Exam for *Foundations of Mathematics* A. Henriques, December 2012.

Problem 1 Prove by contradiction that the set $X := \{\frac{n}{n+1} : n \in \mathbb{N}\}$ does not admit a maximum. Prove that the set X does have a minimum.

Problem 2 State the Collatz conjecture.

Problem 3 Prove the following statements using induction:
• For all $n \geq 1$, the equation

$$\tfrac{1}{1} + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \ldots + \tfrac{1}{2^n} = 2 - 2^{-n}$$

holds.
• For all $n \geq 5$, the inequality

$$\tfrac{1}{1} + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \ldots + \tfrac{1}{n} < \tfrac{1}{2}n$$

holds.
• For all $n \geq 1$, the sum $\displaystyle\sum_{k=1}^{n}(k+2)$ is equal to $(n^2 + 5n)/2$.

Problem 4 Let a_n be the sequence of numbers defined recursively by

$$a_1 = 1,\ a_2 = 1, \text{ and } a_n = a_{n-1} - a_{n-2} \text{ for } n \geq 3.$$

Compute the first few values of this sequence and make a guess for the general formula for a_n. Prove your guess using induction.

Problem 5 .
On the set $\mathbb{Z}$, is $xRy := \begin{cases} \text{True} & \text{if } x+y \text{ is an even number} \\ \text{False} & \text{otherwise} \end{cases}$ an equivalence relation?

On the set $\mathbb{Q}$, is $xRy := \begin{cases} \text{True} & \text{if } x+y \text{ is an even number} \\ \text{False} & \text{otherwise} \end{cases}$ an equivalence relation?

Given an example of a relation that is transitive but not reflexive.