow[dll] \\
A \otimes_R M \arrow[r] & B \otimes_R M \arrow[r] & C \otimes_R M \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
& \cdots & \cdots & \cdots \\
& \mathrm{Ext}_R^2(C, M) \arrow[r] & \mathrm{Ext}_R^2(B, M) \arrow[r] & \mathrm{Ext}_R^2(A, M) \arrow[ull] \\
& \mathrm{Ext}_R^1(C, M) \arrow[r] & \mathrm{Ext}_R^1(B, M) \arrow[r] & \mathrm{Ext}_R^1(A, M) \arrow[ull] \\
0 \arrow[r] & \mathrm{Hom}_R(C, M) \arrow[r] & \mathrm{Hom}_R(B, M) \arrow[r] & \mathrm{Hom}_R(A, M) \arrow[ull]
\end{tikzcd}$$

and

$$\begin{tikzcd}
& \cdots & \cdots & \cdots \\
& \mathrm{Ext}_R^2(N, A) \arrow[r] & \mathrm{Ext}_R^2(N, B) \arrow[r] & \mathrm{Ext}_R^2(N, C) \arrow[ull] \\
& \mathrm{Ext}_R^1(N, A) \arrow[r] & \mathrm{Ext}_R^1(N, B) \arrow[r] & \mathrm{Ext}_R^1(N, C) \arrow[ull] \\
0 \arrow[r] & \mathrm{Hom}_R(N, A) \arrow[r] & \mathrm{Hom}_R(N, B) \arrow[r] & \mathrm{Hom}_R(N, C) \arrow[ull]
\end{tikzcd}$$

The proof of equation (2) goes as follows:

- Start with $0 \to A \to B \to C \to 0$.
- Take projective resolutions while being careful not to destroy the fact that it's a short exact sequence:

$$0 \to P_\bullet \to Q_\bullet \to R_\bullet \to 0.$$

This can be done by virtue of the "horseshoe lemma" (see Kobi's notes: Lemma 6.5 of lecture 6). Note that each $0 \to P_n \to Q_n \to R_n \to 0$ is *split* exact.

- Apply F. Since additive functors preserve split exact sequences (because they are equationally defined), we get a short exact sequence of chain complexes

$$0 \to F(P_\bullet) \to F(Q_\bullet) \to F(R_\bullet) \to 0.$$

• Take homology, and use the fact that the homology of a short exact sequence of chain complexes always yields a long exact sequence of homology groups (see Kobi's notes: Lemma 6.7 of lecture 6).

(week 6) November 17th:

Today, I did a review of direct limits and inverse limits.
Let P be a poset, and let $\big(\{A_\alpha\}_{\alpha\in P}, \{\iota_{\alpha,\beta} : A_\alpha \to A_\beta\}_{\alpha<\beta}\big)$ be a diagram of abelian groups (or R-modules). Then we can form the direct limit

$$\operatorname*{colim}_{\alpha\in P} A_\alpha.$$

This direct limit admits the description

$$\operatorname*{colim}_{\alpha\in P} A_\alpha = \coprod_{\alpha\in P} A_\alpha \Big/ \sim \qquad \begin{array}{l}\text{(where } \sim \text{ is the equivalance relation} \\ \text{generated by } a \sim \iota_{\alpha,\beta}(a) \text{ for } a\in A_\alpha)\end{array}$$

only when P is a directed poset. Otherwise, $\coprod A_\alpha / \sim$ is not even a group (how would you add two elements?), and one needs to use a description of the form $\bigoplus_{\alpha\in P} A_\alpha / \sim$ instead.

Given a poset P and a diagram $\big(\{A_\alpha\}_{\alpha\in P}, \{\pi_{\alpha,\beta} : A_\beta \to A_\alpha\}_{\alpha<\beta\in P}\big)$, the inverse limit admits the following explicit description:

$$\lim_{\alpha\in P} A_\alpha = \Big\{(a_\alpha) \in \prod_{\alpha\in P} A_\alpha \Big|\ \pi_{\alpha,\beta}(a_\beta) = a_\alpha\ \forall \alpha<\beta\in P\Big\}$$

Here are some important examples of limits and colimits:

$$\operatorname{colim}\Big(A \hookrightarrow A^2 \hookrightarrow A^3 \hookrightarrow A^4 \hookrightarrow \dots\Big) = \bigoplus^{\infty} A$$

$$\lim\Big(A \twoheadleftarrow A^2 \twoheadleftarrow A^3 \twoheadleftarrow A^4 \twoheadleftarrow \dots\Big) = \prod^{\infty} A$$

$$\operatorname{colim}\Big(\mathbb{Z}/p \hookrightarrow \mathbb{Z}/p^2 \hookrightarrow \mathbb{Z}/p^3 \hookrightarrow \mathbb{Z}/p^4 \hookrightarrow \dots\Big) = \mathbb{Z}[\tfrac{1}{p}]/\mathbb{Z}$$

$$\lim\Big(\mathbb{Z}/p \twoheadleftarrow \mathbb{Z}/p^2 \twoheadleftarrow \mathbb{Z}/p^3 \twoheadleftarrow \mathbb{Z}/p^4 \twoheadleftarrow \dots\Big) = \mathbb{Z}_p \quad \text{(the } p\text{-adic integers)}$$

$$\operatorname{colim}\Big(\mathbb{Z}/2! \hookrightarrow \mathbb{Z}/3! \hookrightarrow \mathbb{Z}/4! \hookrightarrow \mathbb{Z}/5! \hookrightarrow \dots\Big) = \mathbb{Q}/\mathbb{Z}$$

$$\lim\Big(\mathbb{Z}/2! \twoheadleftarrow \mathbb{Z}/3! \twoheadleftarrow \mathbb{Z}/4! \twoheadleftarrow \mathbb{Z}/5! \twoheadleftarrow \dots\Big) = \widehat{\mathbb{Z}} \quad \text{(the profinite completion of } \mathbb{Z})$$

Here, the p-adic integers is a ring (not just a group) whose elements are formal sums $\sum_{i=0}^\infty a_n p^n$ for $a_n \in \{0,1,2,\dots,p-1\}$. It is useful to think of an element $\sum_{i=0}^\infty a_n p^n$ as an "infinite number written in base p" : $\dots a_4a_3a_2a_1a_0$. If one decides to also allow finitely many p-adic digits after the comma, then one gets the field $\mathbb{Q}_p$ of p-adic numbers. An element of $\mathbb{Q}_p$ is therefore a formal expression of the form $\sum_{i=N}^\infty a_n p^n$ for some $N\in\mathbb{Z}$. The field of p-adic rationals admits the alternative descriptions $\mathbb{Q}_p = \mathbb{Z}_p[\frac{1}{p}]$ and $\mathbb{Q}_p = \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Q}$.

We have isomorphisms:

$$\mathbb{Q}/\mathbb{Z} = \bigoplus_{p:\text{prime}} \mathbb{Z}[\tfrac{1}{p}]/\mathbb{Z}$$

$$\widehat{\mathbb{Z}} = \prod_{p:\text{prime}} \mathbb{Z}_p$$

$$\operatorname{Hom}\big(\mathbb{Z}[\tfrac{1}{p}]/\mathbb{Z},\ \mathbb{Z}[\tfrac{1}{p}]/\mathbb{Z}\big) = \mathbb{Z}_p$$

$$\mathrm{Hom}\big(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}\big) = \widehat{\mathbb{Z}}$$
$$\mathrm{Hom}\big(\mathbb{Z}[\tfrac{1}{p}]\,,\ \mathbb{Z}[\tfrac{1}{p}]/\mathbb{Z}\big) = \mathbb{Q}_p$$
$$\mathrm{Hom}\big(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}\big) = \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$$

which we leave as exercises.

We can use all this to compute $\mathrm{Ext}^1(\mathbb{Q},\mathbb{Z})$. Using an injective resolution of $\mathbb{Z}$, we get:

$$\begin{aligned}\mathrm{Ext}^1(\mathbb{Q},\mathbb{Z}) &= H^1\Big(\,\mathrm{Hom}(\mathbb{Q},\mathbb{Q}) \to \mathrm{Hom}(\mathbb{Q},\mathbb{Q}/\mathbb{Z}) \to 0\Big)\\ &= H^1\Big(\mathbb{Q} \to \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} \to 0\Big) = \frac{\widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}}{\mathbb{Q}} \qquad \text{(a $\mathbb{Q}$-vector space of uncountable dimension)}\end{aligned}$$

November 19th:

Let $\mathcal{C}$ be an abelian category, and let $A \in \mathcal{C}$ be an object. A *generalised element* of A is a morphism $P \to A$, where $P \in \mathcal{C}$ is projective. If a is a generalised element of A and $f : A \to B$ is a morphism in $\mathcal{C}$, we write $f(a)$ for $f \circ a$.

Lemma: A morphism $f : A \to B$ is zero iff every generalized element a of A maps to zero.

Lemma: A morphism $f : A \to B$ is a monomorphism iff every generalized element a of A that maps to zero has the property that it is itself zero.

Lemma: A morphism $f : A \to B$ is an epimorphism iff every for every generalized element b in B $\exists$ a generalized element a of A that maps to it.

Lemma: Let $A \to B \to C$ be two maps that compose to zero. The sequence $A \to B \to C$ is exact in the middle iff for every generalized element b of B that maps to zero in C $\exists$ a generalized element a of A that maps to it.

Then I proved by diagram chasing that

$$\begin{bmatrix}\text{Short exact sequence}\\ \text{of chain complexes}\end{bmatrix} \xrightarrow{\text{Take cohomology}} \begin{bmatrix}\text{Long exact sequence of}\\ \text{cohomology groups}\end{bmatrix}$$

(For a proof, see Proposition 6.7 in lecture 6 of Kobi's notes.)

(week 7) November 24th:

A *bigraded chain complex* is a two-dimensional array $C_{\bullet\bullet} = (C_{n,m})_{n,m\in\mathbb{Z}}$ of modules (or objects of some arbitrary abelian category) equipped with horizontal differentials $d_h : C_{n,m} \to C_{n-1,m}$ and vertical differentials $d_v : C_{n,m} \to C_{n,m-1}$ satisfying $d_h^2 = 0$, $d_v^2 = 0$, and $d_h d_v = -d_v d_h$. Those three equations together are equivalent to the statement that $D := d_h + d_v$ acting on $\mathrm{Tot}(C_{\bullet\bullet})$ satisfies $D^2 = 0$. Here, $\mathrm{Tot}(C_{\bullet\bullet})$ is the chain complex that has $\bigoplus_{p+q=n} C_{p,q}$ in degree n.

Lemma (Lemma 9.3 in Kobi's notes): If $C_{\bullet\bullet}$ is bounded below, and d_h is exact, then D is exact.

November 26th:

I proved that given modules M and N, the two ways of computing $\mathrm{Tor}(M,N)$ agree: you can either resolve M, or resolve N, or both. This is proved by showing that the maps

$$M \otimes Q_\bullet \leftarrow \mathrm{Tot}(P_\bullet \otimes Q_\bullet) \to P_\bullet \otimes N$$

are quasi-isomorhpisms, where $P_\bullet \to M$ is a projective resolution and $Q_\bullet \to N$ is a projective resolution.

Actually, if $P_\bullet \to M$ is a projective resolution and $Q_\bullet \to N$ is any quasi-isomorphism (not necessarily a projective resolution), then we still have that

$$\mathrm{Tot}(P_\bullet \otimes Q_\bullet) \to P_\bullet \otimes N$$

is a quasi-isomorhpism.

The corresponding statement for *Hom* and *Ext* is that

$$\mathrm{Hom}(P_\bullet, N) \to \mathrm{Tot}\,\mathrm{Hom}(P_\bullet, I^\bullet) \leftarrow \mathrm{Hom}(M, I^\bullet)$$

are quasi-isomorhpisms, where $P_\bullet \to M$ is a projective resolution and $N \to I^\bullet$ is an injective resolution.

If we decided to take a projective resolution $N \leftarrow Q_\bullet$ instead, then we have a quasi-isomorphism

$$\mathrm{Hom}(P_\bullet, N) \leftarrow \widehat{\mathrm{Tot}}\,\mathrm{Hom}(P_\bullet, Q_\bullet)$$

provided we replace Tot by $\widehat{\mathrm{Tot}}$: the version of Tot that uses $\prod$ instead of $\bigoplus$.

Definition: The *internal hom* of two chain complexes $A_\bullet$ and $B_\bullet$ is defined to be the chain complex

$$\underline{\mathrm{Hom}}(A_\bullet, B_\bullet) := \widehat{\mathrm{Tot}}\,\mathrm{Hom}(A_\bullet, B_\bullet).$$

Given a chain complex $A_\bullet$, we define $A_\bullet[n]$ to be the same thing, but with its grading shifted by $-n$.

Exercise 23. Let $A_\bullet$ and $B_\bullet$ be two chain complexes. Show that there is a canonical isomorhpism between

$$H_n\big(\underline{\mathrm{Hom}}(A_\bullet, B_\bullet)\big)$$

and the group of chain maps $A_\bullet \to B_\bullet[n]$ modulo chain homotopy.

Exercise 24. Recall that an additive functor between abelian categories is called *exact* is it preserves short exact sequences. Prove that an exact functor preserves (not necessarily short) exact sequences.

Exercises 23–24 should be handed in on Tuesday Dec 1st at 5pm.

(week 8) December 1st:
Ring structure on $\mathrm{Ext}^*_k(M, M)$.

Example: Ring structure on $\mathrm{Ext}^*_{k[x]/x^2}(k, k)$.

Example: Ring structure on $\mathrm{Ext}^*_{k[x]/x^3}(k, k)$.

December 3rd: Group cohomology. (see the last 3 lectures of Kobi's course, last year).
The bar resolution

$$\mathbb{Z}[G]^{G\times G\times G} \to \mathbb{Z}[G]^{G\times G} \to \mathbb{Z}[G]^G \to \mathbb{Z}[G] \to \mathbb{Z}$$

The proof that it's a resolution.

Definition of group cohomology (case when the coefficients don't have any action by G).

Interpretation of $H^1(G, A)$ as homomorphisms $G \to A$.

Interpretation of $H^2(G, A)$ as central extensions of G by A.