Problem 1 Determine which of the following maps are injective, surjective, or bijective:

$$\begin{aligned} f:[0,1]\to[0,1], &\quad f(x)=x^2 \\ g:\mathbb{R}\to\mathbb{R}, &\quad g(x)=x^2 \\ h:\mathbb{Z}_5\to\mathbb{Z}_5, &\quad h(x)=x+1 \\ k:\mathbb{N}\to\mathbb{N}, &\quad k(x)=x+1 \end{aligned}$$

Problem 2 Show that the function

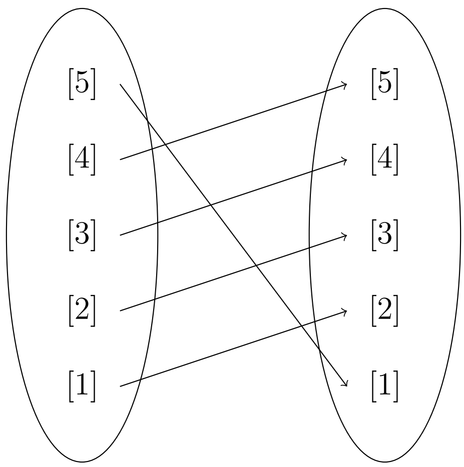
$$f:\mathbb{Z}_5\to\mathbb{Z}_5, \qquad f([a])=\begin{cases}[\frac{a}{2}] & \text{if } a \text{ is even}\\ [\frac{a+5}{2}] & \text{if } a \text{ is odd}\end{cases}$$

is well defined.

Solution: The function f is injective. Indeed, for $x\in[0,1]$, the equation $f(x)=f(y)$ implies $x=\pm y$. But since x and y are both positive, this readily implies $x=y$. The function f is surjective because for $y\in[0,1]$, the equation $f(x)=y$ always has a solution $x\in[0,1]$, namely $x=\sqrt{y}$. The function is both injective and surjective and therefore bijective.

The function g is not injective because it can happen that $g(x)=g(y)$ without necessarily having $x=y$, namely take $x=-1$ and $y=1$. The function g is not sujective because there is no $x\in\mathbb{R}$ such that $g(x)=-1$. g is also not bijective (for being bijective it would need to be both injective and surjective).

The function h is bijective. This is best seen by a little picture:



Every element in the right is hit by exactly one arrow.

The function k is injective: $k(x) = k(y)$ implies $x = y$ as is readily seen by subtracting one from the equation $x + 1 = y + 1$. k is not surjective: there is no element $x \in \mathbb{N}$ such that $k(x) = 1$. It is therefore also not bijective.

Solution: We need to show that if $[a] = [b]$ then $f([a]) = f([b])$. Writing this out explicitly, we need to check that

$$\begin{cases} [\frac{a}{2}] = [\frac{b}{2}] & \text{if } a \text{ and } b \text{ are both even} \\ [\frac{a+5}{2}] = [\frac{b+5}{2}] & \text{if } a \text{ and } b \text{ are both odd} \\ [\frac{a}{2}] = [\frac{b+5}{2}] & \text{if } a \text{ is even and } b \text{ is odd} \\ [\frac{a+5}{2}] = [\frac{b}{2}] & \text{if } a \text{ is odd and } b \text{ is even} \end{cases}$$

By definition, $[a] = [b]$ means that $b = a + 5n$ for some $n \in \mathbb{Z}$.

Case 1: Both a and b are even. Then n is even, and we have

$$\frac{b}{2} = \frac{a + 5n}{2} = \frac{a}{2} + 5 \cdot \frac{n}{2}$$

and so $[\frac{a}{2}] = [\frac{b}{2}]$ (because $\frac{n}{2}$ is an integer).
Case 2: Both a and b are odd. Then n is even, and we have

$$\frac{b+5}{2} = \frac{a + 5n + 5}{2} = \frac{a+5}{2} + 5 \cdot \frac{n}{2}$$

and so $[\frac{a+5}{2}] = [\frac{b+5}{2}]$ (because $\frac{n}{2}$ is an integer).
Case 3: a is even and b is odd. Then n is odd, and we have

$$\frac{b+5}{2} = \frac{a + 5n + 5}{2} = \frac{a}{2} + 5 \cdot \frac{n+1}{2}$$

and so $[\frac{a}{2}] = [\frac{b+5}{2}]$ (because $\frac{n+1}{2}$ is an integer).
Case 4: a is odd and b is even. Then n is odd, and we have

$$\frac{b}{2} = \frac{a + 5n}{2} = \frac{a + 5 + 5(n-1)}{2} = \frac{a+5}{2} + 5 \cdot \frac{n-1}{2}$$

and so $[\frac{a+5}{2}] = [\frac{b}{2}]$ (because $\frac{n-1}{2}$ is an integer).