Chiral conformal field theory

Course notes

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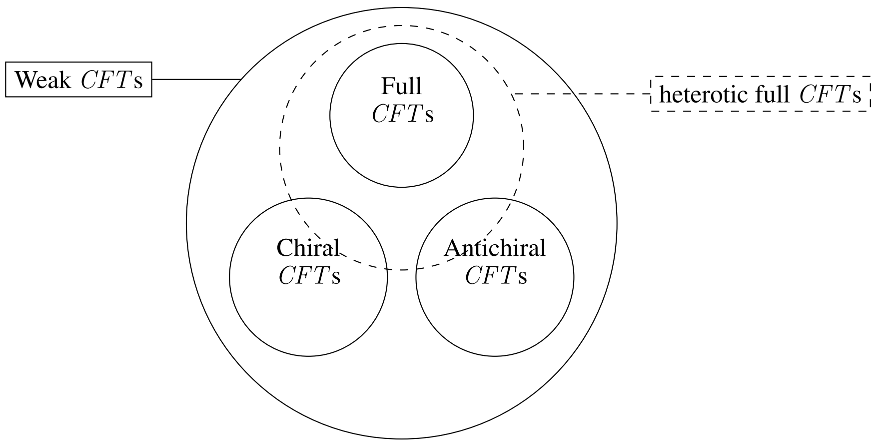
Introduction

These notes are concerned with two-dimensional conformal field theory. There are two things which go by the name "conformal field theory" (CFT), and which are quite distinct: <u>Chiral conformal field theories</u> and <u>Full conformal field theories</u>. A chiral conformal field theory is not a full conformal field theory, and a full conformal field theory is not a chiral conformal field theory. Instead, they are related by constructions

$$
\begin{tikzcd}
\left\{\begin{matrix}\text{Chiral conformal}\\ \text{field theories}\end{matrix}\right\} \arrow[r, "\text{diagonal}"] & \left\{\begin{matrix}\text{Full conformal}\\ \text{field theories}\end{matrix}\right\} \arrow[l, bend right, "\text{antichiral part}"'] \arrow[l, bend left, "\text{chiral part}"] \arrow[loop right, "\text{complex conjugation}"]
\end{tikzcd}
$$

In these notes, we will be essentially only treating chiral conformal field theories.

Note: Graeme Segal uses the term *weak CFT* for what we here call a 'chiral Segal CFT'. We prefer to use the term weak CFT for a more general notion, which contains both chiral and full CFTs as special cases. The following Venn diagram indicates how all these notions fit together. (The classes of full, chiral, and antichiral CFTs are disjoint, with the exception of the trivial CFT which is in the intersection of all three classes, and which is omitted from the diagram.)



Very loosely speaking, a CFT is *chiral* if 'stuff depends holomorphically on the Riemann surfaces', *antichiral* if 'stuff depends anti-holomorphically', *full* if the theory is single-valued on all surfaces and its chiral and antichiral parts are isomorphic, and *heterotic* when that last condition is removed from the definition of full CFT.

There exist three mathematical formalizations of the concept of chiral conformal field theory:

χCFT $\rightarrow$

- Vertex operator algebras,
- Conformal nets, and
- Segal CFTs.

Terminology warning: Whereas the term 'vertex operator algebra' (VOA) unambiguously refers to chiral CFTs, there exist variants of the notions of conformal net and of Segal CFT which model the notion of full CFT. In order to avoid any ambiguity, it is therefore preferable to use the terminology 'chiral conformal net' and 'chiral Segal CFT'. (There also exits a variant of the notion of vertex operator algebra which formalizes full CFTs, and which goes by the name 'full field algebra'.)

The notions of vertex operator algebra, of chiral conformal net, and of chiral Segal CFT are expected/conjectured to be equivalent, provided appropriate qualifiers are added. Note that these notions cannot be completely equivalent because:

- *Unitarity* is built into conformal nets, but not into VOAs, nor Segal CFTs.
- *Rationality* is built into Segal CFTs, but not into VOAs, nor conformal nets (rationality is a certain finiteness condition that a chiral CFT might or might not satisfy).
- There exists a certain equivalence between Segal CFTs called *infinitesimal equivalence*. Infinitesimally equivalent Segal CFTs model the same physics and should therefore be treated as 'the same'.

We propose:

Conjecture 1 *(i) There is a bijection*

$$\textit{unitary VOAs} \quad \Leftrightarrow \quad \textit{chiral conformal nets.}$$

(ii) There is a bijection

$$\textit{rational VOAs} \quad \Leftrightarrow \quad \begin{array}{c}\textit{chiral Segal CFTs up to}\\ \textit{infinitesimal equivalence.}\end{array}$$

(iii) There is a bijection

$$\textit{rational conformal nets} \quad \Leftrightarrow \quad \textit{unitary chiral Segal CFTs.}$$

There exist a couple of constructions in the literature which connect VOAs, chiral conformal nets, and chiral Segal CFTs. But these constructions only work in special cases, and it is fair to say that the above conjecture is wide open.

These notes will be focusing mostly on chiral Segal CFTs, which is the *least developed* of the above three mathematical formalizations of chiral CFT (for example, the only chiral CFTs which have been constructed so far, or proven to exist as Segal CFTs are free field CFTs). But this is also, in some sense, the most powerful one of the above three formalisms, and we expect that it should be easy (in comparison to other constructions) to construct a VOA or a conformal net from a chiral Segal CFT.

Complex cobordisms

The definition of Segal CFT (from now on, 'Segal CFT' = 'chiral Segal CFT') is based on the notion of *complex cobordism*.

Before talking about complex cobordisms, let us first describe the notion of a *Riemann surface with boundary*. Let $\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) \geq 0\}$ be the complex upper half plane, and let $\mathring{\mathbb{H}} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ be its interior.

Definition: *A Riemann surface with boundary is a ringed space* $(\Sigma, \mathcal{O}_\Sigma)$ *which is locally isomorphic to* $(\mathbb{H}, \mathcal{O}_\mathbb{H})$*, where* $\mathcal{O}_\mathbb{H}$ *is the sheaf on* $\mathbb{H}$ *given by*

$$\mathcal{O}_\mathbb{H}(U) := \left\{ f : U \to \mathbb{C} \;\middle|\; \begin{array}{l} f|_{U\cap\mathring{\mathbb{H}}} \text{ is holomorphic,} \\ \exists\, V \subset \mathbb{C} \text{ open and } g \in C^\infty(V) \text{ s.t. } f = g|_U \end{array} \right\}$$

for $U \subset \mathbb{H}$ *an open subset.*

By a classical result known as Borel's lemma, for an open subset $U \subset \mathbb{H}$, the condition that a function $f : U \to \mathbb{C}$ be the restriction a C^∞ function defined on some open $V \subset \mathbb{C}$ is equivalent to f being smooth all the way to the boundary. Here, 'smooth all the way to the boundary' is just the usual notion of smoothness, adapted to the case of manifolds with boundary (when writing down the limits which are used to define the derivative of a function, restrict the domain of the limit to just one side if necessary, so as to not fall outside of the manifold).

An equivalent definition of the sheaf $\mathcal{O}_\mathbb{H}$ is to declare $\mathcal{O}_\mathbb{H}(U)$ to be the set of continuous functions on U which are holomorphic when restricted to $U \cap \mathring{\mathbb{H}}$, and smooth when restricted to $U \cap \partial\mathbb{H}$:

$$\mathcal{O}_\mathbb{H}(U) = \left\{ f \in C^0(U, \mathbb{C}) \;\middle|\; \begin{array}{l} f|_{U\cap\mathring{\mathbb{H}}} \text{ is holomorphic,} \\ f|_{U\cap\partial\mathbb{H}} \text{ is smooth} \end{array} \right\}.$$

The equivalence between the above two definitions of $\mathcal{O}_\mathbb{H}$ will be proven below, in Lemma 2. We first state an important theorem:

Theorem.[1] *(Riemann mapping theorem for simply connected domains with smooth boundary)* Let $D \subset \mathbb{C}$ be a compact simply connected domain with smooth boundary. Then there exists an isomorphism

$$D \;\cong\; \mathbb{D} := \{z \in \mathbb{C} \,:\, |z| \leq 1\}$$

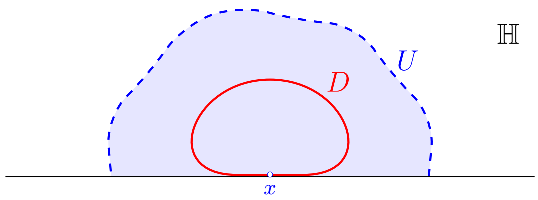
which is holomorphic in the interior, and smooth all the way to the boundary. Moreover, that isomorphism is unique up to an element of

$$\mathrm{Aut}(\mathbb{D}) = PSU(1,1) = \left\{ z \mapsto \tfrac{az+b}{\bar{b}z+\bar{a}} : |a|^2 - |b|^2 = 1 \right\}.$$

Lemma 2 *Let* $U \subset \mathbb{H}$ *be an open subset, and let* $f : U \to \mathbb{C}$ *be a continuous function such that* $f|_{U\cap\mathring{\mathbb{H}}}$ *is holomorphic and* $f|_{U\cap\partial\mathbb{H}}$ *is smooth. Then* f *is smooth all the way to the boundary.*

[1] S. Bell. *Mapping problems in complex analysis and the* $\bar{\partial}$*-problem.* Bull. Amer. Math. Soc., 22(2):233–259, 1990.

Proof. Let $x \in U \cap \partial\mathbb{H}$ be a point, and let $D \subset U$ be a neighbourhood of x which is compact, simply connected, and with smooth boundary:



Let $\psi : \mathbb{D} \to D$ be a uniformizing map. The function $g := \psi^* f$ is continuous, holomorphic in the interior of $\mathbb{D}$, and smooth on the boundary⚠. The Taylor coefficients a_n of $g(z) = \sum a_n z^n$ satisfy

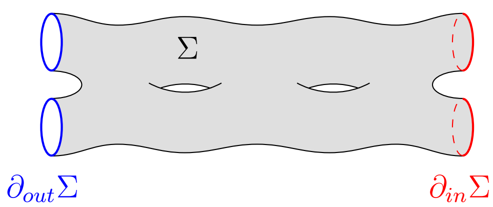
$$\begin{aligned} a_n &= \tfrac{1}{2\pi i} \oint_{|z|=r} g(z) z^{-(n+1)} dz && \text{for any } r < 1 \\ &= \tfrac{1}{2\pi i} \oint_{|z|=1} g(z) z^{-(n+1)} dz && \text{since } g \text{ is continuous} \\ &= \tfrac{\pm 1}{n(n-1)\ldots(n-k+1)} \cdot \tfrac{1}{2\pi i} \oint_{|z|=1} \big(g|_{\partial\mathbb{D}}\big)^{(k)}(z) \cdot z^{-(n-k+1)} dz && \forall k \leq n. \end{aligned}$$

It follows that $|a_n| \leq \frac{1}{n(n-1)\ldots(n-k+1)} \cdot \|(g|_{\partial\mathbb{D}})^{(k)}\|_\infty$. The coefficients a_n decay faster that any power of n, so $g(z)$ is smooth all the way to the boundary. The same therefore holds for f around x.

⚠ There is a gap in the above proof, because we don't know that $f|_{\partial D}$ is smooth at the two boundary points of the interval $[a, b] := \partial D \cap \partial\mathbb{H}$. But we can fix that gap. Let $h \in \mathcal{O}_{\mathbb{H}}(\mathbb{H})$ be an auxiliary function with zeros of infinite order at a and b (for example, $h(z) = e^{-\frac{1+i}{\sqrt{z-a}} - \frac{1+i}{\sqrt{z-b}}}$). We run the same argument with $\tilde{f} := hf$ (the function $\tilde{f}|_{\partial D}$ is now smooth at a and b because it vanishes to infinite order), deduce that $\tilde{f}$ is smooth all the way to the boundary, and divide by h to get the result. $\square$

We'll be distinguishing two types of complex cobordisms. There's the ones which we'll be calling 'thick complex cobordisms', and there's the ones which we'll be calling 'complex cobordisms with thin parts'. Thick complex cobordisms are special cases of complex cobordisms with thin parts.

Definition: *A thick complex cobordism is a Riemann surface with boundary equipped with a decomposition of its boundary into a disjoint union $\partial\Sigma = \partial_{in}\Sigma \sqcup \partial_{out}\Sigma$:*



We equip $\partial_{out}\Sigma$ with the orientation induced by that of Σ, and we equip $\partial_{in}\Sigma$ with the opposite of that orientation.

More generally, given oriented 1*-manifolds* S_1 *and* S_2*, a (thick) complex cobordism from* S_1 *to* S_2 *is a triple* $(\Sigma, \varphi_{in}, \varphi_{out})$ *where* Σ *is a complex cobordism as defined above, and* $\varphi_{in} : S_1 \to \partial_{in}\Sigma$ *and* $\varphi_{out} : S_2 \to \partial_{out}\Sigma$ *are diffeomorphisms.*

One shortcoming with the notion of thick complex cobordism is that it does not allow for *identity cobordisms*. This is addressed by the following variant. We start be describing the local model:

Given smooth functions $a, b : \mathbb{R} \to \mathbb{R}$*,* $a \le b$*, let*

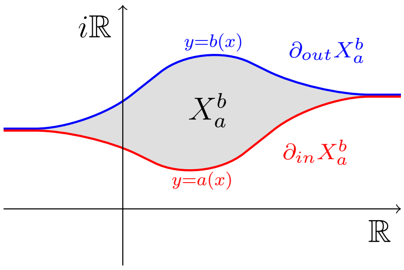
$$X_a^b := \{x + iy \in \mathbb{C} \,|\, a(x) \le y \le b(x)\},$$

equipped with the sheaf

$$\mathcal{O}_{X_a^b}(U) := \left\{ f : U \to \mathbb{C} \;\middle|\; \begin{array}{l} f|_{U \cap \mathring{X}_a^b} \text{ is holomorphic,} \\ \exists V \subset \mathbb{C} \text{ open and } g \in C^\infty(V) \text{ s.t. } f = g|_U \end{array} \right\} \tag{1}$$

where $\mathring{X}_a^b = \{x + iy \in \mathbb{C} \,|\, a(x) < y < b(x)\}$*. Let also*

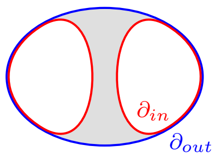
$$\partial_{in} X_a^b = \{x + ia(x) \,|\, x \in \mathbb{R}\} \qquad \text{and} \qquad \partial_{out} X_a^b = \{x + ib(x) \,|\, x \in \mathbb{R}\}.$$



Definition: *A complex cobordism with thin parts is a ringed space* $(\Sigma, \mathcal{O}_\Sigma)$ *equipped with two subspaces* $\partial_{in}\Sigma \subset \Sigma$ *and* $\partial_{out}\Sigma \subset \Sigma$ *(typically not disjoint), which is locally isomorphic to the ringed space* $(X_a^b, \mathcal{O}_{X_a^b})$ *defined above, with its two subspaces* $\partial_{in/out} X_a^b$*, for some functions* $a \le b$*.*

A complex cobordism with thin parts from S_1 *to* S_2 *is a triple* $(\Sigma, \varphi_{in}, \varphi_{out})$ *where* Σ *is as above, and* $\varphi_{in} : S_1 \to \partial_{in}\Sigma$ *and* $\varphi_{out} : S_2 \to \partial_{out}\Sigma$ *are diffeomorphisms.*

The following is an example of a complex cobordism with thin parts:



An alternative definition of the sheaf $\mathcal{O}_{X_a^b}$ is to declare

$$\mathcal{O}_{X_a^b}(U) := \left\{ f \in C^0(U, \mathbb{C}) \;\middle|\; \begin{array}{l} f|_{U \cap \mathring{X}_a^b} \text{ is holomorphic,} \\ f|_{U \cap \partial_{in} X_a^b} \text{ and } f|_{U \cap \partial_{out} X_a^b} \text{ are smooth} \end{array} \right\} \tag{2}$$

The equivalence between these two definitions is the content of the following lemma, whose proof is deferred to the next section:

Lemma 3 *Let Σ be complex cobordisms with thin parts, $U \subset \Sigma$ an open subset, and $U \hookrightarrow \mathbb{C}$ an embedding (a map of ringed spaces).*

Let $f : U \to \mathbb{C}$ be a function satisfying the conditions (2). *Then there exists an open $V \subset \mathbb{C}$, and a function $g \in C^\infty(V)$ such that $f = g|_U$.*

Proof of Lemma 3 (first part). We first note that the problem is local: assuming the existence of an open cover $\{U_i \subset U\}$, and functions $g_i \in C^\infty(V_i)$ whose restrictions to U_i agree with f, we may use a partition of unity $\{\varphi_i : V_i \to \mathbb{R}\}$ to construct the desired function as follows: $g = \sum \varphi_i g_i$.

Without loss of generality, we may therefore assume that $\Sigma = \overline{X}_a^b := X_a^b \cup \{\infty\}$ for two compactly supported functions $a \le b : \mathbb{R} \to \mathbb{R}$, and that the function f is globally defined on $\overline{X}_a^b$. The rest of the proof (the main part) is deferred to the next section. $\boxminus$

For the purposes of chiral CFT, it is important to understand not just individual complex cobordisms but also the *moduli spaces* thereof. Let S_1 and S_2 be closed 1-manifolds. The space[2] $Cob^{\mathrm{conf}}(S_1, S_2)$ that parametrizes complex cobordisms from S_1 to S_2 is an infinite dimensional smooth manifold with boundary[3], which is furthermore equipped with a holomorphic structure.

The moduli space $Cob^{\mathrm{conf}}(S_1, S_2)$ can be written as the closure an open subset of some infinite dimensional complex manifold[4] (see p. 36 for a discussion in the special case of annuli). We will take the point of view that, in order to describe the complex structure (/ the smooth structure / the topology) on $Cob^{\mathrm{conf}}(S_1, S_2)$, it is enough to understand what it means for a map

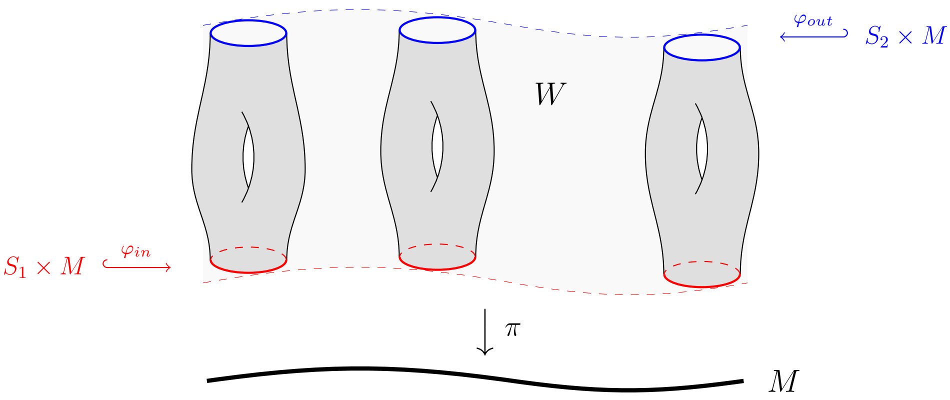
$$M \to Cob^{\mathrm{conf}}(S_1, S_2) \tag{3}$$

from a finite dimensional complex manifold (/ smooth manifold / topological space) to be holomorphic (/ smooth / continuous). Let us first restrict attention to thick cobordisms. By definition, a holomorphic map from an n-dimensional complex manifold M into the thick part of the moduli space is the same thing as a diagram

[2]The part of the moduli space that contains closed components is in fact not a space but a stack. If one insists that all components have non-empty boundary, then the stackiness goes away and the moduli space is just a space.

[3]It is actually worse than just a manifold with boundary. The locus of cobordisms $\Sigma \in Cob^{\mathrm{conf}}(S_1, S_2)$ whose two boundaries $\partial_{in}\Sigma$ and $\partial_{out}\Sigma$ touch n times is a corner of codimension n. More generally, one gets the same singularities as one encounters in the space of non-negative polynomials.

[4]Again, this is only true in the absence of closed components. In the presence of closed components, the statement remains true provided one replaces the word “manifold” by “orbifold”.



where W is an $(n+1)$-dimensional complex manifold with boundary (a space locally isomorphic to $f^{-1}(\mathbb{R}_{\geq 0})$, for $f : \mathbb{C}^{n+1} \to \mathbb{R}$, $f^{-1}(0)$ smooth), π is a proper holomorphic submersion, and $\varphi = \varphi_{in} \sqcup \varphi_{out} : (S_1 \sqcup S_2) \times M \to W$ is an isomorphism onto ∂W satisfying $\pi \circ \varphi = pr_2$. Moreover, and very importantly: *for every point x of S_1 or S_2, the section $\varphi|_{\{x\}\times M} : M \to W$ of π should be holomorphic*.

If we only require M to be a smooth manifold (possibly with boundary, or corners, or even worse) and only require W to be equipped with a complex structure along the fibers of the projection π, then we obtain the notion of a smooth map (3). And if M is a topological space, and the fibers of π are equipped with continuously varying complex structures, then that's what it means for a map (3) to be continuous.

The story with thin parts is analogous. The one notable difference is that W is now locally isomorphic to $f^{-1}(\mathbb{R}_{\geq 0}) \cap g^{-1}(\mathbb{R}_{\leq 0})$, for a pair of smooth functions $f, g : \mathbb{C}^{n+1} \to \mathbb{R}$ satisfying $g \leq f$.

Conformal welding

Defining complex cobordisms via ringed spaces allows for a particularly elegant description of the operation of composition of cobordisms, as a pushout in the category of ringed spaces.

Theorem. *(Conformal welding)* Let Σ_1 and Σ_2 be thick complex cobordisms, and let $\phi : \partial_{in}\Sigma_1 \to \partial_{out}\Sigma_2$ be an orientation preserving diffeomorphism. Then $(\Sigma_1 \cup_\phi \Sigma_2, \mathcal{O}_{\Sigma_1 \cup_\phi \Sigma_2})$ with

$$\partial_{in}(\Sigma_1 \cup_\phi \Sigma_2) = \partial_{in}\Sigma_2, \qquad \partial_{out}(\Sigma_1 \cup_\phi \Sigma_2) = \partial_{out}\Sigma_1,$$
$$\mathcal{O}_{\Sigma_1 \cup_\phi \Sigma_2}(U) := \left\{ f : U \to \mathbb{C} \,\middle|\, f|_{U \cap \Sigma_i} \in \mathcal{O}_{\Sigma_i}(U \cap \Sigma_i), \text{ for } i = 1, 2 \right\} \tag{4}$$

is a thick complex cobordism. (I.e., the ringed space $(\Sigma_1 \cup_\phi \Sigma_2, \mathcal{O}_{\Sigma_1 \cup_\phi \Sigma_2})$ is isomorphic to $(\mathbb{C}, \mathcal{O}_\mathbb{C})$ in a neighbourhood of the image of $\partial_{in}\Sigma_1$.)

Moreover, the image of $\partial_{in}\Sigma_1$ inside $\Sigma_1 \cup_\phi \Sigma_2$ (equivalently, the image of $\partial_{out}\Sigma_2$) is a smooth curve.

Proof. The problem being local, it is enough to treat the case

$$\Sigma_1 = \mathbb{D}_- := \{z \in \mathbb{C} \cup \{\infty\} \, : \, |z| \geq 1\}, \qquad \Sigma_2 = \mathbb{D}_+ := \{z \in \mathbb{C} \, : \, |z| \leq 1\},$$
$$\phi : S^1 = \partial\mathbb{D}_- \xrightarrow{\cong} \partial\mathbb{D}_+ = S^1.$$

We will construct a homeomorphism $f : \mathbb{D}_- \cup_\phi \mathbb{D}_+ \to \mathbb{CP}^1$ that is holomorphic on the interiors $\mathring{\mathbb{D}}_-$ and $\mathring{\mathbb{D}}_+$, and smooth on $\mathbb{D}_-$ and $\mathbb{D}_+$ all the way to the boundary.

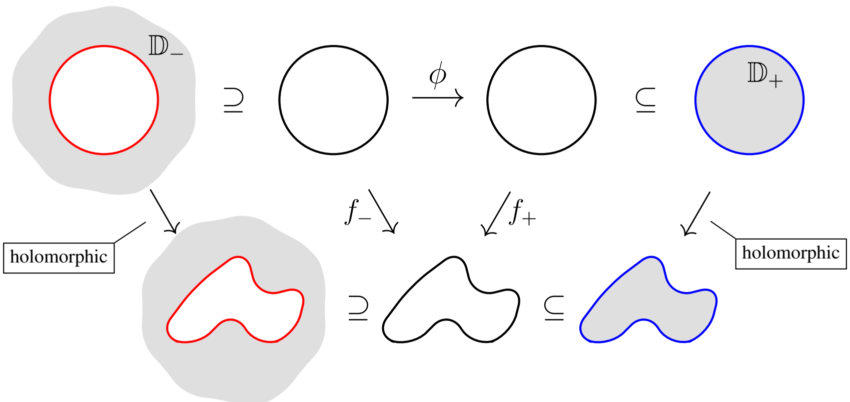
The space of such isomorphisms is three dimensional (corresponding to the fact that $\mathrm{Aut}(\mathbb{CP}^1) = PSL(2, \mathbb{C})$). In order to force the solution to be unique, we will also insist that $f : \infty \in \mathbb{D}_- \mapsto \infty \in \mathbb{CP}^1$, with first derivative $f'(\infty) = 1$, and second derivative $f''(\infty) = 0$.

Remark. When $\phi : S^1 \to S^1$ is real analytic, the problem is easy. Extend ϕ to a biholomorphic map $\tilde{\phi} : U \to V$ from an open neighbourhood U of $S^1 = \partial\mathbb{D}_-$ to an open neighbourhood V of $S^1 = \partial\mathbb{D}_+$. We may then rewrite $\mathbb{D}_- \cup_\phi \mathbb{D}_+$ as $(\mathbb{D}_- \cup U) \cup_{\tilde{\phi}} (\mathbb{D}_+ \cup V)$, which is now obviously a complex manifold. The latter is then isomorphic to $\mathbb{CP}^1$ by the Riemann uniformization theorem. $\diamond$

Letting $f_\pm := f|_{\mathbb{D}_\pm}$, we may rephrase the problem as that of finding a pair of functions $f_+, f_- \in \mathcal{C}^\infty(S^1)$ of the form

$$f_+(z) = a_0 + a_1 z + a_2 z^2 + \ldots \qquad f_-(z) = z + b_1 z^{-1} + b_2 z^{-2} + \ldots$$

that make the following diagram commute:



(5)

Let $\mathcal{H} = \left\{ \sum_{n\geq 0} a_n z^n \right\} \subset \mathcal{C}^\infty(S^1)$ be the Hardy space, and $\mathcal{H}^\perp = \left\{ \sum_{n<0} b_n z^n \right\}$ its orthogonal complement.

Claim. For $f \in \mathcal{C}^\infty(S^1)$

$$Tf := \tfrac{1}{2\pi i} \text{ P.V.} \int_{w \in S^1} \frac{f(w)}{w - z} dw = \begin{cases} \tfrac{1}{2} f & \text{if } f \in \mathcal{H} \\ -\tfrac{1}{2} f & \text{if } f \in \mathcal{H}^\perp \end{cases}$$

where $\text{P.V.} \int_{w \in S^1} := \lim_{\varepsilon \to 0} \int_{w \in S^1, |w-z| \geq \varepsilon}$, and 'P.V.' stand for 'principal value'.

Proof of claim: We prove the claim for f real analytic; the general case follows by continuity. If $f \in \mathcal{H}$, then f extends to a holomorphic function on a neighborhood of $\mathbb{D}_+$. We then have

$$\tfrac{1}{2\pi i}\ \text{P.V.}\int_{S^1} \frac{f(w)dw}{w-z} = \tfrac{1}{2\pi i}\left[\tfrac{1}{2}\left(\oint + \cancel{\oint}\right)\right]\frac{f(w)dw}{w-z}$$

$$= \tfrac{1}{4\pi i}\cdot 2\pi i\,\mathrm{Res}_z \frac{f(w)}{w-z} = \tfrac{1}{2}f(z).$$

If $f \in \mathcal{H}^\perp$, it extends to a holomorphic function on a neighborhood of $\mathbb{D}_-$ and vanishes at infinity. The 1-form $\frac{f(w)dw}{w-z}$ is regular at infinity (dw has a double pole at infinity while $\frac{f(w)}{w-z}$ has at least a double zero) so, by the same argument as above with $\mathbb{D}_-$ instead of $\mathbb{D}_+$, we get

$$\tfrac{1}{2\pi i}\ \text{P.V.}\int_{S^1} \frac{f(w)dw}{w-z} = \tfrac{1}{2\pi i}\left[\tfrac{1}{2}\left(\cancel{\oint} + \oint\right)\right]\frac{f(w)dw}{w-z}$$

$$= \tfrac{1}{4\pi i}\cdot\left(-2\pi i\,\mathrm{Res}_z \frac{f(w)}{w-z}\right) = -\tfrac{1}{2}f(z).$$

The contour runs *clockwise* around $\mathbb{D}_-$

$\diamond$

Going back to conformal welding, let $V_\phi : \mathcal{C}^\infty(S^1) \to \mathcal{C}^\infty(S^1)$ be the operator of precomposition by ϕ. We may rewrite the conditions in (5) as

$$V_\phi f_+ = f_- \qquad T f_+ = \tfrac{1}{2}f_+ \qquad T f_- = z - \tfrac{1}{2}f_-\,. \tag{6}$$

Any solution of these equations must satisfy $V_\phi T V_\phi^{-1} f_- = V_\phi T f_+ = \frac{1}{2}V_\phi f_+ = \frac{1}{2}f_-$. In particular, $(T - V_\phi T V_\phi^{-1})f_- = z - f_-$. Hence

$$\Big[1 - (T - V_\phi T V_\phi^{-1})\Big](f_-) = z. \tag{7}$$

We may perform a variable substitution in the singular integral to get the following general formula:

$$\begin{aligned} V_\phi T V_\phi^{-1} f(z) &= T V_\phi^{-1} f(\phi(z)) \\ &= \tfrac{1}{2\pi i}\ \text{P.V.}\int \frac{V_\phi^{-1} f(w)}{w-\phi(z)}dw \\ &= \tfrac{1}{2\pi i}\ \text{P.V.}\int \frac{f(\phi^{-1}(w))}{w-\phi(z)}dw = \tfrac{1}{2\pi i}\ \text{P.V.}\int \frac{f(u)}{\phi(u)-\phi(z)}\phi'(u)du. \end{aligned}$$

The integral kernel of $T - V_\phi T V_\phi^{-1}$ is therefore given by

$$K(z,w) = \frac{1}{w-z} - \frac{\phi'(w)}{\phi(w)-\phi(z)}.$$

The singularities of $\frac{1}{w-z}$ and $\frac{\phi'(w)}{\phi(w)-\phi(z)}$ exactly cancel out. So, despite its appearance, $K(z,w)$ is in fact a smooth function, which makes $T - V_\phi T V_\phi^{-1}$ a *smoothing operator*[5]! For ϕ close enough to the identity, the operator $1-(T - V_\phi T V_\phi^{-1})$ is therefore invertible, and we may solve Equation (7) by simply writing

$$f_- := \Big[1-(T - V_\phi T V_\phi^{-1})\Big]^{-1}(z).$$

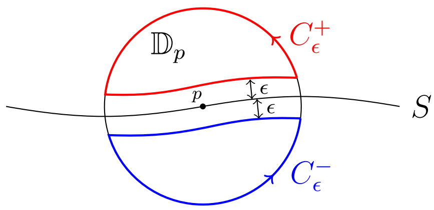
It remains to check that f_- as above together with $f_+ := V_\phi^{-1} f_-$ form a solution of the conformal welding problem (6). This is obvious when ϕ is analytic (because then (5) admits a unique solution, by the Riemann uniformization theorem), and follows by continuity for arbitrary smooth ϕ. This finishes the proof of the theorem for ϕ close to the identity.

When ϕ is not necessarily close to the identity, write it as a composite $\phi = \phi'' \circ \phi'$ of an analytic diffeomorphism ϕ' followed by a diffeomorphism ϕ'' that is close to the identity. The surface $\mathbb{D}_- \cup_\phi \mathbb{D}_+$ can be obtained from $\mathbb{D}_- \cup_{\phi'} \mathbb{D}_+ \cong \mathbb{CP}^1$ by cutting it along an analytically embedded circle $S \subset \mathbb{CP}^1$ (the image of $\partial \mathbb{D}_-$) and regluing the two halves using ϕ''. Locally around S, this is equivalent to the problem of constructing $\mathbb{D}_- \cup_{\phi''} \mathbb{D}_+$, which we have just solved above.

It remains to identify the sheaf (4) with the sheaf of holomorphic functions on the welded surface. This is the content of the next lemma. $\square$

Lemma 4 *Let $S \subset \mathbb{C}$ be a smooth curve, and $U \subset \mathbb{C}$ an open. If $f : U \to \mathbb{C}$ is a continuous function whose restriction to $U \setminus S$ is analytic, then f is analytic.*

Proof: We work in the neighbourhood of a point $p \in S$. Let $\mathbb{D}_p \subset U$ be a disc centred at p. Consider the following contours:



By Cauchy's residue formula, we have

$$f(z) = \frac{1}{2\pi i} \oint_{C_\epsilon^+ \cup C_\epsilon^-} \frac{f(w)}{w-z} dw$$

for every $z \in \mathring{\mathbb{D}}_p$ whose distance from S is at least ϵ. Taking the limit as $\epsilon \to 0$ and noting that the two contributions along S cancel each other, we get

$$f(z) = \frac{1}{2\pi i} \oint_{\partial \mathbb{D}_p} \frac{f(w)}{w-z} dw$$

[5]Such operators act on L^2 functions as compact operators. Indeed, they act as compact operators on any Sobolev space of functions.

for every $z \in \mathring{\mathbb{D}}_p \setminus S$. By continuity, the last formula also holds for $z \in S$. To finish the argument, we note that the right hand side is an analytic function of z, because each of the functions $z \mapsto \frac{f(w)}{w-z}$ is analytic. $\square$

In the presence of thin parts, conformal welding is again just a pushout in the category of ringed spaces:

Proposition. *(Conformal welding for complex cobordism with thin parts)* Let Σ_1 and Σ_2 be complex cobordisms with thin parts, and let $\phi : \partial_{in}\Sigma_1 \to \partial_{out}\Sigma_2$ be an orientation preserving diffeomorphism. Then $(\Sigma_1 \cup_\phi \Sigma_2, \mathcal{O}_{\Sigma_1\cup_\phi\Sigma_2})$ with $\partial_{in}(\Sigma_1 \cup_\phi \Sigma_2) = \partial_{in}\Sigma_2$, $\partial_{out}(\Sigma_1 \cup_\phi \Sigma_2) = \partial_{out}\Sigma_1$, and

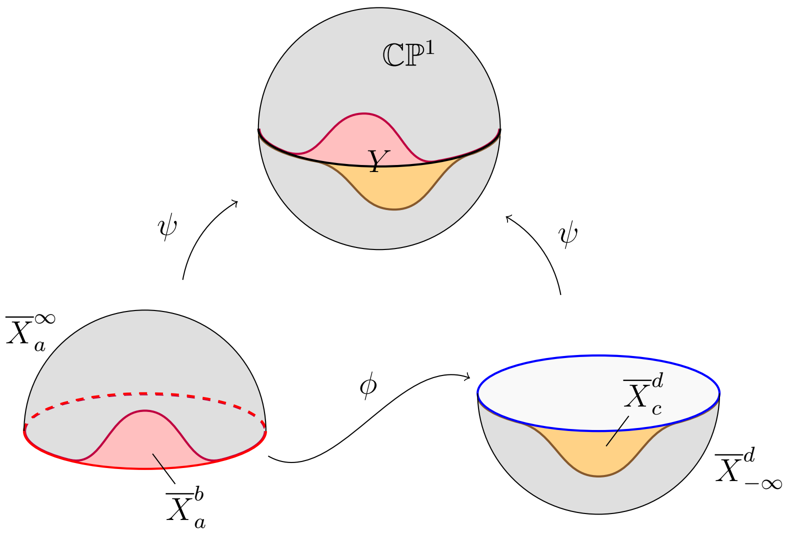
$$\mathcal{O}_{\Sigma_1\cup_\phi\Sigma_2}(U) := \left\{ f : U \to \mathbb{C} \,\middle|\, f|_{U\cap\Sigma_i} \in \mathcal{O}_{\Sigma_i}(U \cap \Sigma_i), \text{ for } i = 1, 2 \right\} \tag{8}$$

is a complex cobordism with thin parts.

Proof. The problem being local, we may assume that $\Sigma_1 = \overline{X}{}_a^b := X_a^b \cup \{\infty\}$ and $\Sigma_2 = \overline{X}{}_c^d := X_c^d \cup \{\infty\}$, for some compactly supported functions $a, b, c, d : \mathbb{R} \to \mathbb{R}$. Write

$$\overline{X}{}_a^b = \overline{X}{}_a^\infty \cap \overline{X}{}_{-\infty}^b \qquad \text{and} \qquad \overline{X}{}_c^d = \overline{X}{}_c^\infty \cap \overline{X}{}_{-\infty}^d.$$

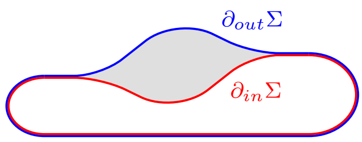
Pick an isomorphism $\psi : \overline{X}{}_a^\infty \cup_\phi \overline{X}{}_{-\infty}^d \to \mathbb{CP}^1$. The image of $\partial_{in}\overline{X}{}_a^\infty$ under ψ (equivalently, the image of $\partial_{out}\overline{X}{}_{-\infty}^d$ under ψ) is a smoothly embedded curve. By the Riemann mapping theorem for simply connected domains with smooth boundary, the map $\psi|_{\overline{X}{}_a^\infty}$ is smooth all the way to the boundary (and holomorphic in the interior). The same holds for $\psi|_{\overline{X}{}_{-\infty}^d}$. It follows that $\psi(\partial_{out}\overline{X}{}_a^b)$ and $\psi(\partial_{in}\overline{X}{}_c^d)$ are smooth curves in $\mathbb{CP}^1$ (being the image of a smooth curve under a smooth map).



The space $\overline{X}{}_a^b \cup_\phi \overline{X}{}_c^d$ can be therefore identified with the subset Y of $\mathbb{CP}^1$ that lies between these two curves.

It remains to identify the sheaf (8) with the set of those functions on Y that are holomorphic in $\mathring{Y}$, and the restriction of a smooth function on an open of $\mathbb{CP}^1$. This is a direct consequence of Lemma 4, once we know that the two descriptions (1) and (2) of the sheaf $\mathcal{O}_{X_a^b}$ are equivalent. That last equivalence is the content of Lemma 3, whose proof we now present. $\square$

Proof of Lemma 3 (second part). Let $f : \Sigma := \overline{X}_a^b \to \mathbb{C}$ be a continuous function which is holomorphic in the interior, and smooth on the boundary. Given an embedding $\Sigma \hookrightarrow \mathbb{C}$

$\Sigma =$  $\subset \mathbb{C}$ (9)

our task is to find a smooth function g on $\mathbb{C}$ that satisfies $g|_\Sigma = f$.

For $c \in \mathbb{R}_{>0}$ large enough, the restriction of $f_1(z) := f(z) + cz$ to $\partial_{in}\Sigma$ is an embedding. Let $D_{out} \subset \mathbb{C} \cup \{\infty\}$ be the closed 'outer' disc bound by $\partial_{in}\Sigma$ (the one containing ∞), and let $D_{in} \subset \mathbb{C}$ be the closed disc bound by $f_1(\partial_{in}\Sigma)$. Let $\phi := f_1|_{\partial D_{out}} : \partial D_{out} \to \partial D_{in}$, and let $\mathbb{CP}^1_\phi := D_{out} \cup_\phi D_{in}$.

Note that $\Sigma \subset D_{out}$. Therefore $\Sigma \cup_\phi D_{in} \subset \mathbb{CP}^1_\phi$.

Pick an isomorphism $\psi : \mathbb{CP}^1_\phi \to \mathbb{CP}^1$. By the Riemann mapping theorem for simply connected domains with smooth boundary, the function $\psi|_{D_{out}}$ is holomorphic in the interior and smooth all the way to the boundary. Let

$$\hat{D} := \psi(\Sigma \cup_\phi D_{in}) \subset \mathbb{C}.$$

The function $f_1 : \Sigma \to \mathbb{C}$ and the inclusion map $\iota : D_{in} \hookrightarrow \mathbb{C}$ assemble to a function $f_1 \cup \iota : \Sigma \cup_\phi D_{in} \to \mathbb{C}$. Let

$$f_2 := (f_1 \cup \iota) \circ \psi^{-1} : \hat{D} \to \mathbb{C}.$$

By Lemmas 2 and 4, the function f_2 is holomorphic in the interior and smooth all the way to the boundary. Pick a C^∞ extension $\hat{f}_2 : \mathbb{CP}^1 \to \mathbb{C}$ of f_2 and let $f_3 := (\hat{f}_2 \circ \psi)|_{D_{out}}$. Note that f_3 agrees with f_2 on Σ, and that it is smooth all the way to the boundary.

Finally, pick a C^∞ extension $\hat{f}_3 : \mathbb{C} \to \mathbb{C}$ of f_3, and let $g(z) := \hat{f}_3(z) - cz$. $\square$

The conformal welding map

$$\boxed{- \cup_{S_2} - \ : \ Cob^{\text{conf}}(S_1, S_2) \times Cob^{\text{conf}}(S_2, S_3) \to Cob^{\text{conf}}(S_1, S_3)}$$

is **smooth**, meaning that a pair of smooth maps

$$M \to Cob^{\text{conf}}(S_1, S_2) \qquad \text{and} \qquad M \to Cob^{\text{conf}}(S_2, S_3)$$

composes to a smooth map

$$M \to Cob^{\text{conf}}(S_1, S_3).$$

The simple reason is that all the ingredients used in this section depend smoothly on parameters. (Specifically, the ingredients are: the Riemann mapping theorem for domains with smooth boundary[6], and taking the inverse of an operator of the form identity plus smoothing operator.)

The conformal welding map is furthermore **holomorphic**, meaning that it maps holomorphic families to holomorphic families. This will be proven later, in Proposition 11 (on p. 38) in the special case of annuli. *[Note that one may not argue as above, as the Riemann mapping theorem does not depend holomorphically on parameters!]*

Now observe that the question of whether holomorphic families of complex cobordisms can be welded into a holomorphic family is a local one ⚠[7]. Any complex cobordism with thin parts is locally isomorphic to an annulus, and ditto for families thereof. The special case of annuli is therefore sufficient to prove the general result.

Full CFT versus chiral CFT

A *concrete linear category* is a pair $(\mathcal{C}, U)$ consisting of a linear category $\mathcal{C}$ together with a faithful functor U from $\mathcal{C}$ to the category of topological vector spaces [*Think:* $\mathcal{C}$ is the category of representations of a group or an algebra, and U the functor which sends a representation to its underlying vector space]. A *concrete functor* $(\mathcal{C}_1, U_1) \to (\mathcal{C}_2, U_2)$ between concrete linear categories is a pair consisting of a linear functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ and a linear natural transformation

$$\begin{tikzcd}
\mathcal{C}_1 \arrow[rr, "F"] \arrow[dr, "U_1"'] & \arrow[d, phantom, "\overset{Z}{\Rrightarrow}"] & \mathcal{C}_2 \arrow[dl, "U_2"] \\
& \mathsf{TopVec} &
\end{tikzcd}$$

A full Segal CFT is a symmetric monoidal functor from the category of complex cobordisms (or rather, a certain central extension of that category by $\mathbb{R}_+$) into the category of topological vector spaces. By contrast, a chiral Segal CFT is a symmetric monoidal functor from the category of complex cobordisms (no central extension) into the category of concrete linear categories. It comes with an extra piece of structure which ensures that the functors (just the functors, not the concrete functors!) associated to annuli are trivial, and there is also a holomorphicity condition.

The word *chiral* in '*chiral CFT*' refers to that last holomorphicity condition.

Let us be precise with what we mean by an "annulus":

Definition 5 *Given a circle S (a manifold diffeomorphic to S^1) let*

$$\mathrm{Ann}(S) := \left\{ \begin{array}{c} \textit{complex cobordisms with thin parts } A \\ + \textit{ orientation preserving diffeomorphisms} \\ \varphi_{in} : S \xrightarrow{\cong} \partial_{in} A, \ \ \varphi_{out} : S \xrightarrow{\cong} \partial_{out} A \end{array} \middle| \begin{array}{c} \partial_{in} A \hookrightarrow A \textit{ and} \\ \partial_{out} A \hookrightarrow A \textit{ are} \\ \textit{htpy equivalences} \end{array} \right\} \Big/ \textit{iso.}$$

[6] S. Bell, *The Cauchy Transform, Potential Theory and Conformal Mapping*, Theorem 28.1.

[7] ⚠ There is a gap in the argument here.

be the semigroup of annuli *with boundary components parametrized by* S*, with operation given by composition of cobordisms (conformal welding).*

The semigroup of annuli admits a certain central extension

$$0 \to \mathbb{C}^{\times} \times \mathbb{Z} \to \tilde{\mathrm{Ann}}(S) \to \mathrm{Ann}(S) \to 0$$

which depends on a number $c \in \mathbb{C}$ called the *central charge* (for rational chiral CFTs, the central charge is always an element of $\mathbb{Q}$).

Definition: *A chiral Segal CFT consists of:*

(1a) For every closed 1-manifold S*, a linear category* $\mathcal{C}(S)$ *isomorphic to* $\mathsf{Vec}_{\mathrm{fd}}^{\oplus r}$ *for some* $r \in \mathbb{N}$*. The assignment* $S \mapsto \mathcal{C}(S)$ *is symmetric monoidal with respect to disjoint union of 1-manifolds, and tensor product of linear categories.*

(1b) For every closed 1*-manifold* S*, a faithful functor* $U : \mathcal{C}(S) \to \mathsf{TopVec}$*. The assignment* $S \mapsto U$ *is a symmetric monoidal transformation from* $S \mapsto \mathcal{C}(S)$ *to the constant 2-functor* $S \mapsto \mathsf{TopVec}$*.*

(2a) For every complex cobordism Σ*, a linear functor* $F_\Sigma : \mathcal{C}(\partial_{\mathrm{in}}\Sigma) \to \mathcal{C}(\partial_{\mathrm{out}}\Sigma)$*. These functors are compatible with the operations of disjoint union, identity cobordisms, and composition of cobordisms.*

(2b) For every complex cobordism Σ*, and every object* $\lambda \in \mathcal{C}(\partial_{\mathrm{in}}\Sigma)$*, a linear map* $Z_\Sigma : U(\lambda) \to U(F_\Sigma(\lambda))$*. The maps* Z_Σ *are compatible with the operations of disjoint union, identity cobordisms, and composition of cobordisms.*

(3a) For every $\tilde{A} \in \tilde{\mathrm{Ann}}(S)$*, a trivialization* $T_{\tilde{A}} : F_A \to \mathrm{id}_{\mathcal{C}(S)}$*. The* $T_{\tilde{A}}$ *are compatible with identities and composition, and the central* $\mathbb{C}^\times$ *acts in a standard way.*

(3b) For every $\lambda \in \mathcal{C}(S)$*, the map which sends* $\tilde{\Sigma}$ *to the composite* $U(T_{\tilde{\Sigma}}) \circ Z_\Sigma : U(V) \to U(F_\Sigma(V)) \to U(V)$ *is continuous on* $\tilde{\mathrm{Ann}}(S)$ *and holomorphic in its interior*[8]*.*

Remark. If we remove the holomorphicity condition in (3b), then one obtains the definition of *weak CFT* mentioned in the introduction. And if one furthermore insists that the categories $\mathcal{C}(S)$ are all trivial (i.e., isomorphic to $\mathsf{Vec}_{\mathrm{fd}}$), then one gets the notion of (heterotic) *full CFT*.

We summarise the above definition of chiral Segal CFT in Table 1.

[8] This is phrased in a somewhat imprecise way. See p. 21 for a more accurate description of the holomorphicity condition.

The items in the first column of that table [items (1a), (2a), (3a)] correspond to the notion of a *modular functor*.[9] In that column, everything is finite dimensional; everything is topological.

The items in the second column [items (1b), (2b), (3b)] correspond to the notion of a *twisted field theory* (the modular functor is the twist). If we were to remove the twist, then we would be left with a single topological vector space for every 1-manifold S, and a single linear map for every complex cobordism.

DEFINITION (SKETCH): CHIRAL SEGAL CFT

(1a) For every 1-manifold S, a category $\mathcal{C}(S)$.	(1b) A 'forgetful' functor $U : \mathcal{C}(S) \to \mathsf{TopVec}$.
(2a) For every cpx cobordism Σ, a functor $F_\Sigma : \mathcal{C}(\partial_{in}\Sigma) \to \mathcal{C}(\partial_{out}\Sigma)$.	(2b) For every $\lambda \in \mathcal{C}(\partial_{in}\Sigma)$, a linear map $Z_\Sigma : U(\lambda) \to U(F_\Sigma(\lambda))$.
(3a) For every $\tilde{A} \in \widetilde{\mathrm{Ann}}(S)$, a trivialization $T_{\tilde{A}} : F_A \to \mathrm{id}_{\mathcal{C}(S)}$.	(3b) For every $\lambda \in \mathcal{C}(S)$, the map $\widetilde{\mathrm{Ann}}(S) \to \mathrm{End}(U(\lambda))$, $\tilde{A} \mapsto U(T_{\tilde{A}}) \circ Z_A$ is holomorphic.

Table 1.

The two items in the first row [items (1a), (1b)] correspond to the idea that, for every 1-manifold S, there is an associated algebra $\mathcal{A}(S)$. That algebra is called the *algebra of observables*, and can be defined as the algebra of endomorphisms of the functor U. In more down-to-earth terms, the algebra of observables is given by

$$\mathcal{A}(S) = \bigoplus_{\substack{\lambda\in\mathcal{C}(S)\\ \lambda \text{ is simple}}} \mathrm{End}(U(\lambda))$$

where the sum ranges of a set of representatives of the isomorphism classes of simple objects of $\mathcal{C}(S)$. Provided we appropriately restrict the class of representations that we allow, we can recover $\mathcal{C}(S)$ as the category of representations of $\mathcal{A}(S)$:

$$\mathcal{C}(S) = \mathrm{Rep}(\mathcal{A}(S)).$$

[9]...at least conjecturally (I say 'conjecturally' because I don't think that this particular definition has ever been compared to the other definitions of modular functor.)

The two items in the second row [items (2a), (2b)] correspond to the idea that, for every complex cobordism Σ, there is an associated $\mathcal{A}(\partial_{out}\Sigma)$-$\mathcal{A}(\partial_{in}\Sigma)$-bimodule H_Σ, equipped with a distinguished 'vacuum vector' $\Omega_\Sigma \in H_\Sigma$. One recovers F_Σ as the functor $H_\Sigma \otimes_{\mathcal{A}(\partial_{in}\Sigma)} -$, and Z_Σ as the operation of tensoring with Ω_Σ:

$$F_\Sigma = H_\Sigma \otimes - \qquad\qquad Z_\Sigma = \Omega_\Sigma \otimes - \,.$$

One may define the bimodule H_Σ and its distinguished vector $\Omega_\Sigma \in H_\Sigma$ it terms of the functor F_Σ and the natural transformation Z_Σ as follows. First of all,

$$H_\Sigma = \bigoplus_{\substack{\lambda\in\mathcal{C}_{in},\,\mu\in\mathcal{C}_{out}\\ \lambda,\mu \text{ simple}}} \mathrm{Hom}\big(\mu, F(\lambda)\big) \otimes \mathrm{Hom}\big(U(\lambda), U(\mu)\big),$$

where, as before, the sum ranges of a set of representatives of the isomorphism classes of simple objects. To construct Ω_Σ, note that for every $\lambda \in \mathcal{C}_{in}$, the map

$$Z_\lambda : U(\lambda) \to U(F(\lambda)) = \bigoplus_{\mu\in\mathcal{C}_{out}} \mathrm{Hom}\big(\mu, F(\lambda)\big) \otimes U(\mu)$$

provides a distinguished vector in $\mathrm{Hom}\big(U(\lambda),\ \bigoplus_{\mu\in\mathcal{C}_{out}} \mathrm{Hom}\big(\mu, F(\lambda)\big) \otimes U(\mu)\big) =$

$$\bigoplus_{\mu\in\mathcal{C}_{out}} \mathrm{Hom}\big(\mu, F(\lambda)\big) \otimes \mathrm{Hom}\big(U(\lambda), U(\mu)\big).$$

The vacuum vector $\Omega_\Sigma \in H_\Sigma$ is the direct sum of all these, indexed over all the simples of $\mathcal{C}_{in}$.

Finally, the two items in the third row [items (3a), (3b)] correspond to the idea that H_Σ depends *topologically* on Σ (this means, in particular, that if Σ_1 and Σ_2 are diffeomorphic cobordisms, then H_{Σ_1} and H_{Σ_2} are isomorphic bimodules), while $\Omega_\Sigma \in H_\Sigma$ depends *holomorphically* on Σ.

Definition: A map $\iota : (\mathcal{C}_1, U_1, F_1, Z_1, T_1) \to (\mathcal{C}_2, U_2, F_2, Z_2, T_2)$ of chiral Segal CFTs is an ***infinitesimal equivalence*** if it's an equivalence at the level of $\mathcal{C}$, F, Z, and for every 1-manifold S and $\lambda \in \mathcal{C}_1(S)$ the comparison map $U_1(\lambda) \to U_2(\iota(\lambda))$ is a dense inclusion.

Remark. Recall that the rationality condition is built into the above definition of chiral Segal CFT. If one wishes to adapt the definition so as to also include *non-rational* theories, then one should: A. Drop the condition that the linear categories $\mathcal{C}(S)$ be isomorphic to $\mathsf{Vec}_{\mathrm{fd}}^{\oplus r}$, B. Only require there to be functors F_Σ associated to open-ended complex cobordisms Σ. Here, a cobordism Σ is said to be *open-ended* if every connected component has non-empty outgoing boundary (equivalently, $\pi_0(\partial_{out}\Sigma) \twoheadrightarrow \pi_0(\Sigma)$).

We do not know how to formulate the condition of infinitesimal equivalence for non-rational chiral Segal CFTs.

The definition of (rational) chiral Segal CFT

In the previous section, we provided a summary of the notion of chiral Segal CFT. Here, we spell out all the details for the convenience of the reader. Recall that rationality is built into the definition of chiral Segal CFT. As before, we organise the definition into six parts, labelled (1a), (1b), (2a), (2b), (3a), (3b).

Main definition

A (rational) chiral Segal CFT of central charge c consists of:

(1a) For every closed (compact, smooth, oriented) 1-manifold S, a category $\mathcal{C}(S)$ isomorphic to $\mathsf{Vec}_{\text{f.d.}}^{\oplus r}$ for some $r \in \mathbb{N}$ which depends on S.

[Think: There is a certain group or algebra associated to S, and $\mathcal{C}(S)$ is the category of representations of that group or algebra (r = number of irreps.)]

For every pair of 1-manifolds S_1, S_2 there is a bilinear functor $\mathcal{C}(S_1) \times \mathcal{C}(S_2) \to \mathcal{C}(S_1 \sqcup S_2) : (\lambda, \mu) \mapsto \lambda \otimes \mu$ which induces an equivalence of categories

$$\mathcal{C}(S_1) \otimes \mathcal{C}(S_2) \xrightarrow{\cong} \mathcal{C}(S_1 \sqcup S_2).$$

Here, given two linear categories $\mathcal{C}$ and $\mathcal{D}$ isomorphic to $\mathsf{Vec}_{\text{f.d.}}^{\oplus r}$, their tensor product $\mathcal{C} \otimes \mathcal{D}$ has objects of the form $\bigoplus c_i \otimes d_i$ for $c_i \in \mathcal{C}$ and $d_i \in \mathcal{D}$, and hom-spaces given by $\mathrm{Hom}_{\mathcal{C}\otimes\mathcal{D}}(\bigoplus c_i \otimes d_i, \bigoplus c'_j \otimes d'_j) = \bigoplus_{ij} \mathrm{Hom}_{\mathcal{C}}(c_i, c'_j) \otimes \mathrm{Hom}_{\mathcal{D}}(d_i, d'_j)$.

We also have an equivalence $\mathsf{Vec}_{\text{f.d.}} \xrightarrow{\cong} \mathcal{C}(\emptyset) : \mathbb{C} \mapsto 1$.

There is an associator $(\lambda \otimes \mu) \otimes \nu \xrightarrow{\cong} \lambda \otimes (\mu \otimes \nu)$, unitors $1 \otimes \lambda \xrightarrow{\cong} \lambda$ and $\lambda \otimes 1 \xrightarrow{\cong} \lambda$, and a braiding $\lambda \otimes \mu \xrightarrow{\cong} \mu \otimes \lambda$ *[we omit the isomorphisms*[10] $(S_1 \sqcup S_2) \sqcup S_3 \cong S_1 \sqcup (S_2 \sqcup S_3)$, $\emptyset \sqcup S \cong S$, $S \sqcup \emptyset \cong S$, *and* $S_1 \sqcup S_2 \cong S_2 \sqcup S_1$*]* which are natural (i.e. for any morphisms $\lambda \to \lambda'$, $\mu \to \mu'$, $\nu \to \nu'$ the following diagrams commute

$$\begin{tikzcd}
(\lambda\otimes\mu)\otimes\nu \arrow[r] \arrow[d] & (\lambda'\otimes\mu')\otimes\nu' \arrow[d] & 1\otimes\lambda \arrow[r] \arrow[d] & 1\otimes\lambda' \arrow[d] & \lambda\otimes 1 \arrow[r] \arrow[d] & \lambda'\otimes 1 \arrow[d] & \lambda\otimes\mu \arrow[r] \arrow[d] & \lambda'\otimes\mu' \arrow[d] \\
\lambda\otimes(\mu\otimes\nu) \arrow[r] & \lambda'\otimes(\mu'\otimes\nu') & \lambda \arrow[r] & \lambda' & \lambda \arrow[r] & \lambda' & \mu\otimes\lambda \arrow[r] & \mu'\otimes\lambda'
\end{tikzcd}$$
)

and subject to the well-known pentagon, triangle, hexagon, and symmetry axioms (the same axioms which appear in the definition of a symmetric monoidal category):

$$\begin{tikzcd}
 & (\lambda\otimes\mu)\otimes(\nu\otimes\rho) \arrow[dr] & \\
((\lambda\otimes\mu)\otimes\nu)\otimes\rho \arrow[ur] \arrow[d] & & \lambda\otimes(\mu\otimes(\nu\otimes\rho)) \\
(\lambda\otimes(\mu\otimes\nu))\otimes\rho \arrow[r] & \lambda\otimes((\mu\otimes\nu)\otimes\rho) \arrow[ur] &
\end{tikzcd}
\quad
\begin{tikzcd}
 & \lambda\otimes\mu \arrow[dr] & \\
(\lambda\otimes 1)\otimes\mu \arrow[ur] \arrow[rr] & & \lambda\otimes(1\otimes\mu)
\end{tikzcd}$$

$$\begin{tikzcd}
 & (\mu\otimes\lambda)\otimes\nu \arrow[r] & \mu\otimes(\lambda\otimes\nu) \arrow[dr] & \\
(\lambda\otimes\mu)\otimes\nu \arrow[ur] \arrow[dr] & & & \mu\otimes(\nu\otimes\lambda) \\
 & \lambda\otimes(\mu\otimes\nu) \arrow[r] & (\mu\otimes\nu)\otimes\lambda \arrow[ur] &
\end{tikzcd}
\quad
\begin{tikzcd}
 & \mu\otimes\lambda \arrow[dr] & \\
\lambda\otimes\mu \arrow[ur] \arrow[rr, "="] & & \lambda\otimes\mu
\end{tikzcd}$$

(1b) For every closed 1-manifold S, a faithful functor $U : \mathcal{C}(S) \to \mathsf{TopVec}$ which equips $\mathcal{C}(S)$ with the structure of a concrete linear category.[11]

[10] These isomorphisms give cobordisms, which give equivalences of catrgories using (2a).

[11] Depending on the type of topological vector spaces one works with, one might want to modify the notion of tensor product accordingly.

If $\mathcal{C}(S) \cong \mathsf{Vec}_{\text{f.d.}}^{\oplus r}$, so that an object can be written as an r-tuple of finite dimensional vector spaces, then the functor U is always of the form $(V_1, \ldots, V_r) \mapsto \bigoplus V_i \otimes W_i$, where the W_i are typically infinite dimensional.

The forgetful functor satisfies $U(\lambda \otimes \mu) = U(\lambda) \otimes U(\mu)$ and $U(1) = \mathbb{C}$, naturally in λ and μ, and compatibly with the associator, unitors, and braiding:

$$
\begin{tikzcd}
U((\lambda \otimes \mu) \otimes \nu) \arrow[r, phantom, "="] \arrow[d] & (U(\lambda) \otimes U(\mu)) \otimes U(\nu) \arrow[d] \\
U(\lambda \otimes (\mu \otimes \nu)) \arrow[r, phantom, "="] & U(\lambda) \otimes (U(\mu) \otimes U(\nu))
\end{tikzcd}
\quad
\begin{tikzcd}
U(1) \otimes U(\lambda) \arrow[r, phantom, "="] \arrow[d, equal] & U(1 \otimes \lambda) \arrow[d] \\
\mathbb{C} \otimes U(\lambda) \arrow[r] & U(\lambda)
\end{tikzcd}
\quad
\begin{tikzcd}
U(\lambda) \otimes U(1) \arrow[r, phantom, "="] \arrow[d, equal] & U(\lambda \otimes 1) \arrow[d] \\
U(\lambda) \otimes \mathbb{C} \arrow[r] & U(\lambda)
\end{tikzcd}
\quad
\begin{tikzcd}
U(\lambda \otimes \mu) \arrow[r, phantom, "="] \arrow[d] & U(\lambda) \otimes U(\mu) \arrow[d] \\
U(\mu \otimes \lambda) \arrow[r, phantom, "="] & U(\mu) \otimes U(\lambda)
\end{tikzcd}
$$

(2a) For every complex cobordism with thin parts Σ from S_1 to S_2, a linear functor $F_\Sigma : \mathcal{C}(S_1) \to \mathcal{C}(S_2)$.

If Σ and Σ' are complex cobordisms from S_1 to S_2, then for every biholomorphic map $\phi : \Sigma \xrightarrow{\cong} \Sigma'$ such that $\phi|_{S_1} = \mathrm{id}$ and $\phi|_{S_2} = \mathrm{id}$, we have an invertible natural transformation $F_\Sigma \cong F_{\Sigma'}$, compatible with composition of maps.

We also have invertible natural transformations $F_{\mathbf{1}_S} \cong \mathrm{id}_{\mathcal{C}(S)}$, $F_{\Sigma_1 \cup \Sigma_2} \cong F_{\Sigma_1} \circ F_{\Sigma_2}$, and $F_{\Sigma_1 \sqcup \Sigma_2} \cong F_{\Sigma_1} \otimes F_{\Sigma_2}$. They are natural with respect to biholomorphic maps of complex cobordisms, and make the following diagrams commute:

$$
\begin{tikzcd}
F_{\mathbf{1}_S \cup \Sigma} \arrow[r] \arrow[d] & F_{\mathbf{1}_S} \circ F_\Sigma \arrow[d] \\
F_\Sigma \arrow[r, phantom, "="] & \mathrm{id}_{\mathcal{C}(S)} \circ F_\Sigma
\end{tikzcd}
\quad
\begin{tikzcd}
F_{\Sigma \cup \mathbf{1}_S} \arrow[r] \arrow[d] & F_\Sigma \circ F_{\mathbf{1}_S} \arrow[d] \\
F_\Sigma \arrow[r, phantom, "="] & F_\Sigma \circ \mathrm{id}_{\mathcal{C}(S)}
\end{tikzcd}
\quad
\begin{tikzcd}
F_{\mathbf{1}_{S_1} \sqcup \mathbf{1}_{S_2}} \arrow[r] \arrow[d] & F_{\mathbf{1}_{S_1}} \otimes F_{\mathbf{1}_{S_2}} \arrow[d] \\
\mathrm{id}_{\mathcal{C}(S_1 \sqcup S_2)} \arrow[r] & \mathrm{id}_{\mathcal{C}(S_1)} \otimes \mathrm{id}_{\mathcal{C}(S_2)}
\end{tikzcd}
\quad
\begin{tikzcd}
F_{\Sigma_1 \cup \Sigma_2 \cup \Sigma_3} \arrow[r] \arrow[d] & F_{\Sigma_1} \circ F_{\Sigma_2 \cup \Sigma_3} \arrow[d] \\
F_{\Sigma_1 \cup \Sigma_2} \circ F_{\Sigma_3} \arrow[r] & F_{\Sigma_1} \circ F_{\Sigma_2} \circ F_{\Sigma_3}
\end{tikzcd}
$$

$$
\begin{tikzcd}
F_{\Sigma \sqcup \emptyset} \arrow[r] \arrow[dr] & F_\Sigma \otimes \mathrm{id}_{\mathsf{Vec}} \arrow[d] \\
& F_\Sigma
\end{tikzcd}
\quad
\begin{tikzcd}
F_{\Sigma_1 \sqcup \Sigma_2 \sqcup \Sigma_3} \arrow[r] \arrow[d] & F_{\Sigma_1} \otimes F_{\Sigma_2 \sqcup \Sigma_3} \arrow[d] \\
F_{\Sigma_1 \sqcup \Sigma_2} \otimes F_{\Sigma_3} \arrow[r] & F_{\Sigma_1} \otimes F_{\Sigma_2} \otimes F_{\Sigma_3}
\end{tikzcd}
\quad
\begin{tikzcd}
F_{(\Sigma_1 \cup \Sigma_2) \sqcup (\Sigma'_1 \cup \Sigma'_2)} \arrow[r] \arrow[d] & F_{\Sigma_1 \cup \Sigma_2} \otimes F_{\Sigma'_1 \cup \Sigma'_2} \arrow[d] \\
F_{\Sigma_1 \sqcup \Sigma'_1} \circ F_{\Sigma_2 \sqcup \Sigma'_2} \arrow[r] & (F_{\Sigma_1} \circ F_{\Sigma_2}) \otimes (F_{\Sigma'_1} \circ F_{\Sigma'_2})
\end{tikzcd}
\quad
\begin{tikzcd}
F_{\Sigma_1 \sqcup \Sigma_2} \arrow[r] \arrow[d] & F_{\Sigma_1} \otimes F_{\Sigma_2} \arrow[d] \\
F_{\Sigma_2 \sqcup \Sigma_1} \arrow[r] & F_{\Sigma_2} \otimes F_{\Sigma_1}
\end{tikzcd}
$$

(The astute reader will have noticed that the above diagrams are a bit sloppy: the functors being compared don't always have the same domain/codomain. Fixing them is not difficult, but would make them very bulky.)

(2b) For every complex cobordism with thin parts Σ from S_1 to S_2 and every object $\lambda \in \mathcal{C}(S_1)$, a continuous linear map $Z_\Sigma : U(\lambda) \to U(F_\Sigma(\lambda))$.

The maps Z_Σ are natural in λ. They're also natural in Σ, meaning that for every biholomorphic map $\phi : \Sigma' \to \Sigma$ fixing S_1 and S_2, and every $\lambda \in \mathcal{C}(S_1)$, we have a commutative diagram

$$
\begin{tikzcd}
U(\lambda) \arrow[r, "Z_\Sigma"] \arrow[d, equal] & U(F_\Sigma(\lambda)) \arrow[d, phantom, "\cong" rotate=90] \\
U(\lambda) \arrow[r, "Z_{\Sigma'}"] & U(F_{\Sigma'}(\lambda))
\end{tikzcd}
$$

We also have $Z_{\mathbf{1_S}} = \mathrm{id}_{U(\lambda)}$, $Z_{\Sigma_1 \cup \Sigma_2} = Z_{\Sigma_1} \circ Z_{\Sigma_2}$, and $Z_{\Sigma_1 \sqcup \Sigma_2} = Z_{\Sigma_1} \otimes Z_{\Sigma_2}$. (Some isomorphisms have been omitted for better readability. For example, the last equality should say that the following diagram is commutative:

$$
\begin{tikzcd}
U(\lambda \otimes \mu) \arrow[r, "Z_{\Sigma_1 \sqcup \Sigma_2}"] \arrow[dd, equal] & U(F_{\Sigma_1 \sqcup \Sigma_2}(\lambda \otimes \mu)) \arrow[d, phantom, "\cong" rotate=90] \\
& U(F_{\Sigma_1}(\lambda) \otimes F_{\Sigma_2}(\mu)) \arrow[d, equal] \\
U(\lambda) \otimes U(\mu) \arrow[r, "Z_{\Sigma_1} \otimes Z_{\Sigma_2}"] & U(F_{\Sigma_1}(\lambda)) \otimes U(F_{\Sigma_2}(\mu))
\end{tikzcd}
$$
)

Before describing items (3a) and (3b) of the definition of chiral Segal CFT, we need a couple of facts about $\mathrm{Diff}(S)$ and its "complexification", the semigroup of annuli $\mathrm{Ann}(S)$.

Let S be a circle (a manifold diffeomorphic to S^1) and let $\mathrm{Diff}(S)$ be its group of orientation preserving diffeomorphisms, and let $\mathrm{Ann}(S)$ be the semigroup of annuli with boundary components parametrized by S. There is an obvious embedding

$$\mathrm{Diff}(S) \hookrightarrow \mathrm{Ann}(S)$$

which sends a diffeomorphism φ to the completely thin annulus $(A{=}S, \varphi_{in}{=}\varphi, \varphi_{out}{=}\,\mathrm{id})$.

We will postpone the proof of the following proposition until after the definition of chiral Segal CFT is completed:

Proposition. *The Lie group* $\mathrm{Diff}(S)$ *admits a universal central extension, whose center is canonically isomorphic to* $i\mathbb{R} \oplus \mathbb{Z}$.

The complex[12] *semigroup* $\mathrm{Ann}(S)$ *admits a universal central extension, whose center is canonically isomorphic to* $\mathbb{C} \oplus \mathbb{Z}$.

Writing ${}^{i\mathbb{R}\oplus\mathbb{Z}}\mathrm{Diff}(S)$ for the universal central extension of $\mathrm{Diff}(S)$, and ${}^{\mathbb{C}\oplus\mathbb{Z}}\mathrm{Ann}(S)$ for the universal central extension of $\mathrm{Ann}(S)$, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & i\mathbb{R}\oplus\mathbb{Z} \arrow[r] \arrow[d] & {}^{i\mathbb{R}\oplus\mathbb{Z}}\mathrm{Diff}(S) \arrow[r] \arrow[d] & \mathrm{Diff}(S) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{C}\oplus\mathbb{Z} \arrow[r] & {}^{\mathbb{C}\oplus\mathbb{Z}}\mathrm{Ann}(S) \arrow[r] & \mathrm{Ann}(S) \arrow[r] & 0
\end{tikzcd}$$

where each vertical arrow is the inclusion of a group into its "complexification".

Given a complex number $c \in \mathbb{C}$ (this will later be the *central charge* of the CFT), we can form the associated central extension

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C}\oplus\mathbb{Z} \arrow[r] \arrow[d, "{(z,n) \mapsto (e^{cz},n)}"'] \arrow[dr, phantom, "\ulcorner"] & {}^{\mathbb{C}\oplus\mathbb{Z}}\mathrm{Ann}(S) \arrow[r] \arrow[d] & \mathrm{Ann}(S) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{C}^\times\oplus\mathbb{Z} \arrow[r] & {}^{\mathbb{C}^\times\oplus\mathbb{Z}}\mathrm{Ann}_c(S) \arrow[r] & \mathrm{Ann}(S) \arrow[r] & 0
\end{tikzcd}$$

where ${}^{\mathbb{C}^\times\oplus\mathbb{Z}}\mathrm{Ann}_c(S)$ is defined as the pushout. Assuming $c \in \mathbb{R}$ (the central charge of a rational CFTs is always a rational number), we can also form the central extension

$$\begin{tikzcd}
0 \arrow[r] & i\mathbb{R}\oplus\mathbb{Z} \arrow[r] \arrow[d, "{(z,n) \mapsto (e^{cz},n)}"'] \arrow[dr, phantom, "\ulcorner"] & {}^{i\mathbb{R}\oplus\mathbb{Z}}\mathrm{Diff}(S^1) \arrow[r] \arrow[d] & \mathrm{Diff}(S^1) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & U(1)\oplus\mathbb{Z} \arrow[r] & {}^{U(1)\oplus\mathbb{Z}}\mathrm{Diff}_c(S^1) \arrow[r] & \mathrm{Diff}(S^1) \arrow[r] & 0
\end{tikzcd}$$

[12] Here, the complex (/ smooth / topological) structures on $\mathrm{Ann}(S)$ are encoded, as in (3), by the data of which maps $M \to \mathrm{Ann}(S)$ are holomorphic (/ smooth / continuous).

which sits as a subgroup ${}^{U(1)\oplus\mathbb{Z}}\mathrm{Diff}_c(S^1) \subset {}^{\mathbb{C}^\times\oplus\mathbb{Z}}\mathrm{Ann}_c(S)$.

We can now finish the definition of chiral Segal CFT of central charge c (the items (1a), (1b), (2a), (2b) didn't depend on c, which is why we only mention it now):

(3a) For every circle S, every annulus $A \in \mathrm{Ann}(S)$, and every lift $\tilde{A} \in {}^{\mathbb{C}^\times\oplus\mathbb{Z}}\mathrm{Ann}_c(S)$, a trivialization $T_{\tilde{A}} : F_A(\lambda) \xrightarrow{\cong} \lambda$.

[Think: 'the map $\Sigma \mapsto F_\Sigma$ is topological.']

The maps $T_{\tilde{A}}$ are natural in λ. They also satisfy $T_{\mathbf{1}_S} = \mathrm{id}$ and $T_{\tilde{A}_1\cup\tilde{A}_2} = T_{\tilde{A}_1} \circ T_{\tilde{A}_2}$ (omitting the isomorphism $F_{A_1\cup A_2} \cong F_{A_1} \circ F_{A_2}$ for better readability). Moreover, the central $\mathbb{C}^\times$ should act in the standard way: $T_{z\tilde{A}} = z \cdot T_{\tilde{A}}$ for every $z \in \mathbb{C}^\times$.

(3b) For every circle S and every object $\lambda \in \mathcal{C}(S)$, the map

$$\begin{array}{ccc} {}^{\mathbb{C}^\times\oplus\mathbb{Z}}\mathrm{Ann}_c(S) & \longrightarrow & \mathrm{End}(U(\lambda)) \\ \Downarrow & & \Downarrow \\ \tilde{A} & \mapsto & \Big(U(\lambda) \xrightarrow{Z_A} U(F_A(\lambda)) \xrightarrow{U(T_{\tilde{A}})} U(\lambda)\Big) \end{array} \tag{10}$$

is holomorphic

[Think: 'the map $A \mapsto Z_A$ is holomorphic.']

Here, $\mathrm{End}(U(\lambda))$ is equipped with the topology of pointwise convergence (the strong operator topology). The map ${}^{\mathbb{C}^\times\oplus\mathbb{Z}}\mathrm{Ann}_c(S) \to \mathrm{End}(U(\lambda))$ required to be continuous, and holomorphic in the sense that for every finite dimensional complex manifold M and every holomorphic map $M \to {}^{\mathbb{C}^\times\oplus\mathbb{Z}}\mathrm{Ann}_c(S)$, the induced map $M \to \mathrm{End}(U(\lambda))$ is holomorphic.

Note that, by a result of Grothendieck, if the $U(\lambda)$ are Banach spaces, a map $M \to \mathrm{End}(U(\lambda))$ is holomorphic in the strong operator topology if and only if it is holomorphic in the norm topology.

Remark. *When the $U(\lambda)$ are more general than Banach spaces, it might makes sense to insist that they be Fréchet spaces, i.e., inverse limits of Banach spaces*

$$U(\lambda) = \varprojlim_{n\in\mathbb{N}} U_n(\lambda).$$

In that case it is probably a good idea to also insist that each U_n be a functor, and that

$$Z = \varprojlim Z_n \qquad \textit{with} \qquad Z_n : U_n(\lambda) \to U_n(F(\lambda)),$$

so that each $(\mathcal{C}, F, T, U_n, Z_n)$ is a chiral Segal CFT in its own right.

Spin CFTs

Spin chiral CFTs are generalizations of chiral CFTs where all manifolds are equipped with **spin structures**, and all vector spaces are replaced by **super vector spaces**.

A *spin structure* on a complex cobordism Σ (possibly with thin parts) is a choice of a square root of its cotangent bundle. More precisely, it consists of a complex line bundle $\mathbb{S}_\Sigma \to \Sigma$, called the *spinor bundle*, together with an isomorphism $\mathbb{S}_\Sigma^{\otimes 2} \cong T^*\Sigma$. Here, the cotangent bundle is defined by $T_p^*\Sigma = \mathfrak{m}_p/\mathfrak{m}_p^2$, were $\mathfrak{m}_p$ is the maximal ideal of functions $f \in \mathcal{O}_\Sigma$ that vanish at the point $p \in \Sigma$. The notion of spin structure extends verbatim to the case of 1-manifolds.

Remark. For a 1-manifold S (always assumed oriented), a square root of its cotangent bundle T^*S is equivalent to a square root of its complexified cotangent bundle $T_\mathbb{C}^*S$. The approach using $T_\mathbb{C}^*S$ makes it easier to understand how a spin structure on a complex cobordism induces a spin structure on its boundary.

Every spin manifold M admits a canonical *spin involution* $s_M : M \to M$. It acts as the identity on the underlying manifold, and acts by -1 on the spinor bundle $\mathbb{S}_M$.

A *super vector space* V is a $\mathbb{Z}/2$-graded vector space $V = V_0 \oplus V_1$, where V_0 is called the *even* (or *bosonic*) part and V_1 is called the *odd* (or *fermionic*) part of V. The category super vector spaces has hom spaces

$$\mathrm{Hom}_{\mathsf{sVec}}(V, W) = \mathrm{Hom}(V_0, W_0) \oplus \mathrm{Hom}(V_1, W_1)$$

and tensor product

$$V \otimes W = \underbrace{\big(V_0 \otimes W_0 \oplus V_1 \otimes W_1\big)}_{\text{even part}} \oplus \underbrace{\big(V_0 \otimes W_1 \oplus V_1 \otimes W_0\big)}_{\text{odd part}}.$$

There is also an internal hom

$$\underline{\mathrm{Hom}}(V, W) = \underbrace{\big(\mathrm{Hom}(V_0, W_0) \oplus \mathrm{Hom}(V_1, W_1)\big)}_{\text{even part}} \oplus \underbrace{\big(\mathrm{Hom}(V_0, W_1) \oplus \mathrm{Hom}(V_1, W_0)\big)}_{\text{odd part}},$$

which is itself a super vector space. The symmetry of the tensor product $V \otimes W \xrightarrow{\cong} W \otimes V$ is given by $v \otimes w \mapsto w \otimes v$ when either v or w is even, and by $v \otimes w \mapsto -w \otimes v$ when both v and w are odd.

A *super linear category* is a linear category $\mathcal{C}$ which is moreover enriched over sVec (i.e., there are of internal homs $\underline{\mathrm{Hom}}(\lambda, \mu) \in$ sVec for any two objects $\lambda, \mu \in \mathcal{C}$) and tensored over sVec (i.e., for any $V \in$ sVec and $\lambda \in \mathcal{C}$, one may form their tensor product $V \otimes \lambda \in \mathcal{C}$).

Every super vector space has a canonical *grading involution* $\gamma_V : V \to V$, which acts by 1 on the even part of V and acts by -1 on the odd part of V. Similarly, every super linear category $\mathcal{C}$ admits a *grading involution* $\gamma_\mathcal{C} : \mathcal{C} \to \mathcal{C}$ which acts as the identity on the set of objects, and is given by $\gamma_{\underline{\mathrm{Hom}}(\lambda,\mu)}$ on morphisms.

The spin variant of the notion of chiral CFTs is very similar to the definition presented in the previous section. It requires the following small number of adjustments. As mentioned already, all 1-manifolds and all cobordisms should be equipped with spin structures, and all vector spaces and all linear categories should be replaced by their super variants. The main novel feature is that:

> *The **spin involution** should always map to the **grading involution**.*

If $s : S \to S$ is the spin involution of a 1-manifold S, then the induced concrete functor

$$\begin{tikzcd} \mathcal{C}(S) \arrow[rr, "F"] \arrow[dr] & \arrow[d, phantom, "Z \Rrightarrow"] & \mathcal{C}(S) \arrow[dl] \\ & \mathsf{Top\,sVec} & \end{tikzcd}$$

should be given by $F = \gamma$ and $Z = \gamma$. More precisely, there should be a specified identification $F \simeq \gamma_{\mathcal{C}(S)}$ such that for any $\lambda \in \mathcal{C}(S)$ the map $Z_\lambda : U(\lambda) \to U(F(\lambda)) \simeq U(\lambda)$ is given by $Z_\lambda = \gamma_{U(\lambda)}$.

We also require that for every spin cobordism Σ from S_1 to S_2, the spin involution $s_\Sigma : \Sigma \cup s_{S_1} \cong s_{S_2} \cup \Sigma$ get sent to the commuting diagram

$$\begin{tikzcd} \mathcal{C}(S_1) \arrow[r, "F_\Sigma"] \arrow[d, "\gamma_{\mathcal{C}(S_1)}"'] & \mathcal{C}(S_2) \arrow[d, "\gamma_{\mathcal{C}(S_2)}"] \\ \mathcal{C}(S_1) \arrow[r, "F_\Sigma"] & \mathcal{C}(S_2) \end{tikzcd}$$

(the natural transformation that fills the square is the identity natural transformation). Note that that the maps $Z_\Sigma : U(\lambda) \to U(F_\Sigma(\lambda))$ are even, ensuring the commutativity of the above diagram.

There is an equivalent way of stating all the above conditions in terms of the algebras of observables $\mathcal{A}(S)$ and the pointed bimodules H_Σ (see page 16). In that language, the spin involution $s_S : S \to S$ of a 1-manifold S should go to the grading involution $\gamma_{\mathcal{A}(S)} : \mathcal{A}(S) \to \mathcal{A}(S)$ of the corresponding algebra of observables, and the spin involution $s_\Sigma : \Sigma \to \Sigma$ of a complex cobordism should induce the grading involution $\gamma_{H_\Sigma} : H_\Sigma \to H_\Sigma$ of the corresponding bimodule. At last, the vacuum vector $\Omega_\Sigma \in H_\Sigma$ should be even.

Remark. Every non-spin chiral Segal CFT $(\mathcal{C}, H, F, Z, T)$ has an associated spin chiral CFT $({}^s\mathcal{C}, {}^sH, {}^sF, {}^sZ, {}^sT)$, where the category associated to a spin 1-manifold S is given by ${}^s\mathcal{C}(S) := \mathsf{sVec} \otimes_{\mathsf{Vec}} \mathcal{C}(S)$.

At first sight, this might seem to contradict the requirement that $s_S \mapsto \gamma_{\mathcal{C}(S)}$. But there is no contradiction as, for categories of the form ${}^s\mathcal{C} = \mathsf{sVec} \otimes_{\mathsf{Vec}} \mathcal{C}$, the grading involution is naturally equivalent to the identity functor. Indeed, the maps $\gamma_H \otimes \mathrm{id}_\lambda : H \otimes \lambda \to H \otimes \lambda$ for $H \otimes \lambda \in \mathsf{sVec} \otimes_{\mathsf{Vec}} \mathcal{C}$ form a natural isomorphism between $\gamma_{{}^s\mathcal{C}}$ and $\mathrm{id}_{{}^s\mathcal{C}}$.

Remark. The condition that the spin involution always map to the grading involution is called the *spin-statistics theorem*. It is possible to consider theories for which the the spin-statistics theorem fails. However, I believe that it is only possible to formulate the condition of unitarity for those spin CFTs that do satisfy the spin-statistics theorem.

The vacuum sector and its symmetries

Let $\mathbb{D} := \{z \in \mathbb{C} \,:\, |z| \le 1\}$, and $S^1 := \partial\mathbb{D}$. Given a chiral Segal CFT, let us define the *unit object*

$$\mathbf{1} \in \mathcal{C}(S^1)$$

to be the image of $1_\emptyset := \mathbb{C} \in \mathsf{Vec}_{\text{f.d.}} = \mathcal{C}(\emptyset)$ under the functor $F_\mathbb{D} : \mathcal{C}(\partial_{in}\mathbb{D}) = \mathcal{C}(\emptyset) \to \mathcal{C}(\partial_{out}\mathbb{D}) = \mathcal{C}(S^1)$. We define the *vacuum sector* H_0 of the CFT to be the underlying vector space of $\mathbf{1}$:

$$H_0 := U(\mathbf{1}).$$

The vacuum sector comes with a *vacuum vector*

$$\Omega := Z_\mathbb{D}(1) \,\in\, H_0$$

defined as the image of $1 \in \mathbb{C}$ under the map $Z_\mathbb{D} : \mathbb{C} = U(1_\emptyset) \to U(F_\mathbb{D}(1_\emptyset)) = U(\mathbf{1})$. More generally, given a complex cobordism Σ with empty incoming boundary, we get a vector space $H_\Sigma := U(F_\Sigma(1_\emptyset))$, and a vacuum vector

$$\Omega_\Sigma := Z_\Sigma(1) \,\in\, H_\Sigma.$$

Remark. In examples of interest, the unit object $\mathbf{1} \in \mathcal{C}(S^1)$ is always simple, equivalently, the vacuum sector H_0 is an irreducible $\mathcal{A}(S^1)$-module, but this property is not guaranteed by the axioms. A chiral Segal CFT with that property is called *irreducible*.

Given a finite collection $(\mathcal{C}_i, U_i, F_i, Z_i, T_i)$ of irreducible Segal CFTs of same central charge, their direct sum is defined on connected manifolds by $S \mapsto (\bigoplus \mathcal{C}_i(S), \bigoplus U_i)$, $\Sigma \mapsto (\bigoplus F_{i,\Sigma}, Z_{i,\Sigma})$, $\tilde{A} \mapsto \bigoplus T_{i,\tilde{A}}$, and is defined on disconnected manifolds to be the tensor product rule of what the theory assigns to each connected component. Every chiral Segal CFT is a direct sum of irreducible ones, and that the direct sum decomposition is canonical. In that sense, the study of chiral Segal CFTs completely reduces to the study of irreducible ones.

Given an object $\lambda \in \mathcal{C}(S^1)$, we sometimes write $H_\lambda := U(\lambda)$. If λ is irreducible and $\lambda \not\cong \mathbf{1}$, we call this a *charged sector* of the CFT.

The vacuum sector depends functorially on the disc $\mathbb{D}$; the automorphism group $\mathrm{Aut}(\mathbb{D}) = PSU(1,1)$ therefore acts of H_0. And since the construction of $\Omega \in H_0$ also depends functorially on $\mathbb{D}$, it is invariant under the action of that group:

$$\Omega \in H_0^{PSU(1,1)}.$$

There are also other (semi)groups which act on H_0. For example, the much bigger semigroup ${}^{\mathbb{C}^\times \oplus \mathbb{Z}}\mathrm{Ann}_c(S)$ acts on H_0, as described in (10). That action does not fix the vacuum vector. There is also a somewhat smaller semigroup which acts H_0 in a way which does fix the vacuum vector:

Definition 6 *The semigroup of univalent maps of the disc is given by*

$$\mathrm{Univ}(\mathbb{D}) := \{\psi : \mathbb{D} \to \mathbb{D} \,|\, \psi \text{ is an embeddings}\}.$$

The inclusion $\mathrm{Univ}(\mathbb{D}) \hookrightarrow \mathrm{Ann}(S^1)$ *which sends* $\psi \in \mathrm{Univ}(\mathbb{D})$ *to the annulus* $A_\psi := \big(\mathbb{D} \setminus \psi(\mathring{\mathbb{D}}), \varphi_{in} = \psi|_{\partial\mathbb{D}}, \varphi_{out} = \mathrm{id}\big)$ *is a semigroup homomorphism:* $A_{\psi_1 \circ \psi_2} = A_{\psi_1} \cup A_{\psi_2}$.

The action of ψ on H_0 is given by

$$H_0 = U(F_\mathbb{D}(1_\emptyset)) \xrightarrow{Z_{A_\psi}} U(F_{A_\psi} F_\mathbb{D}(1_\emptyset)) \cong U(F_{A_\psi \cup \mathbb{D}}(1_\emptyset)) \cong U(F_\mathbb{D}(1_\emptyset)) = H_0 \tag{11}$$

and it indeed fixes the vacuum vector:

$$\Omega = Z_\mathbb{D}(1) \stackrel{Z_{A_\psi}}{\longmapsto} Z_{A_\psi} Z_\mathbb{D}(1) \mapsto Z_{A_\psi \cup \mathbb{D}}(1) \mapsto Z_\mathbb{D}(1) = \Omega.$$

We prove the next lemma under the assumption that the Segal CFT is irreducible (the statement also holds true without that assumption):

Lemma 7 *Let* $\psi : \mathbb{D} \to \mathbb{D}$ *be a univalent map, let* A *be its image in* $\mathrm{Ann}(S)$*, and let* $\tilde{A} \in {}^{\mathbb{C}^\times \times \mathbb{Z}}\mathrm{Ann}_c(S)$ *be an arbitrary lift. Then the actions of* $\tilde{A}$ *and* ψ *on* H_0 *given by* (10) *and* (11) *agree up to scalar.*

Proof. Write S_ψ for the isomorphism $F_{A_\psi} F_\mathbb{D}(1_\emptyset) \xrightarrow{\cong} F_{A_\psi \cup \mathbb{D}}(1_\emptyset) \xrightarrow{\cong} F_\mathbb{D}(1_\emptyset)$. By definition, the actions of $\tilde{A}$ and ψ are given by $U(T_{\tilde{A}}) \circ Z_A$ and $U(S_\psi) \circ Z_A$, respectively. Since $\mathbf{1} = F_\mathbb{D}(1_\emptyset) \in \mathcal{C}(S^1)$ is a simple object, there exists a constant $a \in \mathbb{C}^\times$ such that $T_{\tilde{A}} = a \cdot S_\psi$. It follows that $U(T_{\tilde{A}}) \circ Z_A = a \cdot U(S_\psi) \circ Z_A$. □

Summarizing, we have the following four (semi)groups which all act compatibly on the vacuum sector of a CFT. The ones in the top row act honestly (i.e., without central extension), whereas the ones in the bottom row only act projectively:

$$\begin{tikzcd}
PSU(1,1) \arrow[r, hook] \arrow[d, hook] & \mathrm{Univ}(\mathbb{D}) \arrow[d, hook] & \text{act honestly on } H_0 \arrow[l] \\
\mathrm{Diff}(S^1) \arrow[r, hook] & \mathrm{Ann}(S^1) & \text{only act projectively} \arrow[l]
\end{tikzcd}$$

Let ${}^{\mathbb{Z}}PSU(1,1)$ denote the universal cover of $PSU(1,1)$, which is also its universal central extension. Similarly, let us write ${}^{\mathbb{Z}}\mathrm{Univ}(\mathbb{D})$ for the universal cover of $\mathrm{Univ}(\mathbb{D})$. The inclusion $\mathrm{Univ}(\mathbb{D}) \hookrightarrow \mathrm{Ann}(S)$ induces a map of central extensions

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, "{n \mapsto (0,n)}"'] & {}^{\mathbb{Z}}\mathrm{Univ}(\mathbb{D}) \arrow[r] \arrow[d] & \mathrm{Univ}(\mathbb{D}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{C}^\times \oplus \mathbb{Z} \arrow[r] & {}^{\mathbb{C}^\times \oplus \mathbb{Z}}\mathrm{Ann}_c(S) \arrow[r] & \mathrm{Ann}(S) \arrow[r] & 0
\end{tikzcd}$$

By Lemma 7, the restriction of the action of ${}^{\mathbb{C}^\times\oplus\mathbb{Z}}\mathrm{Ann}_c(S)$ on H_0 to the subsemigroup ${}^{\mathbb{Z}}\mathrm{Univ}(\mathbb{D})$ descends to the quotient $\mathrm{Univ}(\mathbb{D})$. So the action of ${}^{\mathbb{C}^\times\oplus\mathbb{Z}}\mathrm{Ann}_c(S)$ on H_0 descends to

$$ {}^{\mathbb{C}^\times}\mathrm{Ann}_c(S) := {}^{\mathbb{C}^\times\oplus\mathbb{Z}}\mathrm{Ann}_c(S)/\mathbb{Z}. $$

Let also:

$$ {}^{U(1)}\mathrm{Diff}_c(S) := {}^{U(1)\oplus\mathbb{Z}}\mathrm{Diff}_c(S)/\mathbb{Z}. $$

All in all, *the following (semi)groups act on the vacuum sector of any chiral CFT:*

$$\boxed{\begin{array}{ccc} PSU(1,1) & \hookrightarrow & \mathrm{Univ}(\mathbb{D}) \\ \downarrow & & \downarrow \\ {}^{U(1)}\mathrm{Diff}_c(S^1) & \hookrightarrow & {}^{\mathbb{C}^\times}\mathrm{Ann}_c(S^1) \end{array}}$$

Remark. *In the presence of spin structures, all these (semi)groups get replaced by their double covers. In particular, the group $PSU(1,1)$ of Möbius transformations gets replaced by its double cover $SU(1,1)$.*

The above diagrams should be contrasted with the case of charged sectors, where it's only the following (semi)groups which act:

$$\begin{array}{ccc} {}^{\mathbb{Z}}PSU(1,1) & \hookrightarrow & {}^{\mathbb{Z}}\mathrm{Univ}(\mathbb{D}) \\ \downarrow & & \downarrow \\ {}^{U(1)\oplus\mathbb{Z}}\mathrm{Diff}_c(S^1) & \hookrightarrow & {}^{\mathbb{C}^\times\oplus\mathbb{Z}}\mathrm{Ann}_c(S^1) \end{array}$$

Let $\lambda \in \mathcal{C}(S^1)$ be a simple object, and let $H_\lambda = U(\lambda)$ be the corresponding charged sector. For $\tilde{A}$ in the kernel of the map ${}^{\mathbb{C}\oplus\mathbb{Z}}\mathrm{Ann}(S) \to \mathrm{Ann}(S)$,

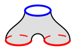
$$ \tilde{A} \in \ker\left({}^{\mathbb{C}\oplus\mathbb{Z}}\mathrm{Ann}(S) \to \mathrm{Ann}(S)\right) \;\cong\; \mathbb{C}\oplus\mathbb{Z}, $$

since F_A and Z_A are trivial, the action (10) of $\tilde{A}$ on H_λ simplifies to $\tilde{A} \mapsto U(T_{\tilde{A}})$, where moreover $T_{\tilde{A}} : \lambda \to \lambda$ is just a scalar. To recapitulate, by (10), for any simple object $\lambda \in \mathcal{C}(S^1)$, the semigroup

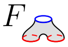
$$ {}^{\mathbb{C}\oplus\mathbb{Z}}\mathrm{Ann}(S) $$

acts on the corresponding charged sector H_λ. The central $\mathbb{C}$ acts via the character $z \mapsto e^{cz}$, where c is the central charge. This is an invariant of the chiral CFT and does not depend on the choice of sector. The central $\mathbb{Z}$ acts via some character $n \mapsto (\theta_\lambda)^n$, where θ_λ is called the conformal spin of the sector (in a rational CFT, the conformal spins are always roots of unity). This number does depend on the sector (for example, the conformal spin of the vacuum sector is always trivial). Let L_0 be the infinitesimal generator of rotations (we'll be more specific about this later). This operator always has spectrum bounded from below, its smallest eigenvalue is denoted h_λ and called the minimal energy. It satisfies $e^{2\pi i h_\lambda} = \theta_\lambda$.

The fusion product

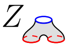
Let $\mathcal{C} := \mathcal{C}(S^1)$. When $\Sigma =$  is a pair of pants, with $\partial_{in} = S^1 \sqcup S^1$ and $\partial_{out} = S^1$, we may identify the linear functor $F_\Sigma : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ with a bilinear functor

$$\boxtimes := F_{\text{pants}} : \begin{array}{ccc} \mathcal{C} \times \mathcal{C} & \to & \mathcal{C} \\ (\lambda,\mu) & \longmapsto & \lambda \boxtimes \mu \end{array}$$



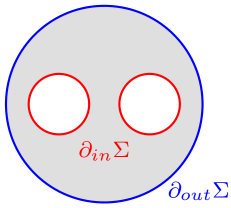
is called the *fusion product*. It comes equipped with a bilinear map

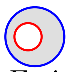
$$Z_{\text{pants}} : \; U(\lambda) \times U(\mu) \;\to\; U(\lambda \boxtimes \mu)$$



between the underling vector spaces (in much the same way as the tensor product $M \otimes_R N$ of two modules M and N over a commutative ring R comes equipped with a bilinear map $M \times N \to M \otimes_R N$).

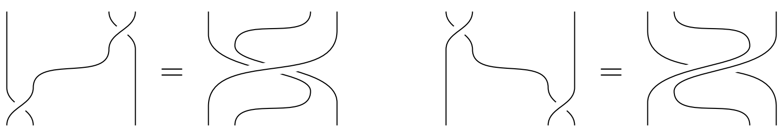
The bilinear map Z_Σ genuinely depends on the complex structure on the pair of pants Σ. But the fusion product F_Σ is essentially independent of that complex structure. For simplicity, let us use pairs of pants Σ embedded in $\mathbb{C}$, where the boundary circles are round, and parametrized by $z \mapsto az + b$ with $a, b \in \mathbb{R}$:

$\Sigma =$  (12)

Note that if $A =$  is an annulus with boundary parametrized by $z \mapsto az + b$ with a and b real, then F_A is canonically trivialized. The trivialization is given by $T_{\tilde{A}}$, where $\tilde{A} \in \mathrm{Univ}(\mathbb{D})^{\mathbb{Z}}$ is the canonical lift of $A \in \mathrm{Univ}(\mathbb{D})$ to an element of the universal cover (using that a and b are real). By composing and un-composing a pair of pants (12) with such annuli, one can reach any other pair of pants of the form (12) in a way which is unique 'up to homotopy'⚠[13]. So the fusion product is well-defined, canonically up to canonical isomorphism.

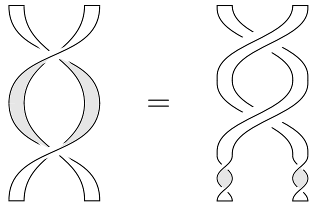
The fusion product is visibly associative and unital, and it endows $\mathcal{C}(S^1)$ with the structure of a monoidal category. But it is more. It's also braided and balanced.

Definition: A monoidal category $(\mathcal{C}, \otimes)$ is *braided* if it's equipped with a family of natural isomorphisms $\beta_{\lambda,\mu} : \lambda \otimes \mu \to \mu \otimes \lambda$ that satisfy the two hexagon axioms:

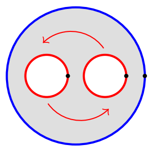
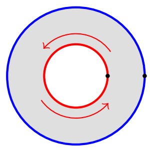


[13]⚠ There is a gap in the argument here.

Definition: A braided monoidal category $(\mathcal{C}, \otimes, \beta)$ is *balanced* if it's equipped with a family of natural isomorphisms $\theta_\lambda : \lambda \to \lambda$ that satisfy $\theta_{\lambda\otimes\mu} = \beta_{\mu,\lambda} \circ \beta_{\lambda,\mu} \circ (\theta_\lambda \otimes \theta_\mu)$. The isomorphism θ_λ is called the *twist*, are is denoted graphically by the full twist of a ribbon. With that graphical notation in mind, the above axiom becomes:



In terms of circles with holes, the braiding β and the twist θ correspond to the motions

 and 

In order to deal with such motions, it's important to relax the condition that the boundary parametrizations be of the form $z \mapsto az + b$ with $a, b \in \mathbb{R}$, and also allow $a, b \in \mathbb{C}$. (The little black dots in the above picture are indicators of where $1 \in S^1$ goes under the boundary parametrizations.)

Let $\mathcal{C} := \mathcal{C}(S^1)$, and let's introduce the following moduli space:

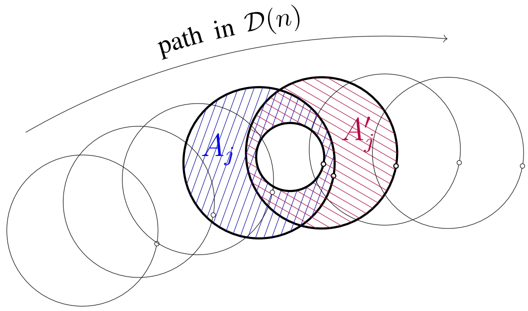
$$\mathcal{D}(n) := \left\{ \begin{matrix} n \text{ non-overlapping round circles in } \mathbb{D} \text{ with} \\ \partial \text{ parametrized by } z \mapsto az + b \text{ with } a, b \in \mathbb{C}. \end{matrix} \right\} \tag{13}$$

For each disc configuration $P \in \mathcal{D}(n)$, we get a functor $F_P : \mathcal{C}^n \to \mathcal{C}$, compatibly with composition. Let us write $P \prec P'$ if every circle of P is contained in the corresponding circle of P'.

Claim: for each homotopy class of path $\gamma : [0, 1] \to \mathcal{D}(n)$ from P_1 to P_2, there is an associated invertible natural transformation $T_\gamma : F_{P_1} \to F_{P_2}$.

The construction of T_γ goes as follows. Subdivide $[0, 1]$ into small intervals $[t_i, t_{i+1}]$ and let $P_i := \gamma(t_i)$. If the subdivision is fine enough, the circles of P_i and of P_{i+1} will have large overlaps. Pick $P'_i \in \mathcal{D}(n)$ such that $P_i \succ P'_i \prec P_{i+1}$, and write $P'_i = P_i \cup (A_1 \sqcup \ldots \sqcup A_n)$ and $P'_i = P_{i+1} \cup (A'_1 \sqcup \ldots \sqcup A'_n)$ for suitable annuli A_j and A'_j. Provided we pick P'_i close enough to P_i and to P_{i+1}, the annuli $A_j, A'_j \in \mathrm{Univ}(\mathbb{D})$ come with preferred lifts $\widetilde{A}_j, \widetilde{A}'_j$ to the universal cover of $\mathrm{Univ}(\mathbb{D})$. We go from F_{P_i} to $F_{P'_i} = F_{P_i} \circ F_{A_1\sqcup\ldots\sqcup A_n}$ by composing with the trivializations $T_{\widetilde{A}_j}$, and we then go back to

$F_{P_{i+1}}$ by composing with the trivializations $T_{\widetilde{A}'_j}$.



(14)

By finely triangulating the domain of a homotopy $h : [0,1]^2 \to \mathcal{D}(n)$ between two paths γ_0 and γ_1 from P_1 to P_2 and playing a similar game as above, we can see that $T_\gamma : F_{P_1} \to F_{P_2}$ only depends on the homotopy class of γ.

The above arguments show that $\mathcal{C}$ is not only monoidal, but also braided, and balanced. But it's even more:

Definition: A monoidal category $(\mathcal{C}, \otimes)$ is called *rigid* if every object $\lambda \in \mathcal{C}$ has a left dual and a right dual. Here, a left dual is an object $\lambda^\vee \in \mathcal{C}$ together with maps $\mathrm{ev} : \lambda^\vee \otimes \lambda \to 1$ and $\mathrm{coev} : 1 \to \lambda \otimes \lambda^\vee$ satisfying $(1_\lambda \otimes \mathrm{ev}) \circ (\mathrm{coev} \otimes 1_\lambda) = 1_\lambda$ and $(\mathrm{ev} \otimes 1_{\lambda^\vee}) \circ (1_{\lambda^\vee} \otimes \mathrm{coev}) = 1_{\lambda^\vee}$. Right duals are defined similarly. Even though this is not obvious from the definition, being rigid is just a property (it's not extra structure). In other words, if an object has a dual (say a left dual), then any two duals are canonically isomorphic.

Definition: A braided tensor category is called *ribbon* if it is balanced, rigid, and for every object $\lambda \in \mathcal{C}$, we have $\mathrm{ev} \circ (\theta_{\lambda^\vee} \otimes 1_\lambda) = \mathrm{ev} \circ (1_{\lambda^\vee} \otimes \theta_\lambda)$.

Definition: A braided tensor category is called *modular* if it is ribbon and the S-matrix $\big[$  $\big]_{\lambda\mu}$ is invertible.

The category $\mathcal{C} = \mathcal{C}(S^1)$ that a chiral Segal CFT assigns to a circle is always modular, but this is hard to prove[14]. Showing that $\mathcal{C}$ is rigid is already very hard. It has been conjectured that the mere fact that $\mathcal{C}$ is part of a modular functor (the left column of the table on page 16) should imply that $\mathcal{C}$ is modular. But this remains an open question.

Remark. The fact that the circles were round in the definition of $\mathcal{D}(n)$ is not so important. What is important is that the parametrizations of the incoming circles extend to holomorphic maps on $\mathbb{D}$, so as to have A_j, A'_j in $\mathrm{Univ}(\mathbb{D})$ in (14).

[14] There is a proof by Y.-Z. Huang in the language of VOAs, and a proof by Kawahigashi–Longo–Mueger in the language of conformal nets.

The Virasoro algebra

One of the wonderful ideas of Graeme Segal, independently due to Y. Neretin, is that the semigroup of annuli, even though it is not a group, is some kind of complexification of $\mathrm{Diff}(S^1)$.

Here, $\mathrm{Diff}(S^1)$ is an infinite dimensional Lie group whose Lie algebra can be identified with the Lie algebra

$$\mathfrak{X}(S^1) := \left\{ f(z)\tfrac{\partial}{\partial z} \,\middle|\, \tfrac{f(z)}{z} \in i\mathbb{R} \right\}$$

of vector fields on S^1. One may similarly associate a Lie algebra to the semigroup of annuli (defined in terms of left invariant vector fields on $\mathrm{Ann}(S^1)$). In the next section, we will check that the Lie algebra of $\mathrm{Ann}(S^1)$ is isomorphic to $\mathfrak{X}_{\mathbb{C}}(S^1)$, the complexification of the Lie algebra of vector fields on S^1. This justifies the claim that $\mathrm{Ann}(S^1)$ behaves like $\mathrm{Diff}_{\mathbb{C}}(S^1)$.

We recall the well-known formula for the Lie bracket of vector fields:

$$\left[f(z)\tfrac{\partial}{\partial z}, g(z)\tfrac{\partial}{\partial z} \right]_{Lie} = (fg' - gf')\tfrac{\partial}{\partial z} \tag{15}$$

Remark. *It is a great annoyance in differential geometry that, for a manifold M, the Lie algebra of* $\mathrm{Diff}(M)$ *is not* $\mathfrak{X}(M)$ *but instead the Lie algebra of vector fields equipped with the opposite of the usual Lie bracket of vector fields.*

In order to avoid this annoyance and the various minus signs that it creates, we will always be endowing $\mathfrak{X}(S^1)$ with the following Lie bracket:

$$\left[f(z)\tfrac{\partial}{\partial z}, g(z)\tfrac{\partial}{\partial z} \right] := \left[f(z)\tfrac{\partial}{\partial z}, g(z)\tfrac{\partial}{\partial z} \right]_{Lie}^{\mathrm{op}} = (gf' - fg')\tfrac{\partial}{\partial z}. \tag{16}$$

The complexification $\mathfrak{X}_{\mathbb{C}}(S^1)$ of $\mathfrak{X}(S^1)$ admits a topological basis given by the vector fields[15]

$$\ell_n := z^{n+1}\tfrac{\partial}{\partial z}.$$

These satisfy an algebra known as the *Witt algebra*:

$$[\ell_m, \ell_n] = (m-n)\ell_{m+n}.$$

Remark. *When acting on the vacuum sector of a CFT, the ℓ_n with $n < 0$ will be acting as creation operators, whereas ℓ_n with $n > 0$ will be acting as annihilation operators. The operator associated to ℓ_0 will have an interpretation as "the energy", and will always have positive spectrum.*

The goal of the next couple sections is to describe the universal central extensions of $\mathrm{Diff}(S)$ and of $\mathrm{Ann}(S)$. We claim that there exist central extensions

$$\begin{gathered} 0 \to i\mathbb{R} \oplus \mathbb{Z} \to {}^{i\mathbb{R}\oplus\mathbb{Z}}\mathrm{Diff}(S) \to \mathrm{Diff}(S) \to 0 \\ 0 \to \mathbb{C} \oplus \mathbb{Z} \to {}^{\mathbb{C}\oplus\mathbb{Z}}\mathrm{Ann}(S) \to \mathrm{Ann}(S) \to 0 \end{gathered} \tag{17}$$

[15] Had we been working with the Lie bracket (15), we would have written $\ell_n = -z^{n+1}\frac{\partial}{\partial z}$.

and that these are universal central extensions.

At the level of Lie algebras (things are always easier at the Lie algebra level), the corresponding claim is that there exist central extensions

$$0 \to i\mathbb{R} \to {}^{i\mathbb{R}}\mathfrak{X}(S) \to \mathfrak{X}(S) \to 0$$
$$0 \to \mathbb{C} \to {}^{\mathbb{C}}\mathfrak{X}_{\mathbb{C}}(S) \to \mathfrak{X}_{\mathbb{C}}(S) \to 0$$

and that these are universal central extensions. If one takes S to be the standard circle S^1, one can be more specific, and write down the cocycle that describes these central extensions. This is the *Virasoro cocycle*:

$$\omega_{Vir}\big(f(z)\tfrac{\partial}{\partial z}, g(z)\tfrac{\partial}{\partial z}\big) = \tfrac{1}{12}\int_{S^1} \tfrac{\partial^3 f}{\partial z^3}(z)\, g(z)\, \tfrac{dz}{2\pi i} \tag{18}$$

equivalently: $$\omega_{Vir}(\ell_m, \ell_n) = \tfrac{1}{12}(m^3 - m)\delta_{m+n,0}.$$

The Witt algebra $\mathbb{W}$ is the algebraic span of the ℓ_n's (the Lie algebra of algebraic vector fields on $\mathbb{C}^\times$), and its universal central extension is called the *Virasoro algebra*. It is standard convention to denote the basis vectors of the Witt algebra ℓ_n, and the corresponding basis vectors of the Virasoro by upper case letters L_n:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{C} & \longrightarrow & Vir & \longrightarrow & \mathbb{W} & \longrightarrow & 0 \\ & & & & \Cup & & \Cup & & \\ & & & & L_n & \mapsto & \ell_n & & \end{array}$$

The commutation relations of the Virasoro algebra are given by

$$\boxed{[L_m, L_n] \;=\; (m-n)L_{m+n} + \tfrac{1}{12}(m^3 - m)\delta_{m+n,0}.}$$

Remark. The Virasoro commutation relations are usually written in a way that includes the central charge:

$$[L_m, L_n] \;=\; (m-n)L_{m+n} + \tfrac{c}{12}(m^3 - m)\delta_{m+n,0}. \tag{19}$$

Let Vir_c be the Lie algebra defined by (19). Provided c and c' are non-zero, there is an isomorphism $Vir_c \cong Vir_{c'}$ that fits in a diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{C} \arrow[r] \arrow[d, "{z \mapsto z\cdot c'/c}"'] & Vir_c \arrow[r] \arrow[d, "\cong"'] & \mathbb{W} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbb{C} \arrow[r] & Vir_{c'} \arrow[r] & \mathbb{W} \arrow[r] & 0 \end{tikzcd}$$

The Lie algebras Vir_c and $Vir_{c'}$ are however distinct *as central extensions of* $\mathbb{W}$ *by* $\mathbb{C}$ (i.e, there's no way to arrange for the left vertical map to be the identity map $\mathrm{id}_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}$).

Also, the notion of a *representation* of Vir_c is distinct from that of a representation of $Vir_{c'}$, because one always includes the requirement that the central element $1 \in \mathbb{C}$ acts by the identity operator.

In order to classify central extensions of the Witt algebra, we will review some notions of Lie algebra cohomology.

Lie algebra cohomology

Let $\mathfrak{g}$ be a Lie algebra, and let A be a vector space.

A 2-cocycle is a bilinear map $\omega : \mathfrak{g} \times \mathfrak{g} \to A$ which is antisymmetric, and satisfies

$$\sum^{3} \omega([X, Y], Z) = 0.$$

Given a 2-cocycle, one can form a central extension $\tilde{\mathfrak{g}} := \mathfrak{g} \oplus A$, with Lie bracket

$$\big[(X, a), (Y, b)\big]_{\tilde{\mathfrak{g}}} := \big([X, Y]_{\mathfrak{g}}, \omega(X, Y)\big)$$

which fits into a central extension $0 \to A \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0$ (an extension such that $A \subset Z(\tilde{\mathfrak{g}})$). If the cocycle can written in the form

$$\omega(X, Y) = \mu([X, Y])$$

for some linear map $\mu : \mathfrak{g} \to A$ (typically not a Lie algebra homomorphism), then we say that ω is a trivial 2-cocycle, and write $\omega = d\mu$.

Theorem. *The second Lie algebra cohomology group*

$$H^2(\mathfrak{g}; A) := \frac{\{\text{2-cocycles}\}}{\{\text{trivial 2-cocycles}\}}$$

is canonically isomorphic to the set of isomorphism classes of central extensions of $\mathfrak{g}$ by A, where two central extensions $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}'$ are called isomorphic *if there exists a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\mathrm{id}_A"'] & \tilde{\mathfrak{g}} \arrow[r] \arrow[d, "\cong"'] & \mathfrak{g} \arrow[r] \arrow[d, "\mathrm{id}_{\mathfrak{g}}"'] & 0 \\
0 \arrow[r] & A \arrow[r] & \tilde{\mathfrak{g}}' \arrow[r] & \mathfrak{g} \arrow[r] & 0
\end{tikzcd}$$

where the two outer vertical maps are identity maps.

Proof outline. ($\Rightarrow$) We already saw how to construct a central extension from a 2-cocycle. Suppose now that $\omega_2 - \omega_1 = d\mu$. Then

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & \tilde{\mathfrak{g}}_1 = \mathfrak{g} \oplus A \arrow[r] \arrow[d, "\left(\begin{smallmatrix} 1 & 0 \\ \mu & 1 \end{smallmatrix}\right)"] & \mathfrak{g} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r] & \tilde{\mathfrak{g}}_2 = \mathfrak{g} \oplus A \arrow[r] & \mathfrak{g} \arrow[r] & 0
\end{tikzcd}$$

is an isomorphism. So the map $\{\text{2-cocycles}\} \to \{\text{central extensions}\}$ descends to a map $H^2(\mathfrak{g}; A) \to \{\text{iso classes of central extensions}\}$.

($\Leftarrow$) Given a central extension of $\mathfrak{g}$ by A, pick a splitting

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & \tilde{\mathfrak{g}} \arrow[r, "r"'] & \mathfrak{g} \arrow[l, bend right, "s"'] \arrow[r] & 0
\end{tikzcd}$$

(usually not a Lie algebra homomorphism) and let $\omega(X, Y) := [s(X), s(Y)] - s([X, Y])$. Given another splitting, we can write it as $s' = s + \mu$ for some $\mu : \mathfrak{g} \to A$. The corresponding cocycles satisfy $\omega' = \omega - d\mu$. So they're equal in $H^2(\mathfrak{g}; A)$. $\square$

We also have:

Proposition 8 *Let $0 \to A_i \to \tilde{\mathfrak{g}}_i \to \mathfrak{g} \to 0$ be central extensions that fit into a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \longrightarrow & \tilde{\mathfrak{g}}_1 & \longrightarrow & \mathfrak{g} & \longrightarrow & 0 \\ & & {\scriptstyle f}\big\downarrow & & \big\downarrow & & \big\Vert & & \\ 0 & \longrightarrow & A_2 & \longrightarrow & \tilde{\mathfrak{g}}_2' & \longrightarrow & \mathfrak{g} & \longrightarrow & 0 \end{array}$$

and let $[\omega_i] \in H^2(\mathfrak{g}_i, A_i)$ be the corresponding cohomology classes. Then $[\omega_2]$ is the image of $[\omega_1]$ under the map $H^2(\mathfrak{g}, A_1) \to H^2(\mathfrak{g}, A_2)$ induced by $f : A_1 \to A_2$.

Theorem 9 *The second cohomology of the Witt algebra is one dimensional $H^2(\mathbb{W}, \mathbb{C}) \cong \mathbb{C}$, and the Virasoro cocycle $[\omega_{Vir}]$ is a generator.*

Let us check that ω_{Vir} is indeed a cocycle. For the purpose of this computation, we rewrite (18) in the following abbreviated (and less precise) form:

$$\omega_{Vir}(f, g) = \oint f'''g = \oint f'g''$$

We then easily compute: $\displaystyle\sum^3 \oint (fg' - f'g)'h'' = \sum^3 \oint \big(\cancel{f'g'} + fg'' - f''g - \cancel{f'g'}\big)h'' = 0.$

The following lemma will be surprisingly useful:

Lemma 10 *Let $\mathfrak{g}$ be a Lie algebra, and let $X \in \mathfrak{g}$ be such that $\mathrm{ad}(X)$ exponentiates to a 1-parameter family of automorphisms of $\mathfrak{g}$* [<u>For us</u>: $\mathfrak{g} = \mathbb{W}$, $X = i\ell_0$, and $\mathrm{ad}(i\ell_0)$ exponentiates to an action of S^1 on $\mathbb{W}$]. *For $\xi \in \mathfrak{g}$, let $\xi_t := \exp\big(t \cdot \mathrm{ad}(X)\big)(\xi)$, so that $\frac{d}{dt}\xi_t = [X, \xi_t]$. Then, for any 2-cocycle ω, we have*

$$[\omega] = [\omega_t] \in H^2(\mathfrak{g}),$$

where $\omega_t(\xi, \eta) := \omega(\xi_t, \eta_t)$.

Proof.

$$\begin{aligned} \omega(\xi_T, \eta_T) - \omega(\xi, \eta) &= \int_0^T \big(\tfrac{d}{dt}\omega(\xi_t, \eta_t)\big)dt \\ &\overset{\frac{d}{dt}\xi_t = [X, \xi_t]}{=} \int_0^T \big(\omega([X, \xi_t], \eta_t) + \omega(\xi_t, [X, \eta_t])\big)dt \\ &\overset{\text{cocycle identity}}{=} \int_0^T \omega(X, [\xi_t, \eta_t])dt \\ &\overset{\xi \mapsto \xi_t \text{ is an automorphism}}{=} \int_0^T \omega(X, [\xi, \eta]_t)dt = \mu([\xi, \eta]) \end{aligned}$$

where $\mu(\xi) := \int_0^T \omega(X, \xi_t)dt$. $\square$

Suppose (as is the case in our example of interest), that $\mathrm{ad}(X)$ exponentiates to an action of S^1 on $\mathfrak{g}$ by Lie algebra automorphisms. Then letting $\mathrm{avg}_{S^1}(\omega) := \int_{S^1} \omega_t dt$, we have

$$\big[\mathrm{avg}_{S^1}(\omega)\big] = \Big[\int_{S^1}\omega_t dt\Big] = \int_{S^1}[\omega_t]dt = \int_{S^1}[\omega]dt = [\omega] \qquad \text{in } \; H^2(\mathfrak{g}, A)$$

for any 2-cocycle ω. Given a linear map $\mu : \mathfrak{g} \to A$, let $\mu_t(\xi) := \mu(\xi_t)$, and let us define $\mathrm{avg}_{S^1}(\mu) := \int_{S^1}\mu_t dt$. If a 2-cocycle ω is trivial, i.e., if there exists μ such that $\omega = d\mu$, then there also exists an S^1-invariant μ with that same property: indeed, letting $\mu' := \mathrm{avg}_{S^1}(\mu)$ we have

$$d\mu' = d\big(\mathrm{avg}_{S^1}(\mu)\big) = \mathrm{avg}_{S^1}(d(\mu)) = \mathrm{avg}_{S^1}(\omega) = \omega.$$

From the above discussion, we deduce that

$$H^2(\mathfrak{g}; A) \;=\; \frac{\big\{\,S^1\text{-invariant 2-cocycles}\,\big\}}{\big\{d\mu \,\big|\, \mu : \mathfrak{g} \to A,\;\; \mu \text{ is } S^1\text{-invariant}\big\}}$$

Remark. *The same argument works with any compact Lie group H in place of S^1. Let $\mathfrak{g}$ be a (typically infinite dimensional) Lie algebra, and let $\mathfrak{h} \subset \mathfrak{g}$ be a finite dimensional subalgebra such that the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ exponentiates to the action of a compact Lie group H on $\mathfrak{g}$. Then $H^2(\mathfrak{g}; A) = \{H\text{-invariant 2-cocycles}\}/\{d\mu \,|\, \mu \text{ is } H\text{-invariant}\}$.*

Armed with the above description of $H^2(\mathfrak{g}; A)$ we can prove the theorem:

Proof of Theorem 9. We'll show that the space of S^1-invariant 2-cocycles $\omega : \mathbb{W}\times\mathbb{W} \to \mathbb{C}$ is two dimensional, spanned by the cocycles

$$\omega_1(\ell_m, \ell_n) := m^3{\cdot}\delta_{m+n,0} \qquad \text{and} \qquad \omega_2(\ell_m, \ell_n) := m{\cdot}\delta_{m+n,0},$$

and that the space of 2-cocycles which are of the form $d\mu$ for some S^1-invariant $\mu : \mathbb{W} \to \mathbb{C}$ is one dimensional, spanned by ω_2. It will follow that $\dim(H^2(\mathbb{W}, \mathbb{C})) = 2 - 1 = 1$.

First of all, the space of S^1-invariant linear functionals $\mathbb{W} \to \mathbb{C}$ is one dimensional, spanned by $\mu : \ell_n \mapsto \delta_{n,0}$. An easy computation yields $d\mu = 2\omega_2$.

Let now ω be an S^1-invariant 2-cocycle. Let $c_{m,n} = \omega(\ell_m, \ell_n)$. S^1-invariance implies that $c_{m,n} = 0$ when $m + n \neq 0$. So let's write $c_n = \omega(\ell_n, \ell_{-n})$. We have $c_{-n} = -c_n$ by antisymmetry. The cocycle identity $\sum^3 \omega([\ell_m, \ell_n], \ell_p) = 0$ is only interesting when $m + n + p = 0$ (otherwise it's trivially satisfied). At the level of the c_n's, it reads

$$(m - n)c_{m+n} + (n - p)c_{n+p} + (p - m)c_{p+m} = 0.$$

Plugging in $p = -m - n$, we get

$$(m - n)c_{m+n} + (2n + m)c_{-m} - (2m + n)c_{-n} = 0.$$

Equivalently,

$$(m - n)c_{m+n} = (2n + m)c_m - (2m + n)c_n.$$

The case $n = 1$ of the above equation reads:

$$(m - 1)c_{m+1} = (2 + m)c_m - (2m + 1)c_1$$

It is a recurrence relation that expresses c_{m+1} in terms of c_m and c_1, provided $m+1 \geq 3$ (otherwise $m-1$ might be zero).

The sequence $\{c_m\}_{m\geq 1}$ is therefore entirely determined by the values of c_1 and of c_2. In particular, the space of S^1-invariant 2-cocycles is at most two dimensional. We already know that that space is at least two dimensional. So it's two dimensional. □

Theorem 9 means that the Virasoro algebra is a universal central extension of the Witt algebra (I should say *the* universal central extension, because universal central extensions are unique up to unique isomorphism). Namely, by the same computation as above, one checks that $H^2(\mathbb{W}, A) = A$ for any vector space A. More precisely, any 2-cocycle is equivalent to $a \cdot \omega_{Vir}$ for some $a \in A$. Effectively, what this produces is, for every central extension $0 \to A \to \tilde{\mathfrak{g}} \to \mathbb{W} \to 0$, a homomorphism of central extensions

$$\begin{array}{cccccccccc} & 0 & \longrightarrow & \mathbb{C} & \longrightarrow & Vir & \longrightarrow & \mathbb{W} & \longrightarrow & 0 \\ \exists! & & & \downarrow & & \downarrow & & \| & & \\ \forall & 0 & \longrightarrow & A & \longrightarrow & \tilde{\mathfrak{g}} & \longrightarrow & \mathbb{W} & \longrightarrow & 0 \end{array}$$

The left vertical map $\mathbb{C} \to A$ is uniquely characterized (by Proposition 8) by the fact that it sends $1 \in H^2(\mathbb{W}, \mathbb{C}) = \mathbb{C}$ to the element $a \in H^2(\mathbb{W}, A) = A$ that classifies the central extension $\tilde{\mathfrak{g}}$. The middle vertical map is also unique, because Vir is spanned by cummutators of lifts of elements of $\mathbb{W}$. That's exactly what it means, by definition, that the Virasoro algebra is the universal central extension of the Witt algebra.

Remark. *The same argument shows that the continuous cohomology $H^2_{cts}(\mathfrak{X}_{\mathbb{C}}(S^1), \mathbb{C})$ (defined in the same way as usual Lie algebra cohomology, except that we now also require the cocycles to be continuous) is one dimensional, generated by ω_{Vir}.*

The same proof can also be adapted to show that $H^2_{cts}(\mathfrak{X}(S^1), i\mathbb{R}) = \mathbb{R}$, generated by ω_{Vir}. (That last statement also follows from the fact that cohomology commutes with complexification: $H^2(\mathfrak{g}, A) \otimes_{\mathbb{R}} \mathbb{C} = H^2(\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}, A \otimes_{\mathbb{R}} \mathbb{C})$.)

Little Fact. Inside the two dimensional space of S^1-invariant 2-cocycles, there is a one dimensional space of $PSU(1,1)$-invariant ones, spanned by ω_{Vir}. That's a good reason to prefer ω_{Vir} as opposed to, say, $(\ell_m, \ell_n) \mapsto m^3 \cdot \delta_{m+n,0}$.

Unfortunately, *the Virasoro cocycle is not coordinate independent*. What this means in practice is that, given a circle S (a manifold diffeomorphic to S^1), there is *no canonical* 2-cocycle on $\mathfrak{X}(S)$ (or on $\mathfrak{X}_{\mathbb{C}}(S)$). But the concept of universal central extension sill makes sense. And the good thing is that, if it exists, a universal central extension is unique up to unique isomorphism. So, even though we don't have a cocycle, we can still talk about *the* universal central extension of $\mathfrak{X}(S)$ (or of $\mathfrak{X}_{\mathbb{C}}(S)$) for any circle S.

The semigroup of annuli as a complexification of $\mathrm{Diff}(S^1)$

Recall that the Lie algebra $\mathfrak{X}_{\mathbb{C}}(S^1)$ admits a topological basis given by the vector fields $\ell_n := z^{n+1}\frac{\partial}{\partial z}$. The subalgebra spanned by ℓ_{-1}, ℓ_0, and ℓ_0 is isomorphic to $\mathfrak{su}(1,1)_{\mathbb{C}} =$

$\mathfrak{sl}(2,\mathbb{C})$ under the isomorphism

$$\ell_{-1} \leftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \ell_0 \leftrightarrow \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \ell_1 \leftrightarrow \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix},$$

and corresponds to the subgroup $PSU(1,1) \subset \mathrm{Diff}(S^1)$.

The Witt algebra $\mathbb{W} = \mathrm{Span}\{\ell_n\}_{n\in\mathbb{Z}}$ is famously known for being a Lie algebra that does not have an associated Lie group. Said otherwise, $\mathrm{Diff}(S^1)$ does not have a complexification (at least not one which is a Lie group). To see that, note that the subalgebra $\mathrm{Span}\{\ell_{-n},\ell_0,\ell_n\} \subset \mathbb{W}$ is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$ via the map

$$\tfrac{1}{n}\ell_{-n} \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \tfrac{1}{n}\ell_0 \mapsto \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \tfrac{1}{n}\ell_n \mapsto \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

If $\mathbb{W}$ or $\mathfrak{X}_\mathbb{C}(S^1)$ were to integrate to a Lie group G, then the above map $\mathfrak{sl}(2,\mathbb{C}) \to \mathbb{W}$ would integrate to a homomorphism $SL(2,\mathbb{C}) \to G$ (since $SL(2,\mathbb{C})$ is simply connected). But the relation $\exp\left(4\pi i\cdot\frac{1}{2}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right) = 1$ holds in $SL(2,\mathbb{C})$. Therefore, for every n, the relation $\exp(4\pi i\cdot\frac{1}{n}\ell_0) = 1$ would have to hold in G. Clearly impossible. $\lightning$

Even though there is no Lie group that complexifies $\mathrm{Diff}(S^1)$, there is a complex semigroup which plays that role: the semigroup of annuli. To see that $\mathrm{Ann}(S^1)$ indeed behaves like $\mathrm{Diff}_\mathbb{C}(S^1)$, we compute its tangent space at the identity $T_1\mathrm{Ann}(S^1)$ and check that it is isomorphic to $\mathfrak{X}_\mathbb{C}(S^1)$.

There is of course a problem, because $\mathrm{Ann}(S^1)$ is not a manifold. It is some kind of infinite dimensional manifold with boundary (or rather with corners), and its identity element sits at the boundary (or rather at some corner of infinite codimension). So it is not clear a priori that the tangent space $T_1\mathrm{Ann}(S^1)$ is a well defined concept. But things are not too bad. The semigroup $\mathrm{Ann}(S^1)$ is the closure of an open subspace of an honest manifold M, which allows up to define its tangent bundle $T\mathrm{Ann}(S^1)$ as the pullback of the tangent bundle of M along the inclusion $\mathrm{Ann}(S^1) \hookrightarrow M$.

Let $\mathbb{D}_- := \{z \in \mathbb{C}\cup\{\infty\} : |z| \geq 1\}$ and $\mathbb{D}_+ := \{z \in \mathbb{C} : |z| \leq 1\}$. Given an annulus A, there is a unique biholomorphic map

$$\psi : \mathbb{D}_- \cup A \cup \mathbb{D}_+ \xrightarrow{\cong} \mathbb{CP}^1$$

sending $\infty \in \mathbb{D}_-$ to $\infty \in \mathbb{CP}^1$, with $\psi'(\infty) = 1$ and $\psi''(\infty) = 0$. Letting $\psi_\pm = \pi|_{\mathbb{D}_\pm}$, this identifies $\mathrm{Ann}(S^1)$ with the space of pairs of embeddings

$$\mathrm{Ann}(S^1) \cong \left\{ \begin{array}{c} \psi_- : \mathbb{D}_- \hookrightarrow \mathbb{CP}^1 \\ \psi_+ : \mathbb{D}_+ \hookrightarrow \mathbb{CP}^1 \end{array} \middle| \begin{array}{c} \psi_+(z) = a_0 + a_1 z + a_2 z^2 + \dots \\ \psi_-(z) = z + b_1 z^{-1} + b_2 z^{-2} + \dots \\ \psi_-(\mathring{\mathbb{D}}_-) \cap \psi_+(\mathring{\mathbb{D}}_+) = \emptyset \end{array} \right\}.$$

Dropping the disjointness condition $\psi_-(\mathring{\mathbb{D}}) \cap \psi_+(\mathring{\mathbb{D}}) = \emptyset$, we obtain a manifold $M := \{\psi_\pm : \mathbb{D}_\pm \hookrightarrow \mathbb{CP}^1 | \psi_-(z) = z + \mathcal{O}(z^{-1})\}$ in which $\mathrm{Ann}(S^1)$ sits as a closed subspace. Moreover, the interior $\mathring{\mathrm{Ann}}(S^1) := \{A \in \mathrm{Ann}(S^1) \,|\, A \text{ is thick}\}$ of $\mathrm{Ann}(S^1)$ is an open subset of M.

The unit element maps to the point $(\mathrm{id}, \mathrm{id}) \in M$, and the tangent space $T_{(\mathrm{id},\mathrm{id})}M$ can be identified with the set of pairs of vector fields (v_-, v_+), $v_\pm \in \mathfrak{X}_{\mathrm{hol}}(\mathbb{D}_\pm)$, where v_- vanishes to third order at $\infty \in \mathbb{D}_-$. Here, $\mathfrak{X}_{\mathrm{hol}}(\mathbb{D}_-) = \mathrm{Span}\{\ell_n\}_{n\leq 1}$ and $\mathfrak{X}_{\mathrm{hol}}(\mathbb{D}_+) = \mathrm{Span}\{\ell_n\}_{n\geq -1}$, with the coefficients of ℓ_n decaying faster than any power as $n \to \infty$. The condition that v_- vanishes to third order at ∞ says that the first three coefficients (the coefficients of ℓ_{-1}, ℓ_0, ℓ_1) are zero. So we get

$$T_{(\mathrm{id},\mathrm{id})}M = \mathrm{Span}\{\ell_n\}_{n<-1} \oplus \mathrm{Span}\{\ell_n\}_{n\geq -1} = \mathfrak{X}_{\mathbb{C}}(S^1)$$

as desired.

Remark. The above computation works equally well at any point $A \in \mathrm{Ann}(S^1)$:

$$T_A\mathrm{Ann}(S) = \big\{(v_\pm) \in \mathfrak{X}_{\mathrm{hol}}(\mathbb{D}_\pm) \,\big|\, v_- \text{ vanishes to third order at } \infty\big\}.$$

But, even though correct, that formula is misleading. ***Yes:*** the tangent bundle $T\mathrm{Ann}(S)$ is trivial, but ***No:*** the left translation map $\ell_A : T_{\mathbf{1}}\mathrm{Ann}(S) \to T_A\mathrm{Ann}(S)$ is not an isomorphism (unless A is completely thin). The map ℓ_A is injective but not surjective.

Here is a descption of the tangent space of $\mathrm{Ann}(S)$ at an arbitrary point $A \in \mathrm{Ann}(S)$ which is well suited to understanding the left translation map $T_{\mathbf{1}}\mathrm{Ann}(S) \to T_A\mathrm{Ann}(S)$:

Lemma.

$$T_A\mathrm{Ann}(S) \;=\; \frac{\mathfrak{X}_{\mathbb{C}}(\partial_{out}A) \oplus \mathfrak{X}_{\mathbb{C}}(\partial_{in}A)}{\mathfrak{X}_{\mathrm{hol}}(A)} \;=\; \frac{\mathfrak{X}_{\mathbb{C}}(S) \oplus \mathfrak{X}_{\mathbb{C}}(S)}{\mathfrak{X}_{\mathrm{hol}}(A)},$$

where $\mathfrak{X}_{\mathrm{hol}}(A)$ denotes the set of vector field on A which are holomorphic in the interior, and smooth all the way to the boundary (as in (1)*).*

Note that when $A = \mathbf{1} \in \mathrm{Ann}(S)$, the lemma recovers our earlier result:

$$T_{\mathbf{1}}\mathrm{Ann}(S) \;=\; \frac{\mathfrak{X}_{\mathbb{C}}(S) \oplus \mathfrak{X}_{\mathbb{C}}(S)}{\mathfrak{X}_{\mathbb{C}}(S)} \;=\; \mathfrak{X}_{\mathbb{C}}(S).$$

Using the above model, the left translation map $\ell_A : T_{\mathbf{1}}\mathrm{Ann}(S) \to T_A\mathrm{Ann}(S)$ sends a vector $v \in \mathfrak{X}_{\mathbb{C}}(S)$ to the class of $(0, v) \in \mathfrak{X}_{\mathbb{C}}(S) \oplus \mathfrak{X}_{\mathbb{C}}(S)$, and the right translation map $r_A : T_{\mathbf{1}}\mathrm{Ann}(S) \to T_A\mathrm{Ann}(S)$ sends v to the class of $(v, 0)$.

Proof. Wlog $S = S^1$ (the standard circle). Let

$$M = \big\{\psi_\pm : \mathbb{D}_\pm \hookrightarrow \mathbb{CP}^1 \,\big|\, \psi_-(z) = z + \mathcal{O}(z^{-1})\big\}$$

be as above, and let

$$M_0 = \big\{(\psi_\pm) \in M \,\big|\, \psi_-(\mathring{\mathbb{D}}_-) \cap \psi_+(\mathring{\mathbb{D}}_+) = \emptyset\big\},$$

so that $M_0 \cong \mathrm{Ann}(S^1)$.

Let $N := \{\gamma_\pm : S^1 \hookrightarrow \mathbb{CP}^1\}$ and

$$N_0 = \big\{(\gamma_\pm) \in N \,\big|\, \gamma_-(S^1) \text{ and } \gamma_+(S^1) \text{ `bound an annulus'}, \\ \text{and } \gamma_-(S^1) \text{ encircles } \infty\big\}.$$

In more precise terms, for

$$\begin{aligned} s: \quad M &\longrightarrow N \\ (\psi_\pm) &\mapsto (\gamma_\pm := \psi_\pm|_{\partial\mathbb{D}_\pm}) \end{aligned}$$

N_0 is the saturation of $s(M_0) \subset N$ under the action of $\mathrm{Diff}(S^1) \times \mathrm{Diff}(S^1)$.

The map s admits a retraction $r : N_0 \to M_0$ which sends a pair $(\gamma_\pm)$ to the annulus bound by $\gamma_-(S^1)$ and $\gamma_+(S^1)$, and then identifies that annulus with an element of M_0. By construction, this map satisfies $s \circ r = \mathrm{id}$. The fiber $r^{-1}(A)$ over a point $A \in \mathrm{Ann}(S^1) \cong M_0$ is the set of embeddings $\sigma : A \hookrightarrow \mathbb{CP}^1$ such that $\partial_{out} A$ encircles ∞.

At the level of tangent spaces, the diagram

$$\begin{tikzcd} r^{-1}(A) \arrow[r] & N_0 \arrow[r, "r"'] & M_0 \arrow[l, bend right, "s"'] \end{tikzcd}$$

induces a split short exact sequence

$$0 \to \Gamma_{\mathrm{hol}}\big(A, \sigma^* T\mathbb{CP}^1\big) \to \Gamma\big(S^1 \sqcup S^1, (\gamma_- \sqcup \gamma_+)^* T\mathbb{CP}^1\big) \to T_A \mathrm{Ann}(S) \to 0.$$

The result follows since $\Gamma_{\mathrm{hol}}(A, \sigma^* T\mathbb{CP}^1) = \Gamma_{\mathrm{hol}}(A, TA) = \mathfrak{X}_{\mathrm{hol}}(A)$, and $\Gamma(S^1 \sqcup S^1, (\gamma_- \sqcup \gamma_+)^* T\mathbb{CP}^1) = \Gamma(S^1, T_{\mathbb{C}} S^1)^{\oplus 2} = \mathfrak{X}_{\mathbb{C}}(S^1) \oplus \mathfrak{X}_{\mathbb{C}}(S^1)$. □

Proposition 11 *The multiplication map* $\mathrm{Ann}(S) \times \mathrm{Ann}(S) \to \mathrm{Ann}(S)$ *is holomorphic. I.e., the map*

$$T_{A_1}\mathrm{Ann}(S) \oplus T_{A_2}\mathrm{Ann}(S) \;\to\; T_{A_1 \cup A_2}\mathrm{Ann}(S)$$

induced by the composition of annuli is complex linear.

Proof. Wlog $S = S^1$. Given $A_1, A_2 \in \mathrm{Ann}$, letting $\psi : \mathbb{D}_- \cup A_1 \cup A_2 \cup \mathbb{D}_+ \to \mathbb{CP}^1$ be the unique isomorphism such that $\psi|_{\mathbb{D}_-} = 1/z + O(z)$, one can write down a map

$$\begin{aligned} \mathrm{Ann}(S^1) \times \mathrm{Ann}(S^1) \;\to\; & \big\{(\gamma_1, \gamma_2, \gamma_3) \,\big|\, \gamma_i : S \hookrightarrow \mathbb{CP}^1\big\} \\ (A_1, A_2) \;\mapsto\; & \big(\psi|_{\partial_{out} A_1}, \psi|_{\partial_{in} A_1} = \psi|_{\partial_{out} A_2}, \psi|_{\partial_{in} A_2}\big). \end{aligned}$$

That map admits a retraction which sends a triple $(\gamma_1, \gamma_2, \gamma_3)$ to the annuli bound by $\gamma_1(S^1)$ and $\gamma_2(S^1)$, and $\gamma_2(S^1)$ and $\gamma_3(S^1)$, respectively.

At the level of tangent spaces, the existence of that retraction means that the vector space

$$\big\{((v_1^{out}, v_1^{in}), (v_2^{out}, v_2^{in})) \in \big(\mathfrak{X}_{\mathbb{C}} S^1 \oplus \mathfrak{X}_{\mathbb{C}} S^1\big) \oplus \big(\mathfrak{X}_{\mathbb{C}} S^1 \oplus \mathfrak{X}_{\mathbb{C}} S^1\big) \,\big|\, v_1^{in} = v_2^{out}\big\}$$

surjects onto

$$T_{A_1}\mathrm{Ann}(S^1) \oplus T_{A_2}\mathrm{Ann}(S^1) \;=\; \frac{\mathfrak{X}_{\mathbb{C}}S^1 \oplus \mathfrak{X}_{\mathbb{C}}S^1}{\mathfrak{X}_{\mathrm{hol}}(A_1)} \oplus \frac{\mathfrak{X}_{\mathbb{C}}S^1 \oplus \mathfrak{X}_{\mathbb{C}}S^1}{\mathfrak{X}_{\mathrm{hol}}(A_2)}\,.$$

So we get a commutative diagram

$$\begin{tikzcd}
\{((v_1^{out},v_1^{in}),(v_2^{out},v_2^{in})) \in (\mathfrak{X}_{\mathbb{C}}S^1 \oplus \mathfrak{X}_{\mathbb{C}}S^1) \oplus (\mathfrak{X}_{\mathbb{C}}S^1 \oplus \mathfrak{X}_{\mathbb{C}}S^1) \,|\, v_1^{in} = v_2^{out}\} \arrow[r] \arrow[d, two heads] & \mathfrak{X}_{\mathbb{C}}S^1 \oplus \mathfrak{X}_{\mathbb{C}}S^1 \arrow[d, two heads] \\
T_{A_1}\mathrm{Ann}(S^1) \oplus T_{A_2}\mathrm{Ann}(S^1) \arrow[r] & T_{A_1\cup A_2}\mathrm{Ann}(S^1)
\end{tikzcd}$$

$$((v_1^{out},v_1^{in}),(v_2^{out},v_2^{in})) \mapsto (v_1^{out},v_2^{in})$$

The top horizontal map and the two vertical maps are visibly complex linear. Therefore so is the bottom map. □

We finish this section by explaining why the Lie algebra of $\mathrm{Ann}(S)$ is the Lie algebra of complexified vector fields on S, with the opposite of the usual bracket of vector fields. We already saw that

$$T_{\mathbf{1}}\mathrm{Ann}(S) \;\cong\; \mathfrak{X}_{\mathbb{C}}(S) \tag{20}$$

as vector spaces. The inclusion $\mathrm{Diff}(S) \hookrightarrow \mathrm{Ann}(S)$ induces an inclusion of Lie algberas $\mathfrak{X}(S) \hookrightarrow T_{\mathbf{1}}\mathrm{Ann}(S)$. The latter being a complex Lie algebra, its bracket is completely determined by what happens on the real subspace $\mathfrak{X}(S)$. The isomorphism (20) is therefore one of Lie algebras.

Central extensions of (semi-)groups

We now address the question of, given a Lie (semi-)group G with Lie algebra $\mathfrak{g}$, and a central extension $0 \to A \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0$, how to build a corresponding central extension of G?

If G is a Lie group, the Lie algebra $\mathfrak{g}$ can be naturally identified with the Lie algebra of left-invariant vector fields on G, equipped with the usual Lie bracket of vector fields. The chain complex which computes Lie algebra cohomology can then be naturally identified with the complex of left-invariant differential forms on G, equipped with the usual de Rham differental. Recall that, given a manifold M and a 2-form $\alpha \in \Omega^2(M)$, its de Rham differential is given by

$$d\alpha(X,Y,Z) = \sum^{3} X \cdot \alpha(Y,Z) - \sum^{3} \alpha([X,Y],Z)$$

where $[X,Y]$ is the Lie bracket of vector fields.

Given an antisymmetric bilinear form $\omega : \mathfrak{g} \times \mathfrak{g} \to A$, let us write $\underline{\omega} \in \Omega^2(G)$ for the corresponding left-invariant form on G. The 2-form $\underline{\omega}$ is closed if and only if ω is a

2-cocycle in the sense introduced before:

$$\begin{aligned} d\underline{\omega} = 0 \quad &\Leftrightarrow \quad d\underline{\omega}(X,Y,Z) = 0, \qquad \forall \text{ left invariant } X,Y,Z \in \mathfrak{X}(G), \\ &\Leftrightarrow \quad \cancel{\sum^3 X \cdot \underbrace{\underline{\omega}(Y,Z)}_{constant}} - \sum^3 \underline{\omega}([X,Y],Z) = 0 \\ &\Leftrightarrow \quad \sum^3 \omega([X,Y],Z) = 0, \qquad \forall X,Y,Z \in \mathfrak{g}. \end{aligned}$$

Here, we've used the fact that, in order to check whether $\underline{\omega}$ is closed, it is enough to evaluate $d\underline{\omega}$ against left-invariant vector fields.

Proposition 12 *1. Given a simply connected Lie group G with Lie algebra $\mathfrak{g}$, and a 2-cocycle $\omega : \mathfrak{g} \times \mathfrak{g} \to A$, let*

$$\tilde{G}_\omega := \left\{ (\gamma, a) \;\middle|\; \begin{matrix} \gamma : [0,1] \to G, \\ \gamma(0) = e,\ a \in A \end{matrix} \right\} \Big/ \begin{matrix} (\gamma, a) \sim \left(\gamma', a + \int_h \underline{\omega}\right) \text{ when} \\ \gamma'(1) = \gamma(1) \text{ and } h \text{ is a homotopy from } \gamma \text{ to } \gamma', \end{matrix}$$

with group operation given by $(\gamma_1, a_1)(\gamma_2, a_2) = (\gamma_1 \cdot \gamma_1(1)\gamma_2, a_1 + a_2)$. Then

$$\tilde{G}_\omega \to G : (\gamma, a) \mapsto \gamma(1)$$

is a central extension of G by $\underline{A} := A/\{\text{periods of } \underline{\omega}\}$.

2. If $\omega' = \omega + d\mu$, then $\tilde{G}_\omega \cong \tilde{G}_{\omega'}$, with isomorphism given by $(\gamma, a) \mapsto (\gamma, a + \int_\gamma \underline{\mu})$.

3. If G is merely connected then, provided $\mathfrak{g}$ has trivial abelianization, $\tilde{G}_\omega$ is a central extension of G by $\underline{A} \times \pi_1(G)$. (If $\mathfrak{g}_{ab} \neq 0$, the kernel of $\tilde{G}_\omega \to G$ might fail to be abelian.)

4. If $\mathfrak{g}_{ab} = 0$ and the central extension associated to ω is universal, then, provided the set of periods of ω is discrete inside A, the central extension

$$1 \longrightarrow \underline{A} \times \pi_1(G) \longrightarrow \tilde{G}_\omega \longrightarrow G \longrightarrow 1$$

is a universal central extension in the category of Lie groups.

Proof. *1.* Since G is simply connected, any element of $K := \ker(\tilde{G}_\omega \to G)$ can be represented by a pair $(*, a)$, where $*$ denotes the constant path. By definition, we then have $(*, a) \sim (*, a + \int_h \underline{\omega})$ for every homotopy from the constant path to itself (also known as a based map $h : S^2 \to G$). The elements $\int_h \underline{\omega} \in A$ are, by definition, the periods of $\underline{\omega}$.

2. The map $(\gamma, a) \mapsto (\gamma, a + \int_\gamma \underline{\mu})$ is well-defined by an application of Stokes' theorem, and is visibly an isomorphism.

3. The projection map $K \to \pi_1(G) : [(\gamma, a)] \mapsto [\gamma]$ fits into a diagram

$$\begin{tikzcd} \underline{A} \arrow[r] \arrow[d, equal] & K \arrow[r] \arrow[d] & \pi_1(G) \arrow[d] \\ \underline{A} \arrow[r] & \tilde{G}_\omega \arrow[r] \arrow[d] & \tilde{G} \arrow[d] \\ & G \arrow[r, equal] & G \end{tikzcd}$$

where all rows and columns are group extensions. The conjugation action $\tilde{G} \circlearrowright \pi_1(G)$ is trivial (since $\tilde{G}$ is connected), so $\pi_1(G)$ is central in $\tilde{G}$. So, given $k \in K$ and $g \in \tilde{G}_\omega$, the commutator $[k, g] \in G_\omega$ maps to $1 \in \tilde{G}$. That commutator therefore lands in $\underline{A}$. The map $[k, -] : \tilde{G}_\omega \to \underline{A}$ is a homomorphism:

$$(kg_1k^{-1}g_1^{-1})(kg_2k^{-1}g_2^{-1}) = kg_1k^{-1}(kg_2k^{-1}g_2^{-1})g_1^{-1} = k(g_1g_2)k^{-1}(g_1g_2)^{-1} \quad ✓$$

and descends to a homomorphism $\tilde{G} \to \underline{A}$. But $\tilde{G}$ is connected and $\mathfrak{g}_{ab} = 0$, so there are no non-trivial homomorphisms from $\tilde{G}$ to an abelian group. Therefore K is central in G and $K \to \tilde{G}_\omega \to G$ is a central extension.

It remains to show that $K \cong \underline{A} \times \pi_1(G)$, i.e., that the sequence $\underline{A} \to K \to \pi_1(G)$ splits. This follows from the general structure theory of abelian groups, using that $\underline{A}$ is divisible and hence an injective abelian group.

4. If $A \to \tilde{\mathfrak{g}}_\omega \to \mathfrak{g}$ is universal, then for any central extension $B \to \tilde{G} \to G$ with associated Lie algebra central extension $\mathfrak{b} \to \tilde{\mathfrak{g}} \to \mathfrak{g}$, there is a unique map

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & \tilde{\mathfrak{g}}_\omega & \longrightarrow & \mathfrak{g} & \longrightarrow & 0 \\
 & & {\scriptstyle f}\downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathfrak{b} & \longrightarrow & \tilde{\mathfrak{g}} & \longrightarrow & \mathfrak{g} & \longrightarrow & 0.
\end{array}$$

Since $\tilde{G}_\omega$ is connected, there is at most one homomorphism $\tilde{G}_\omega \to \tilde{G}$ that integrates the map $\tilde{\mathfrak{g}}_\omega \to \tilde{\mathfrak{g}}$. So all we need to do is construct such a homomorphism.

The canonical splitting $\mathfrak{g} \to \tilde{\mathfrak{g}}_\omega$ induces a splitting $s : \mathfrak{g} \to \tilde{\mathfrak{g}}$. Letting $F : A \to B$ be the homomorphism which integrates f, the map $\tilde{G}_\omega \to \tilde{G}$ is given by

$$[(\gamma, a)] \mapsto \delta(1) \cdot F(a)$$

where $\delta : [0,1] \to \tilde{G}$ is the unique solution of $\delta(t)^{-1}\frac{d}{dt}\delta(t) = s\big(\gamma(t)^{-1}\frac{d}{dt}\gamma(t)\big)$. *[Here, $\gamma(t)^{-1}\frac{d}{dt}\gamma(t)$ denotes the left-translate of $\frac{d}{dt}\gamma(t) \in T_{\gamma(t)}G$ back to $T_eG = \mathfrak{g}$.]* □

Remark. *The splitting of $0 \to \underline{A} \to K \to \pi_1(G) \to 0$ is not canonical, so the center of $\tilde{G}_\omega$ is only non-canonically isomorphic to $\underline{A} \times \pi_1(G)$.*

The above proposition takes care of the central extension of $\mathrm{Diff}(S^1)$ (the top row in (17)). But the case of $\mathrm{Ann}(S^1)$ is more tricky because the various tangent spaces of $\mathrm{Ann}(S^1)$ are no longer all isomorphic (or rather, left translation is not an isomorphism). So we can't talk about the left-invariant 2-form associated to a Lie algebra 2-cocycle. To go around this difficulty, we use a little trick. Given an annlus $A \in \mathrm{Ann}(S^1)$, let

$$\mathrm{Ann}^{\le A} := \big\{A_1 \in \mathrm{Ann}(S^1) \,\big|\, \exists A_2 : A_1A_2 = A\big\} \cong \big\{\gamma : S^1 \hookrightarrow A \,\big|\, \gamma \text{ “wraps around } A\text{”}\big\}.$$

When thinking in terms of maps $\gamma : S^1 \to A$, the tangent space of $\mathrm{Ann}^{\le A}$ is easy to compute, and we see that the map $\mathrm{Ann}^{\le A_2} \to \mathrm{Ann}^{\le A}$ given by $B \mapsto A_1B$ induces an isomorphism of tangent spaces

$$T_1(\mathrm{Ann}^{\le A_2}) = \mathfrak{X}_{\mathbb{C}}(S^1) \xrightarrow{\cong} T_{A_1}(\mathrm{Ann}^{\le A}).$$

So $\underline{\omega_{Vir}}$ makes sense as a 2-form on $\mathrm{Ann}^{\leq A}$, and we can define

$$^{\mathbb{C}\times\mathbb{Z}}\mathrm{Ann}(S^1) := \left\{ (A,\gamma,a) \;\middle|\; \begin{array}{l} A\in \mathrm{Ann}(S^1),\ a\in\mathbb{C} \\ \gamma:[0,1]\to \mathrm{Ann}^{\leq A}, \\ \gamma(0)=1,\ \gamma(1)=A \end{array} \right\} \Big/ \begin{array}{l} (\gamma,a)\sim(\gamma', a+\int_h \underline{\omega_{Vir}}), \\ h \text{ a homotopy from } \gamma \text{ to } \gamma' \end{array}$$

very much like what we did in Proposition 12.

Here's a big chart with all the groups and semigroups related to the Virasoro algebra:

$$\begin{tikzcd}
 & & {}^{i\mathbb{R}\oplus\mathbb{Z}}\mathrm{Diff}(S^1) \arrow[r] \arrow[dd] \arrow[dddd] & {}^{U(1)\oplus\mathbb{Z}}\mathrm{Diff}_c(S^1) \arrow[r, two heads] \arrow[dd] \arrow[dddd] & {}^{\mathbb{Z}}\mathrm{Diff}(S^1) \arrow[dd] \arrow[dddd] \\
{}^{\mathbb{Z}}PSU(1,1) \arrow[urr, dashed] \arrow[urrrr] \arrow[dd] \arrow[ddddr] & & & & \\
 & & {}^{i\mathbb{R}}\mathrm{Diff}(S^1) \arrow[r] \arrow[dddd] & {}^{U(1)}\mathrm{Diff}_c(S^1) \arrow[r, two heads] \arrow[dddd] & \mathrm{Diff}(S^1) \arrow[dddd] \\
PSU(1,1) \arrow[urr] \arrow[urrrr] \arrow[ddddr] & & & & \\
 & & {}^{\mathbb{C}\oplus\mathbb{Z}}\mathrm{Ann}(S^1) \arrow[r] \arrow[dd] & {}^{\mathbb{C}^\times\oplus\mathbb{Z}}\mathrm{Ann}_c(S^1) \arrow[r, two heads] \arrow[dd] & {}^{\mathbb{Z}}\mathrm{Ann}(S^1) \arrow[dd] \\
 & {}^{\mathbb{Z}}\mathrm{Univ}(\mathbb{D}) \arrow[ur] \arrow[urrr] \arrow[dd] & & & \\
 & & {}^{\mathbb{C}}\mathrm{Ann}(S^1) \arrow[r] & {}^{\mathbb{C}^\times}\mathrm{Ann}_c(S^1) \arrow[r, two heads] & \mathrm{Ann}(S^1) \\
 & \mathrm{Univ}(\mathbb{D}) \arrow[ur] \arrow[urrr] & & &
\end{tikzcd}$$

The dotted map exists because the universal cover of $PSU(1,1)$ is also its universal central extension. This allows us to identify a canonical copy of $\mathbb{Z}$ inside the center of $^{i\mathbb{R}\oplus\mathbb{Z}}\mathrm{Diff}(S^1)$, and to define $^{i\mathbb{R}}\mathrm{Diff}(S^1)$ as the quotient by that $\mathbb{Z}$. Similarly, $^{\mathbb{C}}\mathrm{Ann}(S^1)$ is the quotient of $^{\mathbb{C}\oplus\mathbb{Z}}\mathrm{Ann}(S^1)$ by that same copy of $\mathbb{Z}$.

Loop groups

There is another class of infinite dimensional Lie groups which are very important in conformal field theory, and to which Proposition 12 readily applies: loop groups.
Fix:

- A finite dimensional, compact, simple, simply connected Lie group G, called the *gauge group*.
- A positive integer $k\in\mathbb{N}$, called the *level*.

The *loop group* of G is the group of smooth maps from S^1 to G:

$$LG := \mathrm{Map}_{C^\infty}(S^1, G),$$

and its Lie algebra $L\mathfrak{g} = C^\infty(S^1,\mathfrak{g})$ is called the *loop algebra*. Let $\omega : L\mathfrak{g}\times L\mathfrak{g}\to i\mathbb{R}$ be the cocycle given by

$$\omega(f,g) := \tfrac{1}{2\pi i}\int_{S^1} \langle f, dg\rangle, \tag{21}$$

where $\langle\ ,\ \rangle$ is *minus*[16] the *basic inner product* on $\mathfrak{g}$. For $\mathfrak{g} = \mathfrak{su}(2)$ (also for $\mathfrak{su}(n)$), this is given by $\langle X, Y\rangle = -\mathrm{tr}(XY)$. For other simple Lie algebras, it is the smallest G-invariant inner product whose restriction to any $\mathfrak{su}(2) \subset \mathfrak{g}$ is a positive integer multiple of the above inner product of $\mathfrak{su}(2)$.

Here, $\langle f, dg\rangle \in \Omega^1(S^1)$ is a somewhat peculiar notation. It denotes the image of $fdg \in \Omega^1(S^1; \mathfrak{g}\otimes\mathfrak{g})$ under the map $\Omega^1(S^1;\mathfrak{g}\otimes\mathfrak{g}) \to \Omega^1(S^1;\mathbb{R})$ induced by $\langle\ ,\ \rangle : \mathfrak{g}\otimes\mathfrak{g} \to \mathbb{R}$.

Let's quickly check that ω is a 2-cocycle:

$$\oint \langle [f,g], dh\rangle \overset{\text{integration by parts}}{=} -\oint \langle [df,g],h\rangle - \oint \langle [f,dg],h\rangle \underset{\langle\ ,\ \rangle \text{ is } G\text{-invariant} \Leftrightarrow \langle [X,Y],Z\rangle = \langle X,[Y,Z]\rangle\ \forall X,Y,Z\in\mathfrak{g}}{=} -\oint \langle df,[g,h]\rangle + \oint \langle dg,[f,h]\rangle \quad \checkmark$$

This 2-cocycle induces a central extension $\widetilde{L\mathfrak{g}}$ of the loop algebra $L\mathfrak{g}$. The complexification $\widetilde{L\mathfrak{g}_{\mathbb{C}}}$ of this central extension is (the C^∞ completion of) the *affine Lie algebra*. That same Lie algebra goes by various names: it also called the *current algebra*, and also the *affine Kac-Moody algebra*[17].

Theorem 13 *The second continuous cohomology $H^2_{cts}(L\mathfrak{g},\mathbb{R})$ is one dimensional, and the cocycle $(f,g)\mapsto \int_{S^1}\langle f, dg\rangle$ represents a generator.*

As a corollary, we learn that $\widetilde{L\mathfrak{g}}$ is a universal central extension of $L\mathfrak{g}$.

Proof. By the same argument as in the proof of Theorem 9,

$$H^2_{cts}(L\mathfrak{g};\mathbb{R}) = \frac{\{\, G\text{-invariant 2-cocycles}\,\}}{\{d\mu \,|\, \mu : L\mathfrak{g}\to\mathbb{R},\ \ \mu \text{ is } G\text{-invariant}\}}$$

where G acts of $L\mathfrak{g}$ by its adjoint action on $\mathfrak{g}$. The space of G-invariant linear functionals $\mu : L\mathfrak{g}\to\mathbb{R}$ is trivial, so all we need to show is that the space of G-invariant 2-cocycles is one dimensional. We already know that it's at least one dimensional. So we need to show that it's at most one dimensional. At this point, it becomes convenient to complexify. Given $X\in\mathfrak{g}_{\mathbb{C}}$, let us introduce the notation X_n for $Xz^n \in L\mathfrak{g}_{\mathbb{C}}$. The Lie bracket of such elements is given by $[X_m, Y_n] = [X,Y]_{m+n}$.

Let ω be a G-invariant 2-cocycle. By continuity, ω is entirely determined by its restriction to the X_n's. Write $c_{m,n}$ for the map $X, Y\mapsto \omega(X_m, Y_n)$. Since ω is G-invariant, so is $c_{m,n}$. The space of G-invariant bilinear forms $\mathfrak{g}_{\mathbb{C}}\times\mathfrak{g}_{\mathbb{C}}\to\mathbb{C}$ is one dimensional, spanned by (minus) the basic inner product. So the $c_{m,n}$ are multiples of the basic inner product. In particular, they satisfy $c_{m,n}(X,Y) = c_{m,n}(Y,X)$. By the antisymmetry of ω, we then have

[16]The 'basic inner product' is an invariant bilinear form on $\mathfrak{g}_{\mathbb{C}}$ whose restriction to $\mathfrak{g}$ is negative definite. We prefer to write the formula (21) in terms of an inner product on $\mathfrak{g}$ which is positive definite.

[17]To be precise, the term 'affine Kac-Moody algebra' usually refers to the semi-direct product $\widetilde{L\mathfrak{g}_{\mathbb{C}}}\rtimes\mathbb{C}$, and 'current algebra' sometimes refers to the non-centrally-extended algebra.

$$c_{m,n}(X,Y) = \omega(X_m, Y_n) = -\omega(Y_n, X_m) = -c_{n,m}(Y,X) = -c_{n,m}(X,Y).$$

So $c_{m,n} = -c_{n,m}$.

Since ω is a 2-cocycle, the $c_{m,n}$ satisfy

$$\begin{aligned}
& & 0 &= c_{m+n,p}([X,Y],Z) + c_{p+m,n}([Z,X],Y) + c_{n+p,m}([Y,Z],X) \\
&\boxed{\text{because the } c_{m,n} \text{ are } G\text{-invariant}} & &= c_{m+n,p}([X,Y],Z) + c_{p+m,n}([X,Y],Z) + c_{n+p,m}([X,Y],Z) \\
&\boxed{\text{commutators span } \mathfrak{g}_{\mathbb{C}}} & &\Rightarrow c_{m+n,p} + c_{p+m,n} + c_{n+p,m} = 0.
\end{aligned}$$

Setting $m = n = 0$, we get

$$c_{0,p} + 2c_{p,0} = 0 \quad \Rightarrow \quad c_{0,p} = 0, \ \forall p.$$

Setting $n = 1$ and $p = r - (m+1)$, we get

$$c_{m+1,r-(m+1)} = c_{1,r-1} + c_{m,r-m} \quad \Rightarrow \quad c_{m,r-m} = m \cdot c_{1,r-1}, \ \forall m.$$

Setting $m = r$ in the last equation, we get

$$0 = c_{r,r-r} = r \cdot c_{1,r-1} \quad \Rightarrow \quad c_{1,r-1} = 0, \ \forall r \neq 0.$$

So $c_{m,n} = 0$ when $n \neq -m$, and $c_{m,-m} = m \cdot c_{1,-1}$. The cocycle ω is therefore entirely determined by the value of $c_{1,-1}$, and the space of G-invariant 2-cocycles is at most one dimensional. □

We now wish to apply Proposition 12 to the cocycle (21).

Unlike $\mathrm{Diff}(S^1)$, whose fundamental group was non-trivial but whose higher homotopy groups were all trivial, the loop group LG is simply connected but has lots of non-trivial higher homotopy groups. We care about $\pi_2(LG)$. As a manifold, LG is diffeomorphic to $G \times \Omega G$, where ΩG denotes the *based* loop group of G. So

$$\pi_2(LG) = \pi_2(G \times \Omega G) = \underbrace{\pi_2(G)}_{=\pi_1(\Omega G)=0} \times \underbrace{\pi_3(G)}_{=\pi_2(\Omega G)=\mathbb{Z}} = \mathbb{Z}.$$

[The computations $\pi_1(\Omega G) = 0$ and $\pi_2(\Omega G) = \mathbb{Z}$ are rather non-trivial. They can be done by applying Morse theory to ΩG, with respect to a suitable Morse function. (A suitable Morse function can be obtained by taking the energy functional $\gamma \mapsto \int_{S^1} \|\gamma^{-1}\gamma'\|^2$, which is not Morse, and deforming it a bit.) One the other hand, they're fairly easy for $G = SU(n)$ by using the fiber sequences $SU(n-1) \to SU(n) \to S^{2n-1}$ and the fact that $SU(2) = S^3$.]

To go further, we need to compute the group of periods of $\underline{\omega}$. The answer turns out to be that the periods of $\underline{\omega}$ are equal to $2\pi i\mathbb{Z} \subset i\mathbb{R}$ (we'll do that computation in a moment). So, by Proposition 12, we get a central extension of LG by $i\mathbb{R}/2\pi i\mathbb{Z} = U(1)$, which is also its universal central extension. We call it the *level 1 central extension* of the loop group, and denote it $\widetilde{LG}$. The level k central extension is then obtained as a pushout:

$$\begin{tikzcd}
0 \arrow[r] & U(1) \arrow[r] \arrow[d, "z \mapsto z^n"'] \arrow[dr, phantom, "\ulcorner"] & \widetilde{LG} \arrow[r] \arrow[d] & LG \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & U(1) \arrow[r] & \widetilde{LG}_k \arrow[r] & LG \arrow[r] & 0
\end{tikzcd}$$

The cocycle which is adapted to the canonical basis element of the Lie algebra of $U(1) \subset \widetilde{LG}_k$ is given by

$$\omega_k(f,g) := \tfrac{k}{2\pi i} \int_{S^1} \langle f, dg \rangle$$

where $\langle\ ,\ \rangle$ is minus the basic inner product on $\mathfrak{g}$. We write $\widetilde{L\mathfrak{g}}_k$ for the corresponding central extension of $L\mathfrak{g}$. (It is isomorphic to $\widetilde{L\mathfrak{g}}$ as a mere Lie algebra, but not as a central extension of $L\mathfrak{g}$ by $i\mathbb{R}$.)

Let us now compute the periods of $\underline{\omega}$. For any simple group G, there is a homomorphism $SU(2) \to G$ that represents a generator of $\pi_3(G)$ (recall that $SU(2) \cong S^3$). Moreover, the restriction of minus the basic inner product of G is minus the basic inner product of $SU(2)$. So it's enough to deal with the case $G = SU(2)$. Let's identify $G = SU(2)$ with the group $\{q \in \mathbb{H} : |q| = 1\}$ of unit quaternions via (the $\mathbb{R}$-linear extension of)

$$1 \leftrightarrow \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \quad i \leftrightarrow \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right), \quad j \leftrightarrow \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), \quad k \leftrightarrow \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right).$$

The Lie algebra $\mathfrak{g} = \mathfrak{su}(2)$ corresponds to the space of imaginary quaternions, and minus the basic inner product is given by $\langle i,i\rangle = \langle j,j\rangle = \langle k,k\rangle = 2$.

Trying to integrate $\underline{\omega}$ over a generator of $\pi_2(LG)$ is a nightmare of a computation. But integrating it over *twice* a generator turns out to be feasible. Let us model S^2 as the space $\{q = ai + bj + ck \ :\ |q|^2 = 1\}$ of unit imaginary quaternions, and let us take S^1 to be $[0, 2\pi]/\!\sim$. The following map represents twice a generator of $\pi_2(LG)$:

$$h : S^2 \to LG$$
$$h(q) := \big(\theta \mapsto \cos(\theta) + q\sin(\theta)\big).$$

By symmetry considerations, the 2-form $h^*\underline{\omega}$ is a constant multiple of the standard volume form of S^2. So it's enough to consider what happens at the point $i \in S^2$. The tangent space at i is spanned by j and k, and one easily computes $\frac{\partial h}{\partial j}(\theta) = j\sin(\theta)$ and $\frac{\partial h}{\partial k}(\theta) = k\sin(\theta)$. Translating back to the origin, we get

$$h^{-1}\tfrac{\partial h}{\partial j}(\theta) = \big(\cos\theta - i\sin\theta\big)j\sin\theta = \tfrac{1}{2}\big(j\sin 2\theta + k(\cos 2\theta - 1)\big)$$
$$h^{-1}\tfrac{\partial h}{\partial k}(\theta) = \big(\cos\theta - i\sin\theta\big)k\sin\theta = \tfrac{1}{2}\big(k\sin 2\theta + j(1 - \cos 2\theta)\big)$$

So *twice* the smallest period of $2\pi i\underline{\omega}$ is given by

$$\mathrm{vol}(S^2) \cdot \int_0^{2\pi} \Big\langle h^{-1}\tfrac{\partial h}{\partial j}, \tfrac{d}{d\theta}\big(h^{-1}\tfrac{\partial h}{\partial k}\big)\Big\rangle\, d\theta$$
$$= 4\pi \int_0^{2\pi} \Big\langle \tfrac{1}{2}\big(j\sin 2\theta + k(\cos 2\theta - 1)\big), \tfrac{d}{d\theta}\tfrac{1}{2}\big(k\sin 2\theta + j(1 - \cos 2\theta)\big)\Big\rangle$$

$$= 2\pi \int_0^{2\pi} \Big\langle j\sin 2\theta + k(\cos 2\theta - 1), k\cos 2\theta + j\sin 2\theta \Big\rangle$$
$$= 2\pi \int_0^{2\pi} \sin^2(2\theta)\langle j,j\rangle + \cos^2(2\theta)\langle k,k\rangle \; = 2\pi \int_0^{2\pi} 2 = 8\pi^2.$$

So the periods of $2\pi i\underline{\omega}$ are $4\pi^2\mathbb{Z}$, and the periods of $\underline{\omega}$ are $2\pi i\mathbb{Z} \subset i\mathbb{R}$.

Canonical anticommutation relations

Let $S^1 \subset \mathbb{C}$ denote the standard unit circle, equipped with its standard spin structure $\mathbb{S}$ inherited from the trivial spin structure on $\mathbb{C}$. We write $f(z)\sqrt{dz}$ for a section of $\mathbb{S}$ (where $\sqrt{dz}$ is a formal symbol), and the isomorphism $\mathbb{S}^{\otimes 2} \cong T^*S$ is given by $f(z)\sqrt{dz} \otimes g(z)\sqrt{dz} \mapsto f(z)g(z)dz$. Let $\Gamma(\mathbb{S}) = \Gamma(S^1, \mathbb{S})$ denote the space of spinors fields on the circle.

Definition: The algebra $CAR(S^1)$ of **C**anonical **A**nticommutation **R**elations is given by:

Generators:	There is one generator $c(f)$ for every section $f \in \Gamma(\mathbb{S})$ and the symbol $c(f)$ depends linearly on f, namely, $c(f+g) = c(f) + c(g)$ and $c(\lambda f) = \lambda c(f)$ for $\lambda \in \mathbb{C}$.
Relations:	For any sections $f, g \in \Gamma(\mathbb{S})$, we have: $$[c(f), c(g)]_+ = \frac{1}{2\pi i}\int_{S^1} fg$$ where $[\,,\,]_+$ is the anticommutator $[A,B]_+ := AB+BA$. Here, fg is viewed as a 1-form via the isomorphism $\mathbb{S}^{\otimes 2} \cong T^*S$.
$*$-structure:	If we let $f \mapsto \bar{f}$ be the antilinear involution on $\Gamma(\mathbb{S})$ given by $\overline{z^n\sqrt{dz}} := z^{-n-1}\sqrt{dz}$, then we set $c(f)^* := c(\bar{f})$

The way to remember the formula for $f \mapsto \bar{f}$ is to view $\sqrt{dz}$ as some kind of substitute for $z^{\frac{1}{2}}$. The formula $z^n\sqrt{dz} \mapsto z^{-n-1}\sqrt{dz}$ then becomes $z^{n+\frac{1}{2}} \mapsto z^{-(n+\frac{1}{2})}$, which agrees with our intuition about bar for z on the unit circle.

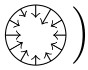
The operation $f \mapsto \bar{f}$ on sections of $\mathbb{S}$ also admits a geometric description, explained in the lemma below. That alternative description makes sense for arbitrary spin 1-manifolds, and thus allows us to define the $*$-structure on $CAR(S)$, for S an arbitrary spin 1-manifold (not just the standard unit circle).

Lemma. *Let $\Gamma(\mathbb{S}) := \Gamma(S^1, \mathbb{S})$. The sections $f \in \Gamma(\mathbb{S})$ that satisfy $\bar{f} = f$ are those whose square pairs positively with every normal outgoing vectors field; the sections $f \in \Gamma(\mathbb{S})$ that satisfy $\bar{f} = -f$ are those whose square pairs positively to every normal ingoing vectors field:*

$$\bar{f} = f \qquad \Leftrightarrow \qquad \underbrace{f^2\Big(\underset{v}{\bigcirc}\Big)} \geq 0 \qquad \forall\, v \;\; \textit{normal outgoing}$$

that's a 1-form under the isomorphism $\mathbb{S}^{\otimes 2} \cong T^*$.

that's a function on S^1

$\bar{f} = -f \qquad \Leftrightarrow \qquad f^2\Big($  $\Big) \geq 0 \qquad \forall\, v$ *normal ingoing*

v

Proof: The condition of pairing positively with normal outgoing vectors defines a ray bundle (a ray is half of a line) inside $T^*\mathbb{D}|_{S^1}$. Its preimage under the squaring map $f \mapsto f^2 : \mathbb{S} \to T^*\mathbb{D}$ is a real line bundle $\mathbb{S}^+ \subset \mathbb{S}$. Similarly, the condition of pairing positively with normal ingoing vectors defines a ray bundle inside $T^*\mathbb{D}|_{S^1}$ (the negative of the previous ray bundle) whose preimage under the squaring map is a real line bundle $\mathbb{S}^- \subset \mathbb{S}$. Since $(\mathbb{S}^-)^2 = -(\mathbb{S}^+)^2$, we have $\mathbb{S}^- = i\mathbb{S}^+$, and in particular $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$.

Now, $z^n\sqrt{dz} + z^{-n-1}\sqrt{dz}$ and $iz^n\sqrt{dz} - iz^{-n-1}\sqrt{dz}$ form a basis of $\{f \in \Gamma(\mathbb{S}) \mid \bar{f} = f\}$. The standard normal outgoing vector field is $z\,\partial_z$, and we check:

$$\big(z^n\sqrt{dz} + z^{-n-1}\sqrt{dz}\big)^2\big(z\,\partial_z\big) = \big(z^{2n+1} + 2 + z^{-(2n+1)}\big) = |1 + z^{2n+1}|^2 \geq 0$$
$$\big(iz^n\sqrt{dz} - iz^{-n-1}\sqrt{dz}\big)^2\big(z\,\partial_z\big) = \big(-z^{2n+1} + 2 - z^{-(2n+1)}\big) = |1 - z^{2n+1}|^2 \geq 0.$$

It follows that $\{\bar{f} = f\} \subseteq \Gamma(\mathbb{S}^+)$. Similarly, $iz^n\sqrt{dz} + iz^{-n-1}\sqrt{dz}$ and $z^n\sqrt{dz} - z^{-n-1}\sqrt{dz}$ form a basis of $\{f \in \Gamma(\mathbb{S}) \mid \bar{f} = -f\}$, and since

$$\big(iz^n\sqrt{dz} + iz^{-n-1}\sqrt{dz}\big)^2\big(-z\,\partial_z\big) = |1 + z^{2n+1}|^2 \geq 0$$
$$\big(z^n\sqrt{dz} - z^{-n-1}\sqrt{dz}\big)^2\big(-z\,\partial_z\big) = |1 - z^{2n+1}|^2 \geq 0,$$

we have $\{\bar{f} = -f\} \subseteq \Gamma(\mathbb{S}^-)$.

Since $\Gamma(\mathbb{S}) = \{\bar{f} = f\} \oplus \{\bar{f} = -f\}$, we conclude that $\{\bar{f} = f\} = \Gamma(\mathbb{S}^+)$ and $\{\bar{f} = -f\} = \Gamma(\mathbb{S}^-)$. $\square$

The algebra of canonical anticommutation relations introduced above is the observables of a spin chiral CFT known as the *Majorana Free Fermion*. There is another, closely related spin CFT known as the *Dirac Free Fermion*. They are given by:

	Majorana	Dirac
generators:	$c(f)$ for $f \in \Gamma(\mathbb{S})$	$a(f)$ and $a^\dagger(f)$ for $f \in \Gamma(\mathbb{S})$
relations:	$[c(f), c(g)]_+ = \frac{1}{2\pi i}\int fg$	$[a(f), a(g)]_+ = [a^\dagger(f), a^\dagger(g)]_+ = 0$, $[a(f), a^\dagger(g)]_+ = \frac{1}{2\pi i}\int fg$
$*$-operation:	$c(f)^* = c(\bar{f})$	$a(f)^* = a^\dagger(\bar{f})$

These two CFTs are related by (Dirac Free Fermion) $\cong$ (Majorana Free Fermion)$^{\otimes 2}$, and the isomorphism is given by

$$\begin{aligned}
a(f) &\longmapsto \tfrac{1}{\sqrt{2}}\big(c(f)\otimes 1 + 1 \otimes ic(f)\big)\\
a^\dagger(f) &\longmapsto \tfrac{1}{\sqrt{2}}\big(c(f)\otimes 1 - 1 \otimes ic(f)\big)\\
\tfrac{1}{\sqrt{2}}\big(a(f) + a^\dagger(f)\big) &\longleftarrow\!\shortmid c(f)\otimes 1\\
\tfrac{1}{i\sqrt{2}}\big(a(f) - a^\dagger(f)\big) &\longleftarrow\!\shortmid 1 \otimes c(f).
\end{aligned}$$

Examples of chiral CFTs

In this section, we introduce three classes of chiral CFTs:

- the unitary *chiral minimal models*
  (there's one such model for every $c = 1 - \frac{6}{m(m+1)}$, for $m = 2, 3, 4, \ldots$)
- the *chiral WZW models*
  (there's one such model for every choice of gauge group G and level k), and
- the chiral *Majorana free fermion*.

At first, we will describe the linear categories that those models assign to 1-manifolds. In the case of a chiral minimal model, the categories $\mathcal{C}(S)$ are given by

$$\mathcal{C}(S) \;=\; \mathrm{Rep}\big(\mathit{Vir}_c(S)\big)$$

where $\mathit{Vir}_c(S)$ denotes the appropriate (universal) central extension of $\mathfrak{X}_{\mathbb{C}}(S)$ by $\mathbb{C}$. (And we insist that the central $\mathbb{C}$ acts in the standard way.)

In the chiral WZW model, these categories are (roughly) given by

$$\mathcal{C}(S) \;=\; \mathrm{Rep}\big(\widetilde{L_S G_k}\big)$$

where $\widetilde{L_S G_k}$ denotes the appropriate central extension of $L_S G := \mathrm{Map}_{C^\infty}(S, G)$ by $U(1)$. (And we insist that the central $U(1)$ acts in the standard way.)

And for the Majorana free fermion, this is the representation category of the algebra of canonical anticommutation relations:

$$\mathcal{C}(S) \;=\; \mathrm{Rep}\big(\mathit{CAR}(S)\big).$$

The above descriptions are not very precise, because we haven't said anything about the class of representations that we're allowing. And without any specifications, those categories are huge. So we *do* need to say a bit more. Let us introduce a couple of technical conditions:

Definition: A representation of Vir_c has *positive energy* if the associated operator L_0 has discrete spectrum, the spectrum is bounded from below, and all the (generalized) eigenspaces are finite dimensional.

In our case of interest, we always have $e^{2\pi i L_0} = \theta_\lambda$ (where θ_λ is the conformal spin). So L_0 is in fact diagonalizable, and there is no need to talk about generalized eigenspaces.

Remark. The operator L_0 is obviously coordinate dependent. However, assuming the action of Vir_c integrates to an action of ${}^{U(1)\oplus\mathbb{Z}}\mathrm{Diff}_c(S^1)$, the property of being positive energy is independent of the choice of coordinate, because one can conjugate any coordinate into any other coordinate by an element of $\mathrm{Diff}(S^1)$.

[I'm guessing that every positive energy representation of Vir_c integrates to a representation of ${}^{U(1)\oplus\mathbb{Z}}\mathrm{Diff}_c(S^1)$, and even to a representation of ${}^{\mathbb{C}^\times\oplus\mathbb{Z}}\mathrm{Ann}_c(S^1)$. But this probably depends on the type of topological vector spaces that one is working with.]

Definition: An irreducible representation of $\widetilde{L\mathfrak{g}}_k$ has *positive energy* if it extends to a representation of $\widetilde{L\mathfrak{g}}_k \rtimes Vir_c$ for some c, and the Virasoro action has positive energy. A positive energy representation of $\widetilde{L\mathfrak{g}}_k$ is a finite direct sum of irreducible positive energy representations of $\widetilde{L\mathfrak{g}}_k$.

Remark 14 When working with Hilbert spaces, one should be aware that the actions of $\widetilde{L_S\mathfrak{g}}_k$ and of $Vir_c(S)$ are by *unbounded* operators.

[A representation of $\widetilde{L\mathfrak{g}}_k$ is called integrable if it integrates to a representation of $\widetilde{LG}_k$. Every integrable positive energy representation of $\widetilde{L\mathfrak{g}}_k$ is infinitesimally equivalent to a unitary representation, and every unitary positive energy representation on a Hilbert space is integrable.]

Definition: Let S be a connected spin 1-manifold. An irreducible representation of $CAR(S)$ has *positive energy* if it extends to a representation of $CAR(S) \rtimes {}^{U(1)\oplus\mathbb{Z}}\mathrm{Diff}_c(S)$ for some c (which is necessarily given by $c = 1/2$), and the $\mathrm{Diff}_c(S)$ action has positive energy. A positive energy representation is a finite direct sum of irreducible ones.

Given the above definitions, we can go back and re-define the above representation categories with a little bit more attention to detail. In place of $\mathrm{Rep}\big(Vir_c(S)\big)$, we should have written

$$\begin{aligned}\mathrm{Rep}^{\text{unitary}}_{\substack{pos\\en.}}\big(Vir_c(S)\big) &:= \big\{\text{positive energy unitary representations of } Vir_c(S)\big\}\\ &= \big\{\text{positive energy unitary representations of } {}^{U(1)\oplus\mathbb{Z}}\mathrm{Diff}_c(S)\big\}.\end{aligned}$$

Similarly, in place of $\mathrm{Rep}\big(\widetilde{L_SG}_k\big)$, we should have written

$$\begin{aligned}\mathrm{Rep}_{\substack{pos\\en.}}\big(\widetilde{L_SG}_k\big) &:= \big\{\text{positive energy unitary representations of } \widetilde{L_SG}_k\big\}\\ &= \big\{\text{positive energy unitary representations of } \widetilde{L_S\mathfrak{g}}_k\big\},\end{aligned}$$

where $\widetilde{L_S\mathfrak{g}}_k$ is the central extension of $L_S\mathfrak{g} := C^\infty(S,\mathfrak{g})$ by $i\mathbb{R}$ defined by the cocycle $\omega_k(f,g) := \frac{k}{2\pi i}\int_S \langle f, dg\rangle$. And finally, for the Majorana free fermion we should have

$$\mathrm{Rep}_{\substack{pos\\en.}}\big(CAR(S)\big) := \big\{\text{positive energy unitary representations of } CAR(S)\big\}.$$

Note that in order to formulate the notion of unitarity, one uses the fact that $CAR(S)$ is a $*$-algebra.

In the case of the WZW and of the free fermion χCFTs, we can also describe the concrete functor F_Σ associated to a complex cobordism, at least when $\partial\Sigma \neq \emptyset$ (more precisely, when each connected component of Σ has non-empty boundary). In the case of the minimal models, we don't yet have a concrete enough description of the Lie algebra $Vir_c(S)$ for us to be able to write something along the same lines as below. Let Σ be a connected complex cobordism with $\partial\Sigma \neq \emptyset$, and let $\mathcal{C}_{in/out} := \mathcal{C}(\partial_{in/out}\Sigma)$, so that

$$F_\Sigma : \mathcal{C}_{in} \longrightarrow \mathcal{C}_{out}.$$

Definition 15 Given an object $(V, \rho_V) \in \mathcal{C}_{in}$, its image $(W, \rho_W) \in \mathcal{C}_{out}$ under the functor F_Σ comes equipped with a linear map $Z_\Sigma : V \to W$ satisfying:

$$\boxed{\forall f \in \left\{ \begin{array}{l} \overset{\text{WZW model}}{\mathcal{O}(\Sigma;\mathfrak{g}_{\mathbb{C}})} \\ \underset{\text{free fermion}}{\Gamma_{hol}(\Sigma,\mathbb{S}_\Sigma)} \end{array} \right. \qquad Z_\Sigma \circ \rho_V(f_{in}) = \rho_W(f_{out}) \circ Z_\Sigma} \tag{22}$$

where $f_{in/out} := f|_{\partial_{in/out}\Sigma}$.

Moreover, (W, ρ_W) and Z_Σ should be universal in the sense that for any $(\tilde{W}, \rho_{\tilde{W}}) \in \mathcal{C}_{out}$ and for any linear map $\tilde{Z} : V \to \tilde{W}$ satisfying the same relations as above, there should exist a unique morphism $\kappa : W \to \tilde{W}$ in $\mathcal{C}_{out}$ that makes the following diagram commute:

$$\begin{tikzcd} & W \arrow[d, "\kappa"] \\ V \arrow[ur, "Z_\Sigma"] \arrow[r, "\tilde{Z}"'] & \tilde{W} \end{tikzcd}$$

The above universal property defines a functor $F_\Sigma : \mathcal{C}_{in} \to \mathcal{C}_{out}$. But, unfortunately, does not guarantee that it has any good formal properties. The following problem has been solved by James Tener in his PhD in the case of the free fermion[18], but remains open in the case of the WZW models:

Open problem: Given composable cobordisms Σ_1 and Σ_2, prove that the natural map $F_{\Sigma_1 \cup \Sigma_2}(\lambda) \to F_{\Sigma_1} \circ F_{\Sigma_2}(\lambda)$ is an isomorphism.

Now, why is this difficult?...
Well... for a universal construction to be well behaved, one needs the category in which it takes place to be "big enough". And, from that point of view, the positive energy condition is very awkward. So what we'd like is to be able to perform the universal construction in a bigger category (one which doesn't include the the positive energy condition), and have the result automatically satisfy the positive energy condition...

If our cobordism is an annulus $A \in \mathrm{Ann}(S)$, then the trivialization

$$T_{\tilde{A}} \;:\; F_A \longrightarrow \mathrm{id}_{\mathcal{C}(S)}$$

associated to a lift $\tilde{A} \in {}^{\mathbb{C}^\times \oplus \mathbb{Z}}\mathrm{Ann}_c(S)$ is constructed as follows. Let $\mathcal{A}(S)$ denote either $\widetilde{L_S\mathfrak{g}_k}$ or $CAR(S)$, depending on which chiral CFT we're treating. For every $\lambda \in \mathcal{C}(S)$, by the positive energy condition, the action $\rho : \mathcal{A}(S) \to \mathrm{End}(U(\lambda))$ extends to an action, again denoted ρ, of $\mathcal{A}(S) \rtimes {}^{U(1)\oplus\mathbb{Z}}\mathrm{Diff}_c(S)$. By definition, this means that we have actions of $\mathcal{A}(S)$ and of ${}^{U(1)\oplus\mathbb{Z}}\mathrm{Diff}_c(S)$ on $U(\lambda)$, satisfying the following covariance relation:

$$\rho({}^{\varphi}f) = \rho(\varphi)\rho(f)\rho(\varphi^{-1}) \qquad \forall f \in \left\{ \begin{array}{l} \mathcal{C}^\infty(S, \mathfrak{g}) \\ \Gamma(\mathbb{S}) \end{array} \right.$$

[18] James Tener only treats the case of thick complex cobordisms.

Here, $f \mapsto {}^{\varphi}f$ denotes the action of (the image of) φ in $\mathrm{Diff}(S)$ on $\mathcal{C}^\infty(S,\mathfrak{g}) \subset \widetilde{L_S\mathfrak{g}_k}$ or on $\Gamma(\mathbb{S}) \subset CAR(S)$.

Since the action of ${}^{U(1)\oplus\mathbb{Z}}\mathrm{Diff}_c(S)$ has positive energy, it extends to a holomorphic representation of ${}^{\mathbb{C}^\times\oplus\mathbb{Z}}\mathrm{Ann}_c(S)$ on $U(\lambda)$.[19] We construct the morphisms $T_{\tilde{A}} : F_A(\lambda) \to \lambda$ making the following diagram commute

$$\begin{tikzcd} & U(F_A(\lambda)) \arrow[d, "U(T_{\tilde{A}})"] \\ U(\lambda) \arrow[ur, "Z_A"] \arrow[r, "\rho(\tilde{A})"'] & U(\lambda) \end{tikzcd}$$

by applying the universal property in Definition 15 to the object $\lambda \in \mathcal{C}(S)$ and to the map $\rho(\tilde{A}) : U(\lambda) \to U(\lambda)$. In order to invoke the universal property, we need to check that the relation (22) holds, namely

$$\rho(\tilde{A})\rho(f_{in}) = \rho(f_{out})\rho(\tilde{A}) \qquad \forall f \in \begin{array}{l} \mathcal{O}(A;\mathfrak{g}_{\mathbb{C}}) \\ \Gamma_{hol}(A,\mathbb{S}_\Sigma). \end{array} \tag{23}$$

Let $\mathrm{Ann}^{\leq A} = \big\{A_1 \in \mathrm{Ann}(S)\,\big|\,\exists A_2 : A_1A_2 = A\big\} = \big\{\gamma : S \hookrightarrow A\,\big|\,\gamma \text{ wraps around } A\big\}$, and let ${}^{\mathbb{C}^\times\oplus\mathbb{Z}}\mathrm{Ann}_c^{\leq A}$ be its pre-image in ${}^{\mathbb{C}^\times\oplus\mathbb{Z}}\mathrm{Ann}_c(S)$.

Claim: The map

$$\begin{aligned} \Phi \;:\; {}^{\mathbb{C}^\times\oplus\mathbb{Z}}\mathrm{Ann}_c^{\leq A} &\longrightarrow \mathrm{End}\big(U(\lambda)\big) \\ \tilde{A}_1 &\mapsto \rho(\tilde{A}_1)\rho(f_\gamma)\rho(\tilde{A}_2) \end{aligned}$$

is constant. Here, $\tilde{A}_2$ is the unique solution of the equation $\tilde{A}_1\tilde{A}_2 = \tilde{A}$, and f_γ is the restriction of f along the map $\gamma : S \to A$.

Equation (23) follows since $\rho(\tilde{A})\rho(f_{in}) = \Phi(\tilde{A})$ and $\rho(f_{out})\rho(\tilde{A}) = \Phi(1)$.

Proof. The map Φ is holomorphic (continuous, and holomorphic in the interior). If A_1 is completely thin, then

$$\Phi(\tilde{A}_1) = \rho(\tilde{A}_1)\rho(f_\gamma)\rho(\tilde{A}_1^{-1})\rho(\tilde{A}) = \rho(f)\rho(\tilde{A})$$

doesn't depend on $\tilde{A}_1$. So Φ is constant on the subset of completely thin A_1's. The map $\mathrm{Diff}(S) = \{\text{thin } A_1\text{'s}\} \hookrightarrow \mathrm{Ann}^{\leq A}$ is the inclusion of a real manifold into a complexification. We finish by noting that a holomorphic function which is constant on the real submanifold is necessarily constant on the complexification. $\square$

[19] This is stated as a theorem in [Y. Neretin. *Holomorphic continuations of representations of the group of diffeomorphisms of the circle*; translation in Math. USSR-Sb. 67 (1990), no. 1, 75–97], but the paper does not include a proof of holomorphicity.

The positive energy condition

In the previous sections, we motivated the introduction of the positive energy condition by saying "otherwise, there's too many representations". But there's a much better reason to include that condition. That's because, in a Segal CFT, that condition is *forced* on you:

Theorem 16 *In a chiral Segal CFT, the action of the Virasoro algebra on any sector always has positive energy.*

The proof will be based on the following result:

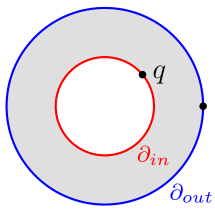
Proposition 17 *If* Σ *is a thick complex cobordism, then* $Z_\Sigma : U(\lambda) \to U(F_\Sigma(\lambda))$ *is a trace-class operator.*

We'll give the definition of trace-class a bit later. For the moment, we just need to know that, for a diagonal operator on a topological vector space, we have

$$\mathrm{Diag}(\alpha_1, \alpha_2, \ldots) \text{ is trace-class} \qquad \Longrightarrow \qquad \lim_{n\to\infty} \alpha_n = 0.$$

When our vector space is a Hilbert space, we have the much stronger result that an operator of the form $\mathrm{Diag}(\alpha_1, \ldots)$ is trace-class iff $\sum |\alpha_n| < \infty$. But things don't work quite as nicely when dealing with more general types of vector spaces. *[The implication* $\sum |\alpha_n| < \infty \Rightarrow \mathrm{Diag}(\alpha_1, \ldots)$ *is trace-class only holds when the diagonal operator is defined w.r.t. an unconditional basis. The implication* $\mathrm{Diag}(\alpha_1, \ldots)$ *trace-class* $\Rightarrow \sum |\alpha_n| < \infty$ *almost never holds.]*

Proof of Theorem 16. For q a complex number, $|q| < 1$, let $A_q := \{z \in \mathbb{C} : |q| \le |z| \le 1\}$, with boundary parametrizations $\varphi_{in} : z \mapsto qz : S^1 \to \partial_{in} A$ and $\varphi_{out} = \mathrm{id}_{S^1}$:

$A_q =$ 

Pick a lift of $\tilde{A}_q \in {}^{\mathbb{Z}}\mathrm{Univ}(\mathbb{D})$ of A_q to the universal cover of $\mathrm{Univ}(\mathbb{D})$. Equivalently, pick a logarithm of q. The corresponding operator on $U(\lambda)$ is then given by

$$\rho(\tilde{A}_q) = U(T_{\tilde{A}_q}) \circ Z_{A_q} = q^{L_0} := e^{\log(q) L_0}.$$

The operator $U(T_{\tilde{A}_q}) \circ Z_{A_q}$ is trace-class by Proposition 17. In particular, its sequence of eigenvalues (counted with multiplicity) tends to zero. This is equivalent to the spectrum of L_0 being discrete, bounded from below, and all its eigenspaces being finite dimensional. $\square$

Before discussing the proof of Proposition 17, we recall some definitions from functional analysis. From now on, we assume that all our vector spaces are complete locally convex topological vector spaces.[20]

[20]A topology is called 'locally convex' if it is generated by a set of (semi-)norms.

Definition: An operator $f : V \to W$ is *trace-class*[21] if it is in the image of the map

$$\mathrm{E} : W \otimes_\pi V' \to \mathcal{L}(V, W).$$

Here, V' is the continuous dual of V (the set of continuous linear maps $V \to \mathbb{C}$), and $\otimes_\pi$ is the projective tensor product of topological vector spaces (defined by the universal property that continuous bi-linear maps out of the product are the same thing as continuous linear maps out of the tensor product). If $f : V \to V$ is trace-class, then its trace $\mathrm{tr}(f) \in \mathbb{C}$ is the image of $\mathrm{E}^{-1}(f)$ under the evaluation map $V \otimes V' \cong V' \otimes V \xrightarrow{ev} \mathbb{C}$.

Warning: *When working with general topological vector spaces, the map* E *can fail to be injective; this can already happen with Banach spaces. When this happens,* $\mathrm{tr}(f)$ *typically fails to be well defined. But everything is ok (i.e., the map* E *is injective) when the spaces have bases. (A subset* $(b_n)_{n\in\mathbb{N}}$ *of a topological vector space* V *is a* basis *if for every* $v \in V$ *there is a unique sequence of numbers* (a_n) *such that* $v = \sum a_n b_n$*.)*

Lemma 18 *A linear map* $f : V \to W$ *is trace-class if and only if there exists a space* X*, and linear maps* $a : \mathbb{C} \to W \otimes_\pi X$*, and* $b : X \otimes_\pi V \to \mathbb{C}$*, such that*

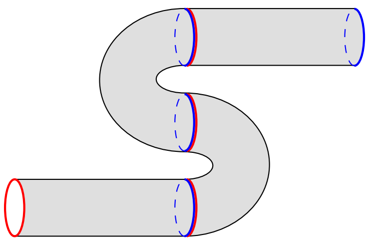
$$f = \left(V \xrightarrow{a\otimes \mathrm{id}} W \otimes X \otimes V \xrightarrow{\mathrm{id}\otimes b} W\right).$$

Proof. $\Rightarrow$: Take $X = V'$, $a = \mathrm{E}^{-1}(f)$, and $b = ev : V' \otimes V \to \mathbb{C}$.
$\Leftarrow$: b induces a map $\tilde{b} : X \to V'$. Then f is the image of $(\mathrm{id}_W \otimes \tilde{b})a \in W \otimes_\pi V'$. □

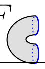
As a corollary, trace-class maps form an ideal: if a map f is trace-class, then so is $f \circ g$, and so is $h \circ f$.

Remark. The statement of Lemma 18 also holds true when V, W, and X are Hilbert spaces, and the projective tensor product is replaced by the Hilbert space tensor product. But the proof is rather different (it relies on the fact that the composition of two Hilbert-Schmidt operators is always trace-class).

The proof of Proposition 17 will be based on the fact that every thick annulus can be decomposed as follows:



(24)

We first compute F and F. Let $1_\emptyset := \mathbb{C}$ be the canonical simple object of $\mathcal{C}(\emptyset) = \mathsf{Vec}_{\mathrm{f.d.}}$.

[21] When working with Hilbert spaces, one typically uses the term 'trace class'. When working with more general topological vector spaces, one typically uses the word 'nuclear' for that same notion.

Proposition. *There exists a canonical involution $\lambda \mapsto \bar{\lambda}$, called* charge conjugation*, on the set of isomorphism classes of simple objects of $\mathcal{C}(S^1)$, such that*

$$F(1_\emptyset) = \bigoplus_\lambda \lambda \otimes \bar{\lambda} \qquad \textit{and} \qquad F(\mu \otimes \nu) = \delta_{\mu,\bar{\nu}} 1_\emptyset.$$

Proof. Write

$$F(1_\emptyset) = \bigoplus_{\lambda,\mu} a_{\lambda,\mu} \lambda \otimes \mu \qquad \text{and} \qquad F(\mu \otimes \nu) = b_{\mu,\nu} 1_\emptyset.$$

The triviality of F means that the matrices $a = (a_{\lambda,\mu})$ and $b = (b_{\mu,\nu})$ satisfy $ab = 1$. Similarly, we have $ba = 1$. Since the entires of a and of b all lie in $\mathbb{N}$, they are permutation matrices.

The composition of with (the cobordism associated to) the diffeomorphism that switches the two boundary circles is isomorphic (rel boundary) to that same cobordism . It follows that $(\text{switch}) \circ F \simeq F_{(\text{switch})} \circ F \simeq F_{(\text{switch})\cup} \simeq F$, and hence that $a_{\lambda,\mu} = a_{\mu,\lambda}$. This tells us that a and b are involutions, and that $a = b$. □

Proof of Proposition 17. Since trace-class maps form an ideal, and since the tensor product of two trace-class maps is again trace-class, it's enough to show that Z_Σ is trace-class when $\Sigma = A$ is an annulus. Cutting A as in (24), we can decompose Z as:

$$\begin{tikzcd}
 & H_\lambda \otimes H_{\bar{\lambda}} \otimes H_\lambda \arrow[dr, dashed] & \\
H_\lambda \arrow[r, "Z"'] & \bigoplus_\mu H_\mu \otimes \underbrace{H_{\bar{\mu}} \otimes H_\lambda}_{\in \ker(F) \text{ unless } \mu=\lambda} \arrow[u, two heads] \arrow[r, "Z"'] & H_\lambda
\end{tikzcd}$$

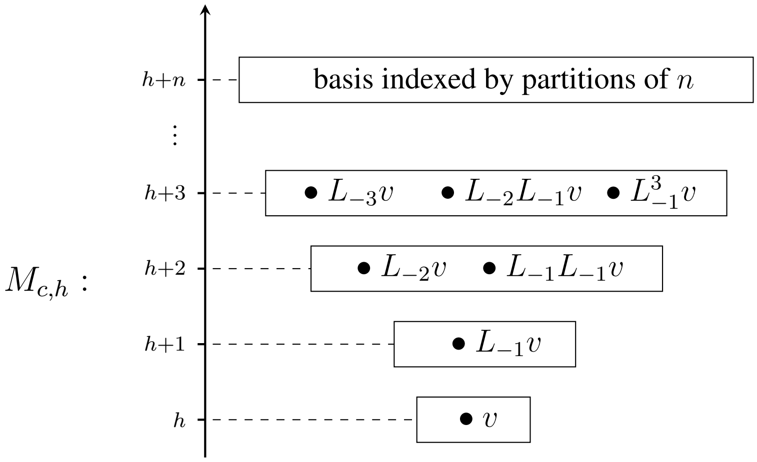
We are then done by Lemma 18, and the fact that trace-class maps form an ideal. □

Unitary representations of the Virasoro algebra

Let us explain a bit what the representation theory of the Virasoro algebra looks like. First of all, for every value c of the central charge and h of the minimal energy, one can form the Verma module

$$M_{c,h} := \mathrm{Ind}_{Vir_c^{\geq 0}}^{Vir_c} \mathbb{C}_{c,h}.$$

Here, $Vir_c^{\geq 0} := \mathrm{Span}\{L_n\}_{n\geq 0} \oplus 1\cdot\mathbb{C}$ acts on the one dimensional module $\mathbb{C}_{c,h}$ by $L_0 \mapsto h$, and $L_n \mapsto 0$ for $n > 0$ (and the central element $1 \in Vir_c^{\geq 0}$ acts by 1). Fix $v \in \mathbb{C}_{c,h}$. The Verma module $M_{c,h}$ is a graded by $\mathbb{N} + h$, with basis given by elements $L_{n_1} L_{n_2} \ldots L_{n_k} v$, for $n_1 \geq \ldots \geq n_k > 0$:



The next step towards constructing unitary representations of the Virasoro algebra is to consider the simple quotients

$$L_{c,h} := M_{c,h}/J_{c,h},$$

where $J_{c,h} \subsetneq M_{c,h}$ is unique the maximal proper submodule of the Verma module. Equivalently, $J_{c,h}$ can be described as the set of all vectors (in degree $\geq h+1$) that cannot be brought back to a non-zero multiple of v by a sequence of L_m's (and, by virtue of the Virasoro Lie algebra relations, it is enough to only consider L_m's with $m > 0$):

$$J_{c,h} = \bigoplus_{\substack{i=h+n,\\ n\in\mathbb{N}_{>0}}} \{\xi \in M_{c,h}(i) \,|\, L_{m_1}\ldots L_{m_k}\xi = 0,\ \ \forall(m_1,\ldots,m_k),\ m_i > 0,\ \textstyle\sum m_i = n\}.$$

Here, $M_{c,h}(i)$ denotes the degree i part of the Verma module. The elements of $J_{c,h}$ are called *null-vectors*.

Example: If $h = 0$, then $L_{-1}v$ is a null-vector.
Proof. We compute $L_1 L_{-1} v = [L_1, L_{-1}]v = 2L_0 v = 0$. $\square$

Definition: An irreducible representation of the Virasoro algebra Vir_c on a Hilbert space[22] is called *unitary* if L_0 is (unbounded) self-adjoint, and $L_n^* = L_{-n}$.

Alternatively, a representation is called unitary if its underlying vector space can be equipped with a positive definite inner product under which $L_n^* = L_{-n}$.

[22]Unbounded operators are only densely defined. So, strictly speaking, a Hilbert space is not a representation. The algebraic direct sum of the L_0-eigenspaces which carries a genuine representations of Vir_c.

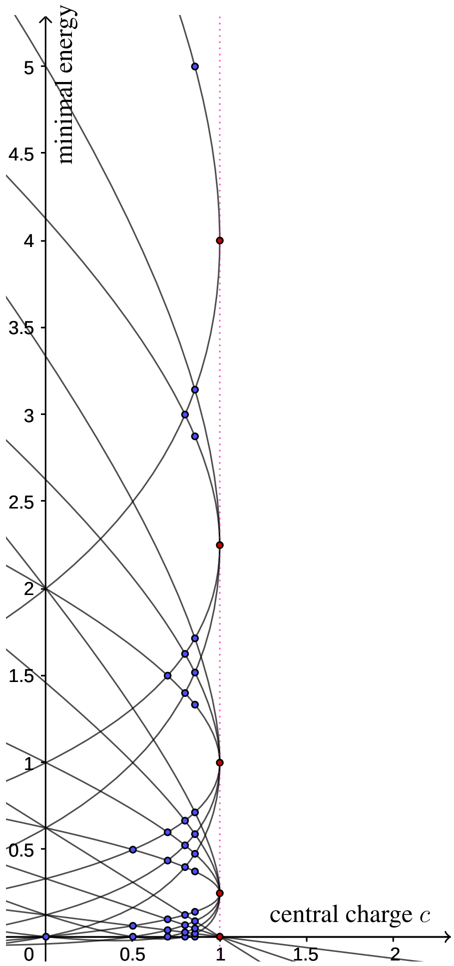
Lemma. *If $L_{c,h}$ is unitary, then $h \geq 0$.*

Proof. We compute $\langle L_{-1}v, L_{-1}v\rangle = \langle L_1 L_{-1}v, v\rangle = \langle 2L_0 v, v\rangle = 2h$. $\square$

It turns out that for $c > 1$ there are no further restrictions. The Verma modules $M_{c,h}$ are simple for all $h > 0$, and the simple quotients $L_{c,h}$ are unitary for all $h \geq 0$. To summarise, for $c > 1$, the irreducible unitary representations of Vir_c are classified by their minimal energy, which can take any value in $\mathbb{R}_{\geq 0}$. The corresponding chiral CFT is not rational, and is called chiral Liouville theory.

If $c = 1$, the chiral CFT is still not rational. It satisfies $M_{1,h} = L_{1,h}$ iff $h \in \mathbb{R}_+ \setminus \{\frac{n^2}{4} | n \in \mathbb{N}\}$, and the simple quotients $L_{1,h}$ are unitary for all $h \geq 0$.

In the range $c < 1$, there exists a discrete set of values of c for which the Virasoro algebra admits unitary representations (outside of that set, Vir_c has no unitary representations). These are the numbers of the form $c = 1 - \frac{6}{m(m+1)}$ for $m = 2, 3, 4, \ldots$

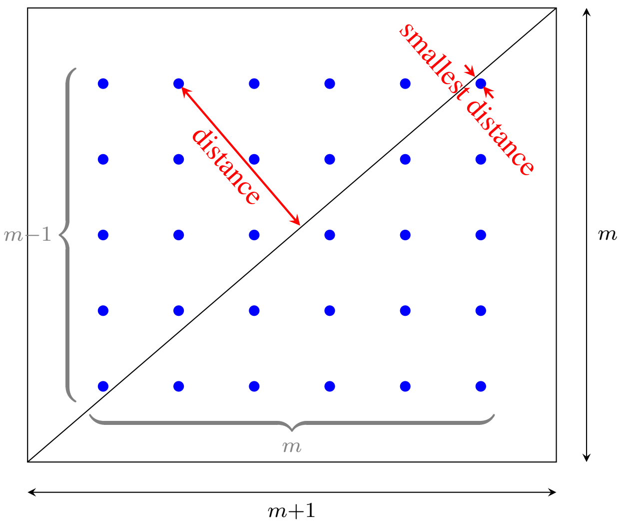


Discrete series unitary irreps of Vir_c.

$$c = 0,\ \tfrac{1}{2},\ \tfrac{7}{10},\ \tfrac{4}{5},\ \tfrac{6}{7},\ \tfrac{25}{28},\ \tfrac{11}{12},\ \tfrac{14}{15},\ \tfrac{52}{55},\ \tfrac{21}{22},\ \tfrac{25}{26},\ \tfrac{88}{91},\ \ldots$$

They correspond to the unitary minimal models. For such a value of the central charge, Vir_c has exactly $m(m-1)/2$ irreducible unitary representations. They are classified by their minimal energy, which can take any value of the form $h_{p,q} := \frac{[(m+1)p-mq]^2-1}{4m(m+1)}$, for $1 \leq p \leq m-1$ and $1 \leq q \leq m$. The curved lines in the above picture are hyperbolas. They are the locus of pairs (c,h) for which the Verma module $M_{c,h}$ admits null-vectors (i.e., such that $J_{c,h} \neq 0$). The blue dots are the unitary simple modules.

A good mnemonic for the above minimal energies $h_{p,q}$ is to note that they're equal to the square-distance to the diagonal minus the smallest square-distance to the diagonal in the following rectangular array of dots:



(normalized so that the smallest square-distance is $\frac{1}{4m(m+1)}$).

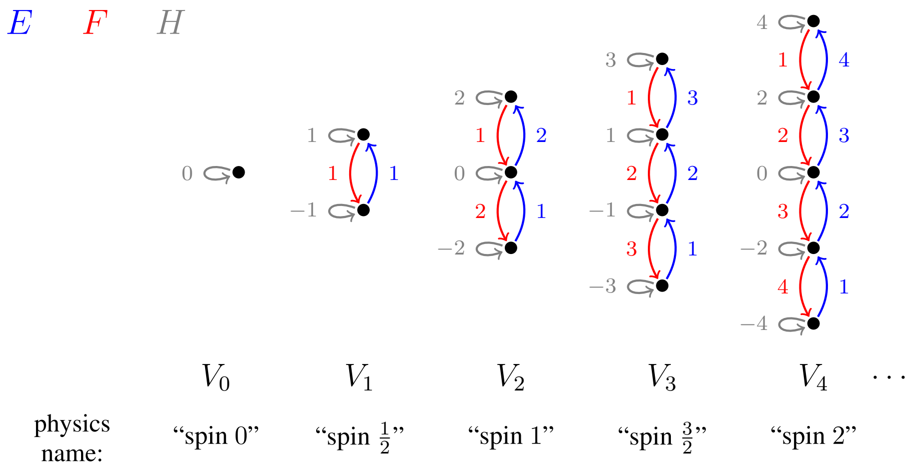
Structure of finite dimensional simple Lie algebras

In this section, we recall various facts from the theory of finite dimensional simple Lie algebras, before treating the more complicated topic of affine Lie algebras. All our Lie algebras will be over $\mathbb{C}$.

We start by analyzing $\mathfrak{sl}(2)$. It is the smallest simple Lie algebra, and has the property that every other simple Lie algebra is build out of copies of it. It has a basis given by the matrices $E := \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $F := \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, and $H := \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, subject to the commutation relations

$$[E, F] = H \qquad [H, E] = 2E \qquad [H, F] = -2F. \tag{25}$$

The irreducible finite dimensional $\mathfrak{sl}(2)$-reps $V_0, V_1, V_2, \ldots$ are classified by their dimension $\dim(V_n) = n + 1$. The next picture is a graphical depiction of these irreps.



Each bullet represents a basis element, and the actions of E, F, H are indicated by the colored arrows. If we call v_i the basis element corresponding to the ith bullet (the top one being v_0) then, for example, a red arrow labelled a between the ith and the $(i+1)$st bullet indicates that $F(v_i) = av_{i+1}$.

Let now $\mathfrak{g}$ be an arbitrary finite dimensional simple Lie algebra over $\mathbb{C}$. A *Cartan subalgebra* is an abelian subalgebra $\mathfrak{h} \subset \mathfrak{g}$ which arises as the Lie algebra of a maximal torus T inside the simply connected Lie group G associated to $\mathfrak{g}$ (such a torus is unique up to conjugacy). The dimension $r := \dim(\mathfrak{h})$ is called the *rank* of $\mathfrak{g}$.

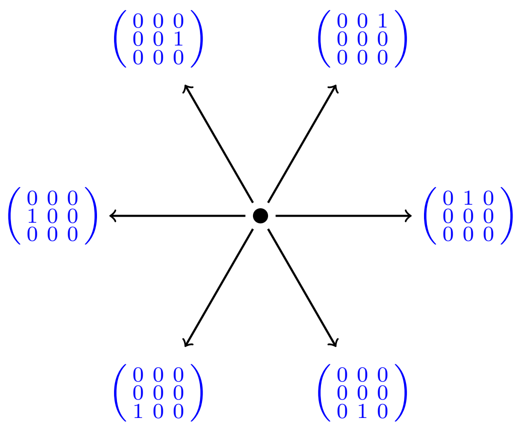
As an $\mathfrak{h}$-representation, $\mathfrak{g}$ decomposes as a direct sum of its Cartan subalgebra $\mathfrak{h}$ and certain one-dimensional subspaces $\mathfrak{g}^\alpha \subset \mathfrak{g}$ called *root spaces*:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}^\alpha. \tag{26}$$

The direct sum is indexed by a finite set $\Phi \subset \mathfrak{h}^*$ called the *root system* of $\mathfrak{g}$, and the elements of Φ are called the *roots* of $\mathfrak{g}$. The root spaces $\mathfrak{g}^\alpha$ are spanned by *root vectors* $E^\alpha \in \mathfrak{g}^\alpha$. The letter satisfy $[H, E^\alpha] = (\alpha, H)E^\alpha$, $\forall H \in \mathfrak{h}$. The root vectors furthermore satisfy $[E^\alpha, E^\beta] \in \mathfrak{g}^{\alpha+\beta}$ when $\alpha + \beta \in \Phi$, and $[E^\alpha, E^\beta] = 0$ otherwise.

We illustrate the notions of root system and root vectors in the case of the Lie algebra $\mathfrak{sl}(3)$ of traceless 3×3 matrices, with its Cartan subalgebra $\mathfrak{h}$ of diagonal matrices:

$\underline{\mathfrak{sl}(3) :}$



The plane in which the above picture is drawn is $\mathfrak{h}^*$, the dual of the Cartan. The six roots form the vertices of a regular hexagon, and the root vectors are 3×3 matrices with a single non-zero off-diagonal entry.

For each root $\alpha \in \Phi$, the root vectors E^α, $F^\alpha := E^{-\alpha}$, and their commutator $H^\alpha := [E^\alpha, F^\alpha] \in \mathfrak{h}$ span a subalgebra $\mathfrak{sl}(2)^\alpha := \mathrm{Span}\{E^\alpha, F^\alpha, H^\alpha\} \subset \mathfrak{g}$ isomorphic to $\mathfrak{sl}(2)$. The elements E^α, F^α, $H^\alpha \in \mathfrak{g}$ can be normalised to satisfy the same commutation relations (25) as $E, F, H \in \mathfrak{sl}(2)$.

The Lie algebra $\mathfrak{g}$ carries a unique invariant bilinear form, up to scalar. Given such a form, we may consider the induced bilinear form $\langle\, , \rangle$ on $\mathfrak{h}^*$. The roots $\alpha \in \mathfrak{h}^*$ either all have the same square-length, in which case the Lie algebra is called *simply laced*, or there are two square-lengths of roots. In the latter case, the roots are divided into 'long roots' and 'short roots'. The basic inner product is the invariant bilinear form on $\mathfrak{g}$ normalised so that either $\langle\alpha,\alpha\rangle = 2$ for every long root (when $\mathfrak{g}$ is not simply laced), or $\langle\alpha,\alpha\rangle = 2$ for every root (when $\mathfrak{g}$ is simply laced). If $\mathfrak{g}$ is not simply laced, let d be the ratio between the square-length of a long root and that of a short root (this number is called the *lacity* of $\mathfrak{g}$, and is either 2 or 3). The element $H^\alpha = [E^\alpha, E^{-\alpha}] \in \mathfrak{h}$ can then be described as

$$H^\alpha = \begin{cases} \langle\alpha,-\rangle & \text{if } \alpha \text{ is a long root or } \mathfrak{g} \text{ is simply laced,} \\ d\langle\alpha,-\rangle & \text{if } \alpha \text{ is a short root} \end{cases} \tag{27}$$

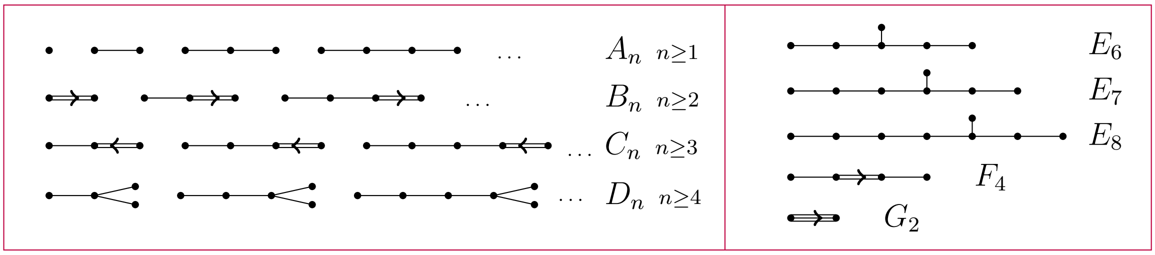
(here, we've identified an element of $\mathfrak{h}$ with the corresponding linear functional on $\mathfrak{h}^*$).

For $\alpha \in \Phi$ a root, let us write

$$\begin{aligned} s_\alpha :\ \mathfrak{h}^* &\to \mathfrak{h}^* \\ \xi &\mapsto \xi - 2\tfrac{\langle\xi,\alpha\rangle}{\|\alpha\|^2}\cdot\alpha \end{aligned}$$

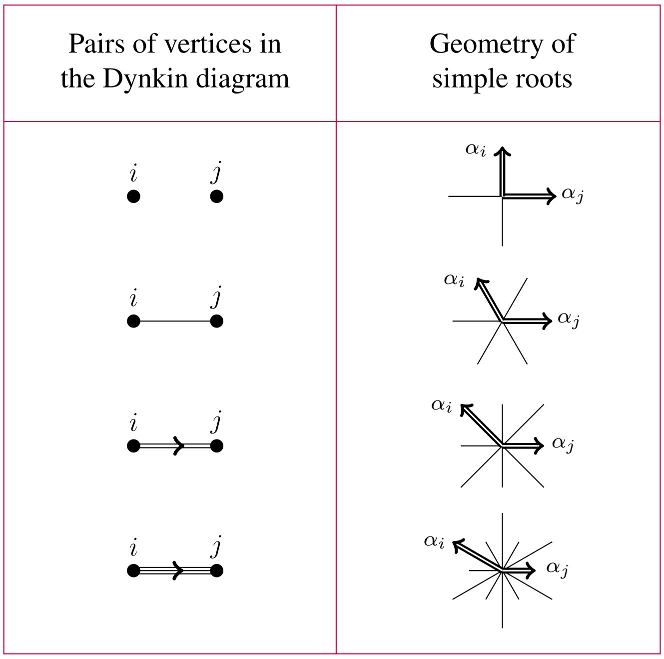
for the reflection across the hyperplane perpendicular to α. These reflections act by symetries of the root system, and generate the *Weyl group* W. Let $\mathfrak{h}^*_{\mathbb{R}}$ be the $\mathbb{R}$-span of the roots (so that $\mathfrak{h}^*$ is the complexification of $\mathfrak{h}^*_{\mathbb{R}}$). The hyperplanes $\{\xi : \langle\xi,\alpha\rangle = 0\}$ partition $\mathfrak{h}^*_{\mathbb{R}}$ into W many chambers; pick one and call it the *dominant Weyl chamber*. It has r many walls (where r is the rank of $\mathfrak{g}$). The roots $\alpha_1, \ldots, \alpha_r \in \Phi$ that are perpendicular to these walls and inwards-pointing are called the *simple roots* of $\mathfrak{g}$.

The *Dynkin diagram* of $\mathfrak{g}$ is a type of graph which encodes the geometry of the simple roots of $\mathfrak{g}$. Each node corresponds to a simple root, and the angles between simple roots as well as their relative lengths are encoded in the way the nodes are connected (except when two nodes are not connected by an edge in which case the relative lengths are not encoded). The Dynkin diagrams that arise from simple finite dimensional Lie algebra are called *of finite type*. There are four infinite families of such Dynkin diagrams, denoted A_n, B_n, C_n, D_n, and five exceptional cases: E_6, E_7, E_8, F_4, G_2 (the subscript denotes the rank of $\mathfrak{g}$):



The Dynkin diagrams of finite type

The type of edge between two vertices of the Dynkin diagram encodes the relative geometry of the corresponding simple roots:



(28)

The Dynkin diagram is a complete invariant of a Lie algebra:

Simple finite dimensional Lie algebras are classified by their Dynkin diagram.

It is in fact possible to write down a presentation of the Lie algebra $\mathfrak{g}$ directly from its Dynkin diagram: there are three sets of generators E^i, F^i, H^i, indexed by the vertices of the Dynkin diagram, with relations

$$
\begin{aligned}
&[H^i, H^j] = 0 \\
&[H^i, E^j] = a_{ij} E^j \\
&[H^i, F^j] = -a_{ij} F^j \\
&[E^i, F^j] = \delta_{ij} H^i \\
&[\underbrace{E^i, \ldots [E^i}_{|a_{ij}|+1}, E^j]\,] = [\underbrace{F^i, \ldots [F^i}_{|a_{ij}|+1}, F^j]\,] = 0,
\end{aligned}
\qquad (29)
$$

where the a_{ij} are prescribed as follows:

$i \quad j$	a_{ij}
$i = j$	2
$\bullet\quad\bullet$	0
$\bullet\!-\!\bullet$	-1
$\bullet\!\Rightarrow\!\bullet$	-1
$\bullet\!\Rrightarrow\!\bullet$	-1
$\bullet\!\Leftarrow\!\bullet$	-2
$\bullet\!\Lleftarrow\!\bullet$	-3

These relations are known as the *Serre relations*, and the matrix (a_{ij}) is known as the *Cartan matrix* associated to the Dynkin diagram. The subalgebra $\mathfrak{b} \subset \mathfrak{g}$ generated by just the E_i and the H_i is called the *Borel subalgebra* of $\mathfrak{g}$.

Recall that the Cartan subalgebra $\mathfrak{h}$ is the Lie algebra of a maximal torus $T \subset G$ inside the simply connected Lie group associated to $\mathfrak{g}$. The set $\Lambda := \mathrm{Hom}(T, \mathbb{C}^\times)$ is a lattice in $\mathfrak{h}^*$ called the *weight lattice*. Every finite dimensional representation V of $\mathfrak{g}$ is a direct sum

$$V = \bigoplus_{\lambda \in \Lambda} V(\lambda),$$

indexed by the weight lattice, of its *weight spaces* $V(\lambda) := \{v \in V : Hv = (\lambda, H)v, \forall H \in \mathfrak{h}\}$. The set of $\lambda \in \Lambda$ such that $V(\lambda) \neq 0$ is called the *set of weights* of V. For example, the set of weights of the adjoint representation is $\Phi \cup \{0\}$.

For any $\lambda \in \mathfrak{h}^*$, we may consider the *Verma module*

$$M_\lambda := \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda,$$

where the action of the Borel on the one-dimensional module $\mathbb{C}_\lambda$ is given by $H \mapsto (\lambda, H)$ for $H \in \mathfrak{h}$, and $E_i \mapsto 0$. The image of $1 \in \mathbb{C}_\lambda$ is called the *highest weight vector* of M_λ. Unlike the Verma module, the *simple quotient*

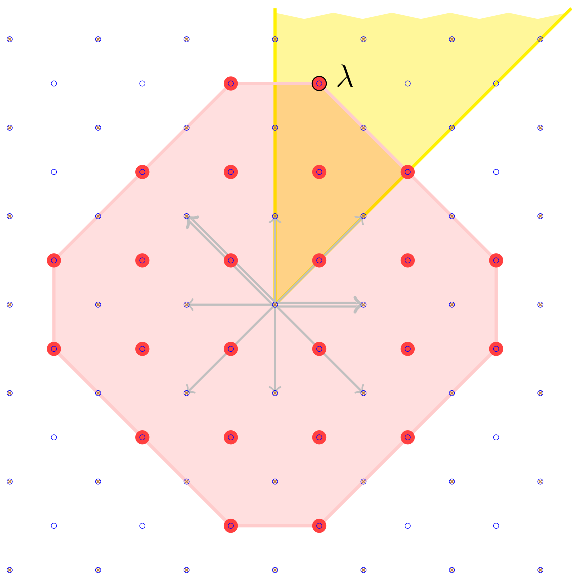
$$L_\lambda := M_\lambda/J_\lambda$$

maximal proper submodule of M_λ

is sometimes finite dimensional. Specifically, L_λ is finite dimensional if and only if λ belongs to the set $\Lambda_+ \subset \Lambda$ of *dominant weights*, defined as the intersection of the weight lattice with the dominant Weyl chamber.

The above construction establishes a one-to-one correspondence between the set of isomorphism classes of finite dimensional irreducible representations, and the set Λ_+ of dominant weights of $\mathfrak{g}$.

The weight λ is called the *highest weight* of L_λ. Let the *root lattice* $\Lambda_{\mathrm{root}} \subset \Lambda$ be the sub-lattice of the weight lattice spanned by the roots. For $\lambda \in \Lambda_+$, the set of weights of L_λ can be then described as the intersection of the shifted root lattice $\Lambda_{\mathrm{root}} + \lambda$ with the *weight polytope* P_λ, where the latter is the convex hull of the orbit of λ under the Weyl group. We illustrate all these notions in the case of an irrep of $\mathfrak{so}(5)$ (the simple Lie algebra of type B_2) associated to some $\lambda \in \Lambda_+$:



The red bullets mark the weights of L_λ. The dominant Weyl chamber is in yellow. The weight polytope P_λ is in pink. The roots are in gray, with the simple roots in bold. The tiny blue circles indicate the weight lattice, and the tiny brown crosses mark the root lattice.

Affine Lie algebras

For $\mathfrak{g}$ a finite dimensional simple Lie algebra over $\mathbb{C}$, let $L\mathfrak{g} := \mathfrak{g}[t, t^{-1}]$ be the algebra of polynomial loops in $\mathfrak{g}$ (algebraic functions on $\mathbb{C}^\times$ with values in $\mathfrak{g}$). For X an element of $\mathfrak{g}$, we write X_n for $Xz^n \in L\mathfrak{g}$, so that the bracket of $L\mathfrak{g}$ is given by $[X_m, Y_n] = [X, Y]_{m+n}$. The *affine Lie algebra*

$$\widetilde{L\mathfrak{g}} \;=\; \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$$

is the central extension of $L\mathfrak{g}$ associated to the cocycle $\omega(f, g) \;=\; \mathrm{Res}_0\langle df, g\rangle$.[23] Its bracket is given by

$$[X_m, Y_n] \;=\; [X, Y]_{m+n} + m\delta_{m+n,0}\langle X, Y\rangle K,$$

and the element K spans its center.

Starting from a Dynkin diagram, we have seen in (29) how to construct a simple Lie algebra $\mathfrak{g}$ by means of the Serre relations. The Lie algebra is finite dimensional if and only if the Dynkin diagram is of finite type (i.e., one of A_n, B_n, C_n, D_n, E_{6-8}, F_4, G_2), but the Serre relations make sense for more general Dynkin diagrams too, leading to a class of infinite dimensional Lie algebras known as *Kac-Moody algebras*. A large portion of the theory of finite dimensional simple Lie algebras passes over essentially unchanged to the more general setup of Kac-Moody algebras.[24] This includes notions such as *roots*, *weights*, the *Weyl group*, and the classification of (certain) simple modules by *dominant weights* $\lambda \in \Lambda_+$, including the fact that the set of weights of L_λ is given by $P_\lambda \cap (\Lambda_{\text{root}} + \lambda)$.

The reflections that generate the Weyl group are given by the same formula $s_\alpha(\xi) = \xi - \frac{2\langle\xi,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha$ as in the finite dimensional case, with the notable difference that the invariant inner product on $\mathfrak{h}^*_{\mathbb{R}}$ is no longer positive definite.

Surprisingly, *affine Lie algebras are (almost) instances of Kac-Moody algebras!* The reason for the 'almost' is that it's not $\widetilde{L\mathfrak{g}}$, but rather the semi-direct product $\hat{\mathfrak{g}} := \widetilde{L\mathfrak{g}} \rtimes \mathbb{C}L_0$ which is a Kac-Moody algebra (where $[X_n, L_0] = nX_n$):[25]

$$\overset{\substack{\text{fin. dim. simple}\\ \text{Lie algebra}}}{\mathfrak{g}} \quad\rightsquigarrow\quad \overbrace{\overbrace{\big(\mathfrak{g}[t, t^{-1}] \oplus \underbrace{\mathbb{C}K}_{\substack{\text{central ext. with}\\ \text{cocycle } \mathrm{Res}_0\langle df, g\rangle}}\big)}^{\text{affine Lie alg.}} \rtimes \mathbb{C}L_0}^{\text{affine Kac-Moody algebra } \hat{\mathfrak{g}}}$$

The Cartan subalgebra of $\hat{\mathfrak{g}}$ is given by $\hat{\mathfrak{h}} := \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}L_0$, with invariant inner product given by $\langle\ ,\ \rangle_{\hat{\mathfrak{h}}} := \langle\ ,\ \rangle_{\mathfrak{h}} \oplus \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The Weyl group of $\hat{\mathfrak{g}}$ is called the *affine Weyl group* and denoted $\hat{W}$. It acts on $\hat{\mathfrak{h}}^*$ and contains the finite Weyl group W as a subgroup.

[23] Here, unlike in (21), $\langle\ ,\ \rangle$ denotes the basic inner product on $\mathfrak{g}$.

[24] More precisely, this holds for the so-called *symmetrizable Kac-Moody algebras*. This class includes all the affine Kac-Moody algebras. It excludes Dynkin diagrams such as , for which it is not possible to consistently assign lengths to the simple roots while respecting the rules (28).

[25] It is customary to alter slightly the Serre relations when the Cartan matrix has zero determinant. In that case, one adds an extra generator to the Cartan subalgebra to ensure that the simple roots are linearly independent as elements of $\mathfrak{h}^*$.

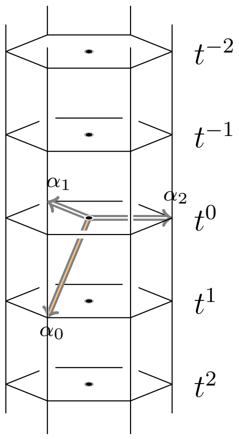
The Kac-Moody algebras obtained by the above construction are called *affine Kac-Moody algebras*.

To see that $\hat{\mathfrak{g}} = \widetilde{L\mathfrak{g}} \rtimes \mathbb{C}L_0$ is an instance of a Kac-Moody algebra, we first note that the weights of its adjoint representation are given by

$$\Psi := \big(\Phi \cup \{0\}\big) \times \mathbb{Z} \subset \big\{\xi \in \hat{\mathfrak{h}}^* : (\xi, K) = 0\big\} \subset \hat{\mathfrak{h}}^*.$$

Let $\alpha_1, \ldots, \alpha_r \in \Phi$ be the simple roots of $\mathfrak{g}$, which we identify with their images $(\alpha_i, 0) \in \Psi$, and let $\alpha_0 := (\alpha_{\min}, 1) \in \Psi$ be the so-called *affine root*. Here, $\alpha_{\min} \in \Phi$ denotes the *lowest root* of $\mathfrak{g}$, characterised by the property that $\alpha_{\min} - \alpha_i \notin \Phi$ for any simple root α_i. We illustrate the case of affine $\mathfrak{su}(3)$:

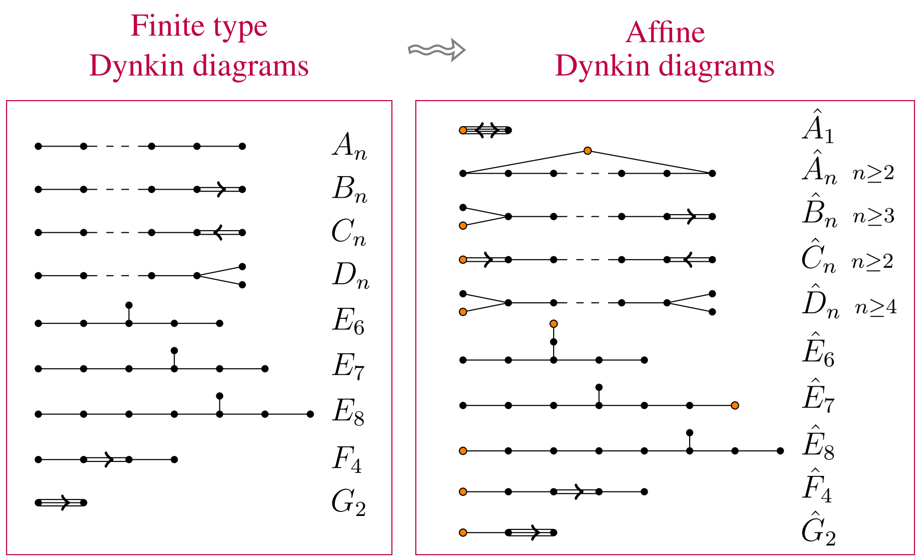
Ψ :



The crucial observation is that the standard generators E^i, F^i, H^i of $\mathfrak{g}$ (viewed as elements of $\hat{\mathfrak{g}}$) along with

$$E^0 := (\text{root vector for } \alpha_{\min}) \otimes t, \quad F^0 := (\text{root vector for } -\alpha_{\min}) \otimes t^{-1}, \quad H^0 := [E^0, F^0]$$

(normalised so that $H^0 = \langle \alpha_{\min}, - \rangle + K \in \hat{\mathfrak{h}}$) satisfy the Serre relations for an extended Dynkin diagram known as the *affine Dynkin diagram* associated to $\mathfrak{g}$.



Starting from the Dynkin diagram of $\mathfrak{g}$, one constructs the associated affine Dynkin diagram by adding one extra vertex (corresponding to the affine root α_0). The angles and relative lengths of $\alpha_{\min}$ to the other simple roots tell us, using (28), how to connect the new vertex to the old ones[26]. (Note that, for the purpose of computing angles and lengths, α_0 and $\alpha_{\min}$ are interchangeable, as the invariant inner product on $\hat{\mathfrak{h}}_0^* := \{\xi \in \hat{\mathfrak{h}}^* : (\xi, K) = 0\} \subset \hat{\mathfrak{h}}^*$ is degenerate, with $\alpha_0 - \alpha_{\min}$ in its kernel.)

A representation of $\hat{\mathfrak{g}}$ is said to have *level k* if the central element $K \in \hat{\mathfrak{g}}$ acts by the scalar k. This can equivalently be formulated by saying that the set of weights of the representation is contained in the hyperplane $\hat{\mathfrak{h}}_k^* := \{\xi \in \hat{\mathfrak{h}}^* : (\xi, K) = k\} \subset \hat{\mathfrak{h}}^*$. The $\hat{\mathfrak{g}}$-modules we will be interested in are the *level k integrable positive energy representations*. Here, a $\hat{\mathfrak{g}}$-module V is called *integrable* if for each subalgebra $\mathfrak{sl}(2)^i := \mathrm{Span}\{E^i, F^i, H^i\} \subset \hat{\mathfrak{g}}$, $i \in \{0, \ldots, r\}$, V decomposes as a direct sum of finite dimensional $\mathfrak{sl}(2)^i$-reps.[27] And an integrable representation has *positive energy* if L_0 is diagonalizable, its spectrum is discrete and bounded below, and its eigenspaces are finite dimensional (let us abbreviate these conditions by 'L_0 has positive energy'). The integrable positive energy representations of $\hat{\mathfrak{g}}$ are the analogs of the finite dimensional representations of a finite dimensional simple Lie algebra. As in the finite dimensional case, such representations are classified by their highest weight (and for the representaiton to be level k, the highest weight λ must satisfy $(\lambda, K) = k$). And the set of weights of the simple module L_λ with highest weight λ is given by $P_\lambda \cap (\Lambda_{\text{root}} + \lambda)$.

[We had previously defined a representation of $\widetilde{L\mathfrak{g}}$ to have positive energy if it extends to a representation of $\widetilde{L\mathfrak{g}} \rtimes \widetilde{Vir}$ for which L_0 has positive energy[28]. Alternatively, one may define a positive energy of $\widetilde{L\mathfrak{g}}$ to be one that extends to a representation of $\hat{\mathfrak{g}} = \widetilde{L\mathfrak{g}} \rtimes \mathbb{C}L_0$ for which L_0 has positive energy. In the next section, we'll see that these two definitions are in fact equivalent, even thought the first one seems a priori much stronger.]

Recall that the affine Weyl group $\hat{W}$ is generated by the reflections $s_0, \ldots, s_r$ associated to the simple roots $\alpha_1, \ldots, \alpha_r$ plus the affine root α_0. The weight polytope P_λ (which is the convex hull of the $\hat{W}$-orbit of λ) is contained in the affine space

$$\hat{\mathfrak{h}}_{k,\mathbb{R}}^* := \mathrm{Span}_{\mathbb{R}}\{\alpha_0, \ldots, \alpha_r\} + \lambda.$$

Let us analyse the action of the affine Weyl group on that space. First note that $\hat{\mathfrak{h}}_{k,\mathbb{R}}^*$ is canonically isomorphic to $\mathfrak{h}_{\mathbb{R}}^* \oplus \mathbb{R}$. The subgroup W generated by $s_1, \ldots, s_r$ acts in the usual way on $\mathfrak{h}_{\mathbb{R}}^*$ and trivially on $\mathbb{R}$. The extra reflection $s_0 : \xi \mapsto \xi - \frac{2\langle \xi, \alpha_0 \rangle}{\langle \alpha_0, \alpha_0 \rangle}\alpha_0 = \xi - \langle \xi, \alpha_0 \rangle \alpha_0$ is then characterised by the fact that it fixes the hyperplane

$$\begin{aligned} \ker(H_0) \cap \hat{\mathfrak{h}}_{k,\mathbb{R}}^* &= \{\xi : \langle \alpha_{\min}, \xi \rangle + k = 0\} \\ &= \{\xi : \langle \alpha_{\min}, \xi \rangle = -k\} \\ &= \{\xi : \langle \alpha_{\max}, \xi \rangle = k\}, \end{aligned}$$

here, we view H_0 as a linear functional on $\hat{\mathfrak{h}}^*$

[26]The case of affine $\mathfrak{su}(2)$ is exceptional as the two roots α_1 and $\alpha_{\min}$ form an angle of 180°, which does appear in (28). We solve this by inventing a new type of edge to denote that configuration, namely .

[27]Note that unitary $\hat{\mathfrak{g}}$-modules are automatically integrable.

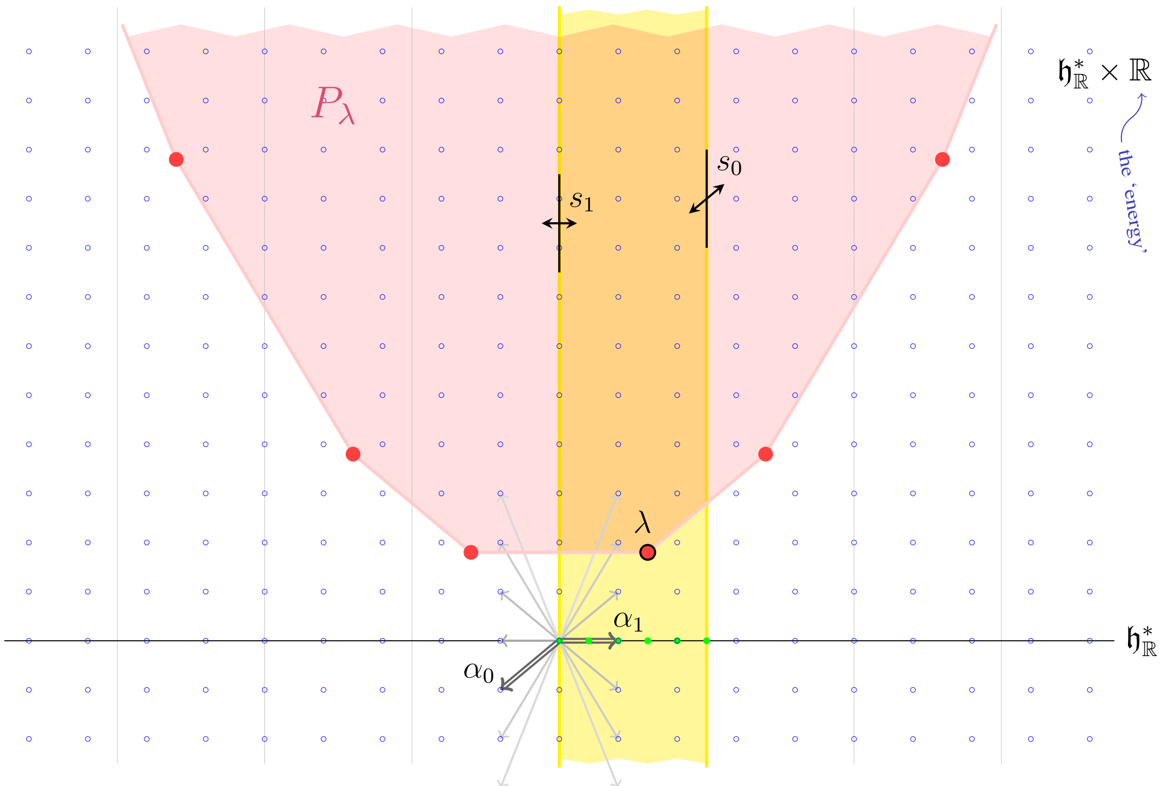
[28]For non-integrable representations, the positive energy condition should not insist that L_0 be diagonalizable. It should only require its generalised eigenspaces to be finite dimensional.

and displaces every vector by a multiple of α_0 (where $\alpha_{\max} = -\alpha_{\min}$ denotes the highest root of $\mathfrak{g}$). The fundamental domain for the action of $\hat{W}$ on $\hat{\mathfrak{h}}^*_{k,\mathbb{R}} \cong \mathfrak{h}^*_{\mathbb{R}} \oplus \mathbb{R}$ is delimited by the hyperplanes $\langle \xi, \alpha_i \rangle = 0$ and $\langle \xi, \alpha_{\max} \rangle = k$. It is isomorphic to $k\mathsf{A} \times \mathbb{R}$, where $k\mathsf{A}$ denotes the *Weyl alcove* of $\mathfrak{g}$ scaled by a factor of k:

Definition: *The Weyl alcove is the r-dimensional simplex $\mathsf{A} \subset \mathfrak{h}^*_{\mathbb{R}}$ bound by the walls of the Weyl chamber, and by the hyperplane $\langle \xi, \alpha_{\max} \rangle = 1$ (the hyperplane that bisects the segment $[0, \alpha_{\max}]$):*

$$\mathsf{A} := \big\{ \xi \in \mathfrak{h}^* : \langle \xi, \alpha_i \rangle \geq 0 \textit{ for } i \in \{1, \ldots, r\} \textit{ and } \langle \xi, \alpha_{\max} \rangle \leq 1 \big\}.$$

We illustrate all the above in the case of a representation of $\hat{\mathfrak{sl}}(2)$ at level $k = 5$:



The reflections s_1 and s_0 generate the affine Weyl group of $\mathfrak{sl}(2)$. Its fundamental domain on $\hat{\mathfrak{h}}^*_{k,\mathbb{R}}$ is the yellow strip $[0,k] \times \mathbb{R}$ (an instance of $k\mathsf{A} \times \mathbb{R}$). The $\hat{W}$-orbit of λ is in red. The tiny blue circles mark the root lattice (or rather, the image of the root lattice under the standard identification $\hat{\mathfrak{h}}^*_{0,\mathbb{R}} \cong \hat{\mathfrak{h}}^*_{k,\mathbb{R}}$). Some roots are drawn in gray. The green dots mark the set $k\mathsf{A} \cap \Lambda$.

It is important to remember that a positive energy representation of $\widetilde{L\mathfrak{g}}$ is not a representation of $\hat{\mathfrak{g}} = \widetilde{L\mathfrak{g}} \rtimes \mathbb{C}L_0$. It is a representation of $\widetilde{L\mathfrak{g}}$ *that extends* to a representation of $\widetilde{L\mathfrak{g}} \rtimes \mathbb{C}L_0$ but the extension is *not part of the data*. Moreover, such extensions are never unique: a representation of $\widetilde{L\mathfrak{g}} \rtimes \mathbb{C}L_0$ can always be modified by a character of

$\mathbb{C}L_0$ without changing the way in which $\widetilde{L\mathfrak{g}}$ acts. This operation has the effect of shifting all the weights of the representation by a certain (arbitrary) amount in the $\mathbb{C}$ direction of $\hat{\mathfrak{h}}_k^* \cong \mathfrak{h}^* \oplus \mathbb{C}$. The upshot is that level k integrable positive energy representations of $\widetilde{L\mathfrak{g}}$ are *not* classified by dominant weights $\lambda \in \hat{\mathfrak{h}}_k^*$. They are instead classified by their images under the projection $\hat{\mathfrak{h}}_k^* \twoheadrightarrow \mathfrak{h}^*$. The set of possible such projections is $k\mathsf{A} \cap \Lambda \subset \mathfrak{h}^*$, the intersection of the weight lattice with the scaled Weyl alcove:

Theorem. *The set of irreducible level k integrable positive energy representations of the affine Lie algebra $\widetilde{L\mathfrak{g}}$ is in canonical bijection with the finite set*

$$\mathsf{A}_k := k\mathsf{A} \cap \Lambda = \big\{\lambda \in \Lambda_+ : \langle \lambda, \alpha_{\max}\rangle \le k\big\}.$$

The correspondence sends a representation to the highest weight of its lowest energy subspace (L_0-eigenspace with lowest eigenvalue).

We record a couple of equivalent descriptions of the category of level k integrable positive energy representations of $\widetilde{L\mathfrak{g}}$:

$$\mathrm{Rep}^{\text{integrable}}_{\substack{pos\\en.}}\big(\widetilde{L\mathfrak{g}}_k\big) \;=\; \mathrm{Rep}^{\text{unitary}}_{\substack{pos\\en.}}\big(\widetilde{L\mathfrak{g}}_k\big) \;=\; \mathrm{Rep}_{\substack{pos\\en.}}\big(\widetilde{LG}_k\big)$$

(here G is the compact simply connected Lie group associated to $\mathfrak{g}$). We illustrate with some examples the set A_k of simple objects of that category:

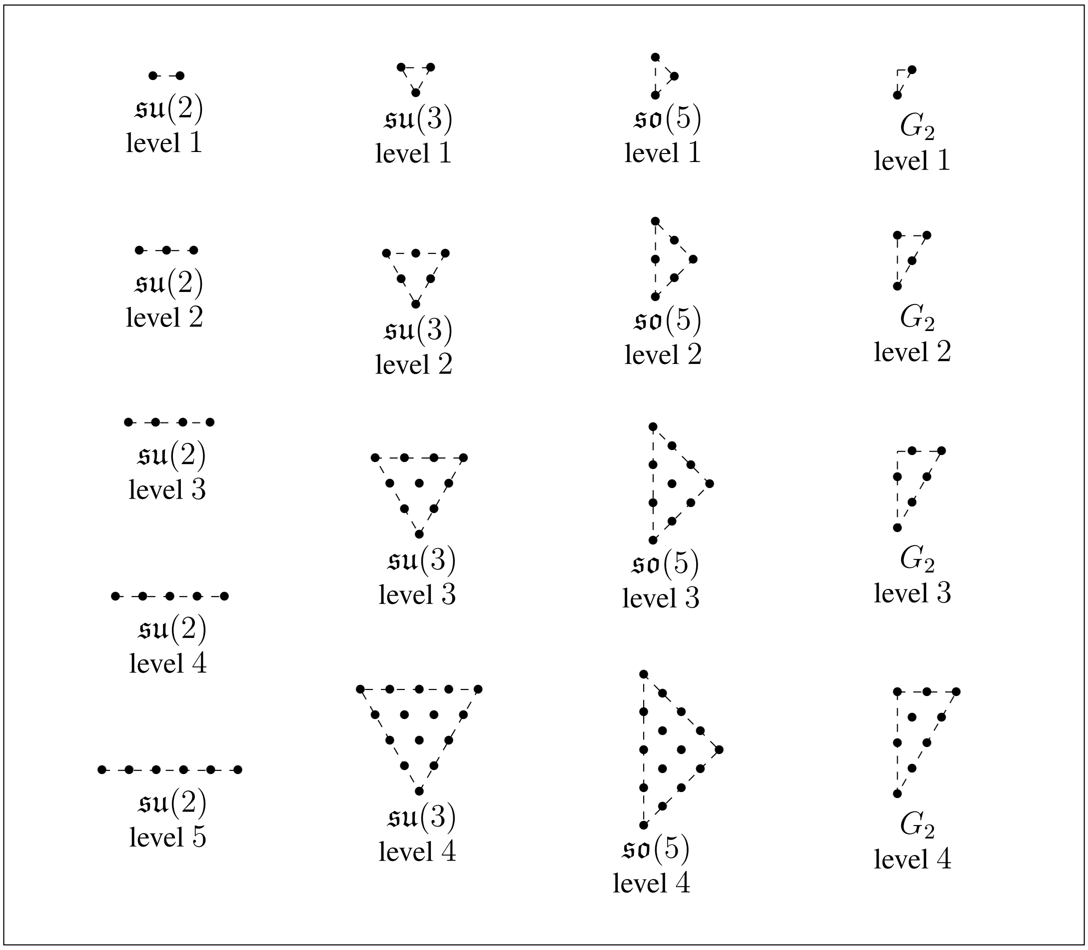


Table 2.

The Segal-Sugawara construction

Recall that a positive energy representation of the affine Lie algebra $\widetilde{L\mathfrak{g}}$ is one that extends to a representation of $\widetilde{L\mathfrak{g}} \rtimes \mathbb{C}L_0$ for which L_0 has positive energy. Let V be an irreducible such representation. Remarkably, the action of $\widetilde{L\mathfrak{g}}$ then automatically extends to the larger Lie algebra $\widetilde{L\mathfrak{g}} \rtimes \mathit{Vir}$ (and again L_0 has positive energy).

More precisely, if V has level k, then there exists a unique central charge $c \geq 0$, depending on k, such that the representation extends, uniquely, to a representation of $\widetilde{L\mathfrak{g}}_k \rtimes \mathit{Vir}_c$. The central charge is given by the formula $c = \frac{k\cdot\dim(\mathfrak{g})}{k+h^\vee}$, where $h^\vee$ denotes the *dual Coxeter number* of $\mathfrak{g}$, defined on the next page.

Warning: A positive energy representation of $\hat{\mathfrak{g}} = \widetilde{L\mathfrak{g}} \rtimes \mathbb{C}L_0$ usually *doesn't* extend to a representation of $\widetilde{L\mathfrak{g}} \rtimes \mathit{Vir}$: one typically needs to change the action of $\mathbb{C}L_0$ for it to extend to $\widetilde{L\mathfrak{g}} \rtimes \mathit{Vir}$.

We summarise the situation in the following diagram. All the arrows are given by restriction (and all the categories are semisimple):

in the definition of this category, we require that the simples remain simple when restricted to $\widetilde{L\mathfrak{g}}_k$

$$\begin{tikzcd}
\mathrm{Rep}^{\text{integrable}}_{\substack{pos\\en.}}\left(\widetilde{L\mathfrak{g}}_k \rtimes \mathit{Vir}_c\right) \arrow[d] \arrow[drr] & & \\
\mathrm{Rep}^{\text{integrable}}_{\substack{pos\\en.}}\left(\widetilde{L\mathfrak{g}}_k \rtimes \mathbb{C}L_0\right) \arrow[rr, two heads] & & \mathrm{Rep}^{\text{integrable}}_{\substack{pos\\en.}}\left(\widetilde{L\mathfrak{g}}_k\right)
\end{tikzcd}$$

simples: $\begin{cases} \mathsf{A}_k & \text{if } c = \frac{k\cdot\dim(\mathfrak{g})}{k+h^\vee} \\ \emptyset & \text{otherwise} \end{cases}$

this functor provides a canonical normalization of the minimal energy

that's an equivalence of categories when $c = \frac{k\cdot\dim(\mathfrak{g})}{k+h^\vee}$

simples: $\mathsf{A}_k \times \mathbb{C} \twoheadrightarrow \mathsf{A}_k$

here, the minimal energy is arbitrary

We recall that the commutation relations of $\widetilde{L\mathfrak{g}}_k \rtimes \mathit{Vir}_c$ are given by:

$$\begin{aligned}
[X_m, Y_n] &= [X,Y]_{m+n} + kn\langle X, Y\rangle \delta_{m+n,0} \\
[X_m, L_n] &= mX_{m+n} \\
[L_m, L_n] &= (m-n)L_{m+n} + \tfrac{c}{12}(m^3-m)\delta_{m+n,0}
\end{aligned} \tag{30}$$

since $z^{n+1}\frac{\partial}{\partial z}z^m = mz^{m+n}$ [29]

where, as before, X_n stands for $Xz^n \in L\mathfrak{g} \subset \widetilde{L\mathfrak{g}}_k$. Note that, as a consequence of the relation $[X_m, L_0] = mX_m$, the operator X_m lowers energy by m (where 'energy' is synonym of L_0-eigenvalue).

[29] This uses the Lie bracket (16) (the opposite of the Lie bracket of vector fields), which is well-adapted to *right* actions.

Before going on, we must explain what dual Coxeter number is. This invariant of the Lie algebra $\mathfrak{g}$ can be defined in a variety of ways. We present here a list, without any attempt at showing that these definitions are all equivalent:

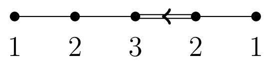
<u>Digression on the dual Coxeter number $h^\vee$</u>

- One standard way of defining the dual Coxeter number is by declaring $2h^\vee$ to be the ratio between the Killing form and the basic inner product. Here, $\langle X, Y\rangle_{\text{Killing}} := \mathrm{Tr}\big(\mathrm{ad}(X)\circ\mathrm{ad}(Y)\big)$, and the basic inner product is characterised by the fact that the long roots (all roots when $\mathfrak{g}$ is simply laced) have square-norm 2.

- Letting $r = \dim(\mathfrak{h})$ be the rank of $\mathfrak{g}$, the dual Coxeter number is also characterised by the formula
$$r{\cdot}h^\vee = \#\{\text{long roots}\} + \tfrac{1}{d}\,\#\{\text{short roots}\}.$$
Here, d is the lacity of $\mathfrak{g}$ (and all roots count as long when $\mathfrak{g}$ is simply laced).

- Here's a method for computing $h^\vee$. Take the affine Dynkin diagram associated to $\mathfrak{g}$. If the Lie algebra is not simply laced, reverse the direction of all the arrows. Call the result $D^\vee$. That's the so-called *dual affine Dynkin diagram*. A labelling of its nodes by positive integers is called *harmonic* if $\forall v \in D^\vee$ we have
$$2 \times \text{label of } v \;=\; \sum \text{labels of neighbours of } v$$
with the extra rule that 'big neighbours count with multiplicity'. Then $h^\vee$ is the sum of all the labels in the minimal harmonic labelling of $D^\vee$.

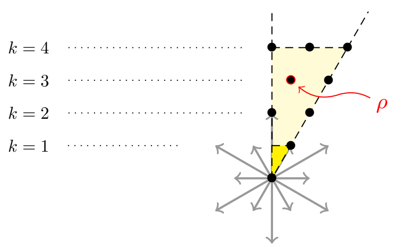
  We illustrate this method for the Lie algebra F_4. The minimal harmonic labelling of the dual affine Dynkin diagram looks as follows:



<u>check harmonic condition</u>:
$2\times 1 = 2$ ✓
$2\times 2 = 1+3$ ✓
$2\times 3 = 2+2\cdot 2$ ✓
$2\times 2 = 3+1$ ✓
$2\times 1 = 2$ ✓

giving us $h^\vee = 1+2+3+2+1 = 9$.

- Yet another way to characterize the dual Coxeter number is to say that $h^\vee$ is the smallest level k such that kA contains an element of the weight lattice in its interior. We illustrate this in the case of the Lie algebra G_2:



By the way, that element in the interior is called the *Weyl vector*. It's always denoted ρ, and is given by the formula $\rho = \frac{1}{2}\sum_{\alpha\in\Phi_+}\alpha$. (Here, $\Phi_+ := \Phi \cap \mathrm{Span}_{\mathbb{N}}\{\alpha_1,\dots,\alpha_r\}$ is the set of positive roots.)

Let us now go back to the main goal of this section, which is to prove the following theorem. Its proof will occupy the rest of the section.

Theorem 19 *For $V \in \mathrm{Rep}^{\text{integrable}}_{\substack{pos\\en.}}(\widetilde{L\mathfrak{g}}_k)$ irreducible, there exists a unique $c \geq 0$, and a unique extension of V to a representation of $\widetilde{L\mathfrak{g}}_k \rtimes Vir_c$. The Virasoro generators are given by the Segal-Sugawara formulas:*

$$\boxed{L_n := \frac{1}{2(k+h^\vee)} \sum_{X\in\mathcal{B}} \left(\sum_{m<0} X_m X_{n-m} + \sum_{m\geq 0} X_{n-m} X_m \right),}$$

where $\mathcal{B}$ is an orthonormal basis of $\mathfrak{g}$ with respect to the basic inner product, and $h^\vee$ is the dual Coxeter number. The central charge is given by

$$c = \frac{k \cdot \dim(\mathfrak{g})}{k + h^\vee}$$

and the minimal energy of V is

$$h = \frac{\|\lambda+\rho\|^2 - \|\rho\|^2}{2(k+h^\vee)},$$

where $\lambda \in \mathsf{A}_k \subset \mathfrak{h}^*$ is the highest weight of the lowest energy subspace of V.

Remark. *The Segal Sugawara formula involves an infinite sum. However, since X_m lowers energy by m, on any given vector $v \in V$ there's only finitely many terms which act in a non-zero way. So the expression in fact makes sense, despite the infinite sum.*

We first show that when (V, π) is an irreducible representation of $\widetilde{L\mathfrak{g}}_k$, then the action of the L_m and the value of the central charge c are uniquely determined, provided they exist. We prove this using the following two lemmas:

Lemma 20 *The equation $[\pi(X_m), \pi(L_n)] = m\pi(X_{m+n})$ uniquely determines $\pi(L_n)$ up to the addition of a scalar.*

Proof. Let $\pi(L_n)$ and $\pi'(L_n)$ be two solutions. Then $[\pi(X_m), \pi(L_n) - \pi'(L_n)] = 0$. So $\pi(L_n) - \pi'(L_n) : V \to V$ is a morphism of $\widetilde{L\mathfrak{g}}_k$-representations. By Schur's lemma, $\pi'(L_n) = \pi(L_n) + \text{cst}$. ☐

Lemma 21 *If the operators $\pi(L_n)$ satisfy then,* $[\pi(L_m), \pi(L_n)] = \pi([L_m, L_n]) + \text{cst}$.

Proof. $$[\pi(X_r), [\pi(L_m), \pi(L_n)]] = \underbrace{[\underbrace{[\pi(X_r), \pi(L_m)]}_{r\cdot\pi(X_{r+m})}, \pi(L_n)]}_{(r+m)r\cdot\pi(X_{r+m+n})} - \underbrace{[\underbrace{[\pi(X_r), \pi(L_n)]}_{r\cdot\pi(X_{r+n})}, \pi(L_m)]}_{(r+n)r\cdot\pi(X_{r+m+n})} =$$
$= (m-n)r\pi(X_{r+m+n})$. So $[\pi(L_m), \pi(L_n)]$ satisfies the same commutation relations as $\pi([L_m, L_n])$, and we're done by the previous lemma. ☐

If we make a random pick for the $\pi(L_m)$'s, subject to the relations $[\pi(X_m), \pi(L_n)] = m\pi(X_{m+n})$, then we won't quite get a representation of the Witt algebra. (The constants which appear in the last lemma are the 2-cocycle that measures the failure of π being a representation.) What we get instead is a representation of a central extension $\widetilde{\mathbb{W}} \to \mathbb{W}$. Since Vir is the universal central extension of $\mathbb{W}$, we get a unique Lie algebra homomorphism $Vir \to \widetilde{\mathbb{W}}$ that commutes with the projection to $\mathbb{W}$. The central charge is where the central element of Vir goes.

The Segal-Sugawara operators are affine Lie algebra analogs of the *quadratic Casimir*

$$C := \sum_{X \in \mathcal{B}} XX,$$

which lives in the universal enveloping algebra $U\mathfrak{g}$ of $\mathfrak{g}$. Before tackling Segal-Sugawara, we'll need to understand the action of C on irreps of $\mathfrak{g}$

Note that, in the definition of the quadratic Casimir, in place of the orthonormal basis $\mathcal{B}$, we could have used any pair of bases $\{X_i\}$ and $\{X^i\}$ satisfying $\langle X_i, X^j\rangle = \delta_i^j$. Bases like that are called *dual bases* of $\mathfrak{g}$:

Lemma 22 *Let $\{X_i\}$ and $\{X^i\}$ be dual bases of $\mathfrak{g}$. Then $C = \sum_i X_i X^i$.*

Proof. We claim that if $(\{X_i\},\{X^i\})$ and $(\{Y_j\},\{Y^j\})$ are two pairs of dual bases, then $\sum X_i X^i = \sum Y_j Y^j$. Let $A = (a_j^i)$ and $B = (b_i^j)$ be the matrices specified by $Y_j = \sum_i a_j^i X_i$ and $Y^j = \sum_i b_i^j X^i$. Then A and B are each other's inverses. If follows that

$$\sum_j Y_j Y^j = \sum_{ijk} X_i a_j^i b_k^j X^k = \sum_{ik} \delta_k^i X_i X^k = \sum_i X_i X^i. \qquad \square$$

The quadratic Casimir is independent of the choice of dual bases. It is therefore invariant under the action of $\mathrm{Aut}(\mathfrak{g})$ on $U\mathfrak{g}$. In particular, it is invariant under the adjoint action of G on $U\mathfrak{g}$, hence invariant under the adjoint action of $\mathfrak{g}$ on $U\mathfrak{g}$. This means:

$$\forall X \in \mathfrak{g} \;\; [X, C] = 0 \quad \text{in} \quad U\mathfrak{g}.$$

As $U\mathfrak{g}$ is generated by $\mathfrak{g}$, it follows that $C \in Z(U\mathfrak{g})$. Hence, by Schur's lemma, C acts as a scalar on any irrep of $\mathfrak{g}$. It will be important to compute those scalars.

To perform the computation, we make use the freedom provided by Lemma 22 to choose a convenient set of dual bases of $\mathfrak{g}$. Let $E^\alpha \in \mathfrak{g}^\alpha$ be root vectors, normalized so that E^α, $E^{-\alpha}$, $H^\alpha := [E^\alpha, E^{-\alpha}]$ satisfy the $\mathfrak{sl}(2)$-relations (25). The basic inner product pairs $\mathfrak{g}^\alpha$ with $\mathfrak{g}^{-\alpha}$ and $\mathfrak{h}$ with itself. By (27), we have

$$\begin{aligned}\langle E^\alpha, E^{-\alpha}\rangle &= \tfrac{1}{2}\langle [H^\alpha, E^\alpha], E^{-\alpha}\rangle \\ &= \tfrac{1}{2}\langle H^\alpha, [E^\alpha, E^{-\alpha}]\rangle \\ &= \tfrac{1}{2}\langle H^\alpha, H^\alpha\rangle = \begin{cases} \tfrac{1}{2}\langle \alpha, \alpha\rangle \;=\; 1 & \text{if } \alpha \text{ is a long root or } \mathfrak{g} \text{ is simply laced,} \\ \tfrac{1}{2}\langle d\alpha, d\alpha\rangle = d & \text{if } \alpha \text{ is a short root} \end{cases}\end{aligned}$$

where d is the lacity of $\mathfrak{g}$. We can thus write C as:

$$C \;=\; \sum_{\substack{\alpha\in\Phi,\\ \alpha\text{ long}}} E^\alpha E^{-\alpha} \;+\; \tfrac{1}{d} \sum_{\substack{\alpha\in\Phi,\\ \alpha\text{ short}}} E^\alpha E^{-\alpha} \;+\; \sum H_i H^i,$$

where $\{H_i\}$ and $\{H^i\}$ be dual bases of $\mathfrak{h}$ (and all roots count as long when $\mathfrak{g}$ is simply laced). Let us abbreviate this as

$$C \;=\; {\sum_{\alpha\in\Phi}}' E^\alpha E^{-\alpha} \;+\; \sum H_i H^i, \tag{31}$$

where the prime means 'add a factor of $\frac{1}{d}$ if the root is short'.

Proposition. *Let L_λ be the irreducible representation of $\mathfrak{g}$ of highest weights $\lambda \in \Lambda_+$. Then C acts on L_λ by the scalar*

$$\langle \lambda, \lambda + 2\rho\rangle = \|\lambda+\rho\|^2 - \|\rho\|^2.$$

Proof. We compute the scalar via its action on the highest weight vector $v \in L_\lambda$. Note that $\sum H_i H^i v$ is independent of the choice of dual bases of $\mathfrak{h}$. By picking $\{H_i\} = \{H^i\}$ an orthonormal basis, we readily see that $\sum H_i H^i v = \|\lambda\|^2 v$. Now, since $E^\alpha v = 0$ for all $\alpha \in \Phi_+$, half of the terms in the expression $\sum' E^\alpha E^{-\alpha} v$ are zero. This gives us:

$$\begin{aligned} Cv \;&=\; {\sum_{\alpha\in\Phi_+}}' E^\alpha E^{-\alpha} v \;+\; \sum H_i H^i v \\ &=\; {\sum_{\alpha\in\Phi_+}}' [E^\alpha, E^{-\alpha}] v \;+\; \sum H_i H^i v \\ &=\; \sum_{\alpha\in\Phi_+} \langle \lambda, \alpha\rangle v \;+\; \|\lambda\|^2 v \;=\; \big(\langle\lambda, 2\rho\rangle + \langle\lambda,\alpha\rangle\big) v, \end{aligned}$$

where the third equality follows from (27). □

Corollary: *Let $V \in \mathrm{Rep}^{\mathit{integrable}}_{\substack{pos\\ en.}}\big(\widetilde{L\mathfrak{g}}_k\big)$. Then the Segal-Sugawara operator L_0 acts on the lowest energy subspace of V by the scalar $\frac{\|\lambda+\rho\|^2-\|\rho\|^2}{2(k+h^\vee)}$.*

Proof. The operators X_m for $m > 0$ lower energy by m, and thus annihilate the lowest energy subspace of V. Therefore, on that subspace, L_0 acts like $\frac{1}{2(k+h^\vee)} \cdot C$. □

Let us go back to the main statements in Theorem 19. We begin by introducing the *unnormalised Segal-Sugawara operators*:

$$T_n := \sum_{X\in\mathcal{B}} \Bigg(\sum_{m<0} X_m X_{n-m} + \sum_{m\geq 0} X_{n-m} X_m \Bigg).$$

Equivalently:

$$\boxed{\begin{aligned} T_0 &= \sum_{X\in\mathcal{B}} \Big(X_0X_0 + 2\sum_{n>0} X_{-n}X_n\Big) \\ T_n &= \sum_{X\in\mathcal{B}} \sum_{m\in\mathbb{Z}} X_m X_{n-m} \qquad\qquad \text{for } n\neq 0 \end{aligned}}$$

Our next main goal is to prove that the Segal-Sugawara operators satisfy the second equation listed in (30). Equivalently:

Proposition 23 *The un-normalised Segal-Sugawara operators satisfy*

$$[X_m, T_n] = 2m(k+h^\vee)X_{m+n}.$$

Proof. **Step 1.** We first prove the $n=0$ case: $[X_m,T_0] = 2m(k+h^\vee)X_m$.
– *Case* $m=0$. This holds because T_0 is G-invariant.
– *Case* $m=1$ and $X\in\mathfrak{h}$. That's the heart of the proof. We'll do it later, in Lemma 24.
– *Case* $m=-1$ and $X\in\mathfrak{h}$, assuming the case $m=1$. Let's write H in place of X. Consider the involution $*:\mathfrak{g}\to\mathfrak{g}$ given by -1 on the Lie algebra $\mathfrak{g}_\mathbb{R}$ of the compact Lie group associated to $\mathfrak{g}$, and $+1$ on its orthogonal complement. Extend this to an involution $Y_m\mapsto (Y^*)_{-m}$ of $\widetilde{L\mathfrak{g}}$. Assuming w.l.o.g. that $H^*=H$, we have:

$$[H_{-1},T_0] = [H_1^*,T_0^*] = -\underbrace{[H_1,T_0]}_{=2(k+h^\vee)H_1 \text{ by assumption}}{}^* = -2(k+h^\vee)H_{-1}.$$

We now assemble the above three cases. As both $[-,T_0]$ and $X_m\mapsto 2m(k+h^\vee)X_m$ are derivations, and since they agree on a set of generators of $\widetilde{L\mathfrak{g}}_k$, they agree everywhere.

Step 2. Let us now assume that $n\neq 0$. Let $\mathbf{U}\big(\widetilde{L\mathfrak{g}}\big)$ denote the version of the universal algebra of $\widetilde{L\mathfrak{g}}$ where we allow infinite sums, as long as they become finite when evaluated on any vector of any positive energy representaiton. For example, $T_0 = \sum_{X\in\mathcal{B}}(X_0X_0 + 2\sum_{m>0}X_{-m}X_m)\in\mathbf{U}\big(\widetilde{L\mathfrak{g}}\big)$. The action $\ell_n\cdot X_m = -mX_{m+n}$ of the Witt algebra on $\widetilde{L\mathfrak{g}}$ extends to an action on $\mathbf{U}\big(\widetilde{L\mathfrak{g}}\big)$. For $n\neq 0$, we have

$$\begin{aligned} \ell_n\cdot T_0 &= 2\sum_{X\in\mathcal{B}}\sum_{m>0}\Big(\underbrace{\underbrace{(\ell_n\cdot X_{-m})}_{mX_{n-m}}X_m}_{mX_mX_{n-m}} + \underbrace{X_{-m}\underbrace{(\ell_n\cdot X_m)}_{-mX_{n+m}}}_{-mX_{-m}X_{n+m}}\Big) \\ &= 2\sum_{X\in\mathcal{B}}\sum_{m\in\mathbb{Z}} mX_mX_{n-m} \\ &= \sum_{X\in\mathcal{B}}\sum_{m\in\mathbb{Z}}\Big(mX_mX_{n-m} + \underbrace{mX_{n-m}X_m}_{(n-m')X_{m'}X_{n-m'}}\Big) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{X\in\mathcal{B}}\sum_{m\in\mathbb{Z}} nX_mX_{n-m} \\
&= nT_n
\end{aligned}
$$

It follows that

$$
\begin{aligned}
n[X_m, T_n] = [X_m, \ell_n\cdot T_0] &= [\underbrace{-\ell_n\cdot X_m}_{mX_{m+n}}, T_0] \;+\; \ell_n\cdot\underbrace{[X_m, T_0]}_{2m(k+h^\vee)X_m} \\
&= 2\big(m(m+n) - m^2\big)(k+h^\vee)X_{m+n} \\
&= 2mn(k+h^\vee)X_{m+n}
\end{aligned}
$$

Now divide by n to obtain the desired equation. $\square$

At last, the following lemma contains the heart of the proof of Proposition 23:

Lemma 24 *For $H\in\mathfrak{h}$, we have $[H_1, T_0] = 2(k+h^\vee)H_1$.*

Proof. Let us assume, without loss of generality, that $\|H\|^2 = 1$. It will be convenient to use the following way of writing T_0, analogous to the formula (31) for the Casimir:

$$
T_0 \;=\; {\sum_{\alpha\in\Phi}}' E_0^\alpha E_0^{-\alpha} + \sum_i H_0^iH_0^i \;+\; 2\sum_{n>0}\Big({\sum_{\alpha\in\Phi}}' E_{-n}^\alpha E_n^{-\alpha} + \sum_i H_{-n}^iH_n^i\Big).
$$

Here, $H^1,\ldots,H^r$ form an orthonormal basis of $\mathfrak{h}$, and the $\sum'$ notation is as in (31). Let us also assume, without loss of generality, that $H^1 = H$. We may now compute:

$$
\begin{aligned}
[H_1, T_0] = &{\sum_{\alpha\in\Phi}}'(\alpha,H)E_1^\alpha E_0^{-\alpha} - {\sum_{\alpha\in\Phi}}'(\alpha,H)E_0^\alpha E_1^{-\alpha} \\
&+ \underbrace{2\sum_{n>0}{\sum_{\alpha\in\Phi}}'(\alpha,H)E_{-n+1}^\alpha E_n^{-\alpha}}_{2\sum_{n\geq 0}\sum'_{\alpha\in\Phi}(\alpha,H)E_{-n}^\alpha E_{n+1}^{-\alpha}} - 2\sum_{n>0}{\sum_{\alpha\in\Phi}}'(\alpha,H)E_{-n}^\alpha E_{n+1}^{-\alpha} + 2[H_1, H_{-1}]H_1
\end{aligned}
$$

the terms for $n>0$ cancel out

$$
\begin{aligned}
&= \underbrace{{\sum_{\alpha\in\Phi}}'(\alpha,H)E_1^\alpha E_0^{-\alpha} + \underbrace{{\sum_{\alpha\in\Phi}}'(\alpha,H)E_0^\alpha E_1^{-\alpha}}_{-\sum'_{\alpha\in\Phi}(\alpha,H)E_0^{-\alpha}E_1^\alpha}}_{\sum'_{\alpha\in\Phi}(\alpha,H)[E_1^\alpha, E_0^{-\alpha}]} + 2kH_1 \\
&= {\sum_{\alpha\in\Phi}}'(\alpha,H)H_1^\alpha + 2kH_1 \\
&= \Big(\sum_{\alpha\in\Phi}(\alpha,H)^2 + 2k\Big)H_1
\end{aligned}
$$

By (27), this element of $\mathfrak{h}$ is equal to

$$\sum_{\alpha\in\Phi}(\alpha,H)\langle\alpha,-\rangle.$$

By symmetry reasons, it belongs to the line spanned by H. We may thus replace each $\langle\alpha,-\rangle$ by its projection $(\alpha,H)H$.

The quantity $\sum_{\alpha\in\Phi}(\alpha,H)^2$ is equal to $\mathrm{Tr}\big(\mathrm{ad}(H)^2\big) = \|H\|^2_{\text{Killing}} = \frac{\|H\|^2_{\text{Killing}}}{\|H\|^2} = 2h^\vee$. It follows that $[H_1, T_0] = 2(k+h^\vee)H_1$, as desired. $\square$

The results in Proposition 23 and Lemma 21 together imply that the Segal-Sugawara operators satisfy $[L_m, L_n] = (m-n)L_{m+n} + \mathrm{cst}$. It remains to determine the constant (hence the central charge of the Virasoro algebra). Note that when $m+n \neq 0$ the constant is zero for degree reasons, so we may focus on the case $m+n=0$:

Proposition 25 *The un-normalised Segal-Sugawara operators satisfy*

$$[T_n, T_{-n}] \;=\; 2(k+h^\vee)\Big(2nT_0 + k\dim(\mathfrak{g})\frac{n^3-n}{6}\Big).$$

Proof.

$$\begin{aligned}
[T_n, T_{-n}] &= \Big[\sum_{X\in\mathcal{B}}\Big(\sum_{m<0} X_m X_{n-m} + \sum_{m\geq 0} X_{n-m}X_m\Big), T_{-n}\Big] \\
&= 2(k+h^\vee)\sum_{X\in\mathcal{B}}\Big(\underbrace{\sum_{m<0} mX_{m-n}X_{n-m}}_{\sum_{m>n}(n-m)X_{-m}X_m} + \underbrace{\sum_{m<0}(n\cancel{-m})X_mX_{-m} + \sum_{m\geq 0}(n\cancel{-m})X_{-m}X_m}_{\boxed{=nT_0}} + \underbrace{\sum_{m\geq 0} mX_{n-m}X_{m-n}}_{\sum_{m\geq -n}(m+n)X_{-m}X_m}\Big) \\
&= 2(k+h^\vee)\Big(2nT_0 + \sum_{X\in\mathcal{B}}\underbrace{\Big(\underbrace{\sum_{m=-n}^{-1}(m+n)X_{-m}X_m}_{\sum_{m=1}^{n}(n-m)X_mX_{-m}} - \sum_{m=1}^{n}(n-m)X_{-m}X_m\Big)}_{\sum_{m=1}^{n}(n-m)\underbrace{[X_m,X_{-m}]}_{=k\cdot m}}\Big) \\
&= 2(k+h^\vee)\Big(2nT_0 + k\dim(\mathfrak{g})\underbrace{\sum_{m=1}^{n} m(n-m)}_{=\frac{n^3-n}{6}}\Big)
\end{aligned}$$

That's a cubic in n. To check that it's $\frac{n^3-n}{6}$, just evaluate it at four distinct values of n.

$\square$

Corollary: *The Segal-Sugawara operators $L_n = \frac{1}{2(k+h^\vee)}T_n$ satisfy*

$$[L_m, L_n] \;=\; (m-n)L_{m+n} + \tfrac{c}{12}(m^3-m)\delta_{m+n,0}$$

with $c = \frac{k\cdot\dim(\mathfrak{g})}{k+h^\vee}$.

This finishes the proof of Theorem 19. $\square$

Primary fields

Let $\Delta \in \mathbb{Z}$ be an integer.

Definition: *A primary field of conformal dimension Δ is a gadget φ that assigns to every complex cobordism Σ equipped with:*

- *points $z_1, \ldots, z_n \in \Sigma$, and*
- *tangent vectors $v_i \in T_{z_i}\Sigma$,*

and to every object $\lambda \in \mathcal{C}(\partial_{in}\Sigma)$, a linear map

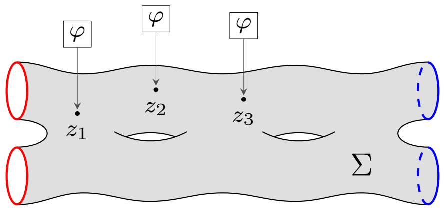
$$Z_{\Sigma,\varphi(z_1;v_1),\ldots,\varphi(z_n;v_n)} : U(\lambda) \to U(F_\Sigma(\lambda)).$$

These maps are homogeneous of degree Δ in the v_i's:

$$Z_{\Sigma,\varphi(z_1;v_1),\ldots,\varphi(z_i;av_i),\ldots,\varphi(z_n;v_n)} = a^\Delta Z_{\Sigma,\varphi(z_1;v_1),\ldots,\varphi(z_n;v_n)} \qquad \forall a \in \mathbb{C}, \tag{32}$$

and agree with Z_Σ when $n = 0$. Moreover, they satisfy the same axioms that the Z_Σ satisfy (naturality in λ and in Σ, compatibility with disjoint union, and with composition of cobordisms).

The map $Z_{\Sigma,\varphi(z_1;v_1),\ldots,\varphi(z_n;v_n)}$ is called a *propagator with field insertions*:



$$Z_{\Sigma,\varphi(z_1;v_1),\ldots,\varphi(z_n;v_n)} : U(\lambda) \longrightarrow U(F_\Sigma(\lambda))$$

Example: The *vacuum field* Ω

$$Z_{\Sigma,\Omega(z_1;v_1),\ldots,\Omega(z_n;v_n)} := Z_\Sigma$$

is a primary field of conformal dimension zero.

We will often suppress the vectors v_i from the notation, and write $Z_{\Sigma,\varphi(z_1),\ldots,\varphi(z_n)}$ instead of $Z_{\Sigma,\varphi(z_1;v_1),\ldots,\varphi(z_n;v_n)}$.

$\boxed{1 \in \mathbb{C} = U(1_\emptyset)}$

When Σ is a closed surface and $\lambda = 1_\emptyset$, then $Z_{\Sigma,\varphi(z_1),\ldots,\varphi(z_n)}(1)$ is called a *correlator*, and denoted

$$\big\langle \varphi(z_1), \ldots, \varphi(z_n) \big\rangle_\Sigma \in H_\Sigma := U(F_\Sigma(1_\emptyset)).$$

A linear functional $\mathcal{B} : H_\Sigma \to \mathbb{C}$ is called a *conformal block*[30], and we write

$$\big\langle \varphi(z_1), \ldots, \varphi(z_n) \big\rangle_{\Sigma,\mathcal{B}} \in \mathbb{C} \tag{33}$$

for the image of $\langle \varphi(z_1), \ldots, \varphi(z_n) \rangle_\Sigma$ under $\mathcal{B}$. When thought of as a function of the z_i's, the expression (33) is also called a *correlation function*.

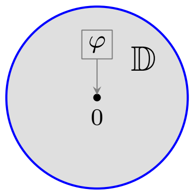
[30]Some people would call the expression (33) (viewed as a function of the z_i) the 'conformal block'.

Theorem. (State-field correspondence) *There is a natural bijection*

$$\left\{\begin{array}{c}\text{Primary fields of}\\ \text{conformal dimension } \Delta\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{States } \xi\in H_0 \text{ such that}\\ L_0(\xi)=\Delta\xi \text{ and } L_n(\xi)=0\,\forall n>0\end{array}\right\}.$$

Proof. Given a field φ, the corresponding state $\xi \in H_0$ is given by

$$\xi :=$$



$$= Z_{\mathbb{D},\varphi(0;\underset{1\in T_0\mathbb{D}=\mathbb{C}}{1})}(\overset{1\in\mathbb{C}=U(1_\emptyset)}{1}) \in H_0.$$

We need to show that

$$L_0(\xi) = \Delta\xi \qquad \text{and} \qquad L_n(\xi) = 0\,\forall n > 0. \tag{34}$$

Let $\mathrm{Univ}_0(\mathbb{D}) := \{f \in \mathrm{Univ}(\mathbb{D}) \,|\, f(0) = 0\}$ be the semigroup associated to the Lie algebra $Vir_{\geq 0} := \mathrm{Span}\{L_n\}_{n\geq 0}$. The conditions (34) are equivalent to

$$Z_A\xi = f'(0)^\Delta\xi \qquad \forall f \in \mathrm{Univ}_0(\mathbb{D}),\ A = \mathbb{D}\setminus f(\mathring{\mathbb{D}}).$$

We can then compute:

$$Z_A\xi = Z_A Z_{\mathbb{D},\varphi(0;1)}(1) = Z_{A\cup\mathbb{D},\varphi(0;1)}(1) = Z_{\mathbb{D},\varphi(0;f'(0))}(1) = f'(0)^\Delta\xi.$$

Conversely, starting from a vector $\xi \in H_0$ that satisfies the equations (34), we proceed as follows. Given a complex cobordism Σ together with points $z_1, \ldots, z_n$ and tangent vectors $v_i \in T_{z_i}\Sigma$, choose disjoint embeddings $f_i : \mathbb{D} \to \Sigma$, $f_i(0) = z_i$, and let

$$\Sigma^0 := \Sigma \setminus (f_1(\mathring{\mathbb{D}}) \sqcup \ldots \sqcup f_n(\mathring{\mathbb{D}})).$$

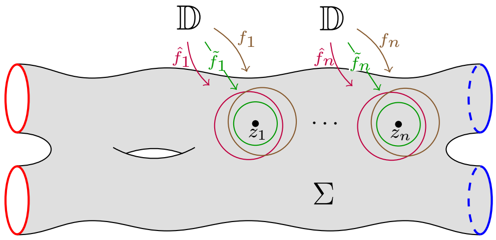
We then define

$$Z_{\Sigma,\varphi(z_1),\ldots,\varphi(z_n)} := \prod\Big(\frac{v_i}{f_i'(0)}\Big)^\Delta Z_{\Sigma^0}\big(\xi\otimes\ldots\otimes\xi\otimes -\big). \tag{35}$$

Let's check that this map lands in the right place:

$$\begin{aligned} Z_{\Sigma^0} : \underset{\underset{\xi}{\cup}}{H_0} \otimes \ldots \otimes \underset{\underset{\xi}{\cup}}{H_0} \otimes U(\lambda) &= U\big(F_{\mathbb{D}}(1)\otimes\ldots\otimes F_{\mathbb{D}}(1)\otimes\lambda\big)\\ &\longrightarrow U\big(F_{\Sigma^0}(F_{\mathbb{D}}(1)\otimes\ldots\otimes F_{\mathbb{D}}(1)\otimes\lambda)\big)\\ &= U\big(F_{\Sigma^0\cup(\mathbb{D}\sqcup\ldots\sqcup\mathbb{D})}(1\otimes\ldots\otimes1\otimes\lambda)\big) = U(F_\Sigma(\lambda)). \quad \checkmark \end{aligned}$$

We need to show that the map (35) is independent of the choice of f_i. Let $\hat{f} : \mathbb{D} \to \Sigma$, $\hat{f}_i(0) = z_i$, be another set of embeddings, and let $\hat{Z}_{\Sigma,\varphi(z_1),\ldots,\varphi(z_n)} : U(\lambda) \to U(F_\Sigma(\lambda))$ be the corresponding map, defined as in (35). We wish to show that $\hat{Z}_{\Sigma,\ldots} = Z_{\Sigma,\ldots}$. In order to do so, we introduce yet another set of embeddings $\tilde{f} : \mathbb{D} \to \Sigma$, $\tilde{f}_i(0) = z_i$, that satisfy $\tilde{f}_i(\mathbb{D}) \subset f_i(\mathbb{D}) \cap \hat{f}_i(\mathbb{D})$. Let $\tilde{Z}_{\Sigma,\ldots}$ be the corresponding map. We will show that $\hat{Z}_{\Sigma,\ldots} = \tilde{Z}_{\Sigma,\ldots} = Z_{\Sigma,\ldots}$.



(36)

Clearly, it's enough to show that $\tilde{Z}_{\Sigma,\ldots} = Z_{\Sigma,\ldots}$ (the other equality follows by symmetry). Let

$$\psi_i := f_i^{-1} \circ \tilde{f}_i \qquad \text{and let} \qquad A_i := \mathbb{D} \setminus \psi_i(\mathring{\mathbb{D}}) = f_i(\mathbb{D}) \setminus \tilde{f}_i(\mathring{\mathbb{D}})$$

be the corresponding annuli. We then have $Z_{A_i}(\xi) = \psi_i'(0)^\Delta \xi = \big(\frac{\tilde{f}_i'(0)}{f_i'(0)}\big)^\Delta \xi$, from which we get:

$$\begin{aligned}
\tilde{Z}_{\Sigma,\varphi(z_1),\ldots,\varphi(z_n)} &= \prod \Big(\frac{v_i}{\tilde{f}_i'(0)}\Big)^\Delta Z_{\tilde{\Sigma}^0}\big(\xi \otimes \ldots \otimes \xi \otimes -\big) \\
&= \prod \Big(\frac{v_i}{\tilde{f}_i'(0)}\Big)^\Delta Z_{\Sigma^0}\big(Z_{A_1}(\xi) \otimes \ldots \otimes Z_{A_n}(\xi) \otimes -\big) \\
&= \prod \Big(\frac{v_i}{\tilde{f}_i'(0)}\Big)^\Delta \prod \Big(\frac{\tilde{f}_i'(0)}{f_i'(0)}\Big)^\Delta Z_{\Sigma^0}\big(\xi \otimes \ldots \otimes \xi \otimes -\big) = Z_{\Sigma,\varphi(z_1),\ldots,\varphi(z_n)}.
\end{aligned} \tag{37}$$

□

Given a bunch of primary fields $\varphi_1, \ldots, \varphi_n$, with corresponding vectors $\xi_1, \ldots, \xi_n \in H_0$, it's now easy to adapt the definition (35):

$$Z_{\Sigma,\varphi_1(z_1;v_i),\ldots,\varphi_n(z_n;v_n)} := \prod \Big(\frac{v_i}{f_i'(0)}\Big)^{\Delta_i} Z_{\Sigma^0}\big(\xi_1 \otimes \ldots \otimes \xi_n \otimes -\big). \tag{38}$$

Here, as before, $\Sigma^0 = \Sigma \setminus (f_1(\mathring{\mathbb{D}}) \sqcup \ldots \sqcup f_n(\mathbb{D}))$ for some $f_i : \mathbb{D} \to \Sigma$ satisfying $f_i(0) = z_i$.

It is also fruitful to allow the ξ_i to take their values in other sectors than the vacuum sector. The corresponding fields are called *charged fields*. Given irreducible objects $\mu_i \in \mathcal{C}(S^1)$, and vectors $\xi_i \in U(\mu_i)$ satisfying the same conditions (34) as before, the definition (38) still makes sense, even thought it's no longer a map $U(\lambda) \to U(F_\Sigma(\lambda)$. Instead, it's a map:

$$Z_{\Sigma,\varphi_1(z_1),\ldots,\varphi_n(z_n)} \,:\, U(\lambda) \,\to\, U\big(F_{\Sigma,\mu_1(z_1),\ldots,\mu_n(z_n)}(\lambda)\big),$$

where $F_{\Sigma,\mu_1(z_1),\ldots,\mu_n(z_n)}(\lambda) := F_{\Sigma^0}(\mu_1 \otimes \ldots \otimes \mu_n \otimes \lambda)$.

As before, $Z_{\Sigma,\varphi_1(z_1),\ldots,\varphi_n(z_n)}$ depends on the choice of tangent vectors $v_i \in T_{z_i}\Sigma$. Similarly, the functor

$$F_{\Sigma,\mu_1(z_1),\ldots,\mu_n(z_n)} : \mathcal{C}(\partial_{in}\Sigma) \to \mathcal{C}(\partial_{out}\Sigma)$$

depends on some extra choices at the points z_i. But what it depends on is somewhat weaker than tangent vectors: the functor $F_{\Sigma,\ldots}$ only depends on rays $\rho_i \subset T_{z_i\Sigma}$ [a *ray* in a vector space V is an element of the quotient $(V \setminus \{0\})/\mathbb{R}_+]$. Also, when defining Σ^0, it was important to have used embeddings $f_i : \mathbb{D} \to \Sigma$ which satisfied $f_i'(0) \in \rho_i$.

Let us show that $F_{\Sigma,\mu_1(z_1),\ldots,\mu_n(z_n)}$ doesn't depend of the choice of f_i (up to canonical iso). Let $\hat{f}$ be another choice, and let $\hat{F}_{\Sigma,\ldots}$ be the corresponding functor. To compare $\hat{F}_{\Sigma,\ldots}$ and $F_{\Sigma,\ldots}$, we pick a third set of maps $\tilde{f}$ as in (36), and let $\tilde{F}_{\Sigma,\ldots}$ be the corresponding functor. It's enough to identify $\tilde{F}_{\Sigma,\ldots}$ with $F_{\Sigma,\ldots}$. Let $\psi_i = f_i^{-1} \circ \tilde{f}_i$ and $A_i = \mathbb{D} \setminus \psi_i(\mathring{\mathbb{D}})$. Since $\psi_i' \in \mathbb{R}_+$, the annulus A_i comes with a canonical lift $\tilde{A}_i$ to the universal cover of $\mathrm{Univ}_0(\mathbb{D})$. The desired identification is then given by:

$$\begin{array}{rl}
\tilde{F}_{\Sigma,\mu_1(z_1),\ldots,\mu_n(z_n)} & = F_{\tilde{\Sigma}^0}\big(\mu_1 \otimes \ldots \otimes \mu_n \otimes -\big) \\
& = F_{\Sigma^0}\big(F_{A_1}(\mu_1) \otimes \ldots \otimes F_{A_n}(\mu_n) \otimes -\big) \\
& \qquad\quad \downarrow T_{\tilde{A}_1} \qquad\qquad\quad \downarrow T_{\tilde{A}_n} \\
& \quad F_{\Sigma^0}\big(\quad \mu_1 \quad \otimes \quad \ldots \quad \otimes \quad \mu_n \quad \otimes \; - \big) \; = \; F_{\Sigma,\mu_1(z_1),\ldots,\mu_n(z_n)}.
\end{array}$$

Descendant fields

Our next goal is to generalize (38) to the case when the condition $Vir_{>0}\,\xi = 0$ is no longer satisfied. These are called *descendant fields*. More precisely, I'd call (the field associated to) $\xi \in H_0$ a *descendant* (of ξ_0) if $\xi = L_{m_1} \ldots L_{m_k}\xi_0$ for some $m_1, \ldots, m_k \le 0$, and some primary ξ_0. When dealing with descendant fields, we need to replace the vectors v_i by elements of the *jet space*:

Definition: Let Σ be a Riemann surface. The *jet space* of order d of Σ is given by:

$$J_z^d\Sigma := \left\{ j : \underset{0\ \mapsto\ z}{\mathbb{C} \dashrightarrow \Sigma} \;\middle|\; \begin{array}{c} j \text{ is holomorphic,} \\ j'(0) \neq 0 \end{array} \right\} \Big/ j_1 \sim j_2 \text{ if } j_1(z) = j_2(z) + o(z^d),$$

$$J^d\Sigma := \bigcup\nolimits_{z\in\Sigma} J_z^d\Sigma\,.$$

Here, the notation $j : \mathbb{C} \dashrightarrow \Sigma$ means that j is only defined in a neighbourhood of 0.

Consider the tower of Lie algebras

$$Vir_{[0]} \twoheadleftarrow Vir_{[0,1]} \twoheadleftarrow Vir_{[0,2]} \twoheadleftarrow Vir_{[0,3]} \twoheadleftarrow \ldots\ldots \qquad \leftarrow Vir_{\ge 0},$$

where $Vir_{[0,d-1]} := Vir_{\ge 0}/\,Vir_{\ge d} = \mathrm{Span}\{L_0, \ldots, L_{d-1}\}$. They integrate to a tower of Lie groups:

$$\mathbb{C}^\times = G_1 \twoheadleftarrow G_2 \twoheadleftarrow G_3 \twoheadleftarrow G_4 \twoheadleftarrow \ldots\ldots \qquad \leftarrow \mathrm{Univ}_0(\mathbb{D}),$$

where $G_d = J_0^d\mathbb{C}$, with group operation given by composition of functions. In other words:

$$G_d = \left\{ \begin{array}{l} \text{changes of coordinate} \\ \text{defined up to degree } d \end{array} \right\}.$$

One can also describe that group more algebraically, as $\mathrm{Aut}(\mathbb{C}[z]/z^{d+1})$.

Definition: A vector $\xi \in H_0$ is called a *finite energy vector* if it is a finite linear combination of eigenvectors of L_0.

We now generalize (38) to the case when ξ_i are arbitrary finite energy vectors. By the positive energy condition, since the L_n for $n > 0$ are lowering operators, the action of $Vir_{\geq 0}$ on ξ_i generates a finite dimensional subspace. Call it $V_i \subset H_0$. The action of $Vir_{\geq 0}$ on V_i factors through a finite quotient $Vir_{[0,d_i-1]}$, and integrates to an action of G_{d_i}. [A priori, one might expect the action to only integrate to an action of the universal cover $\widetilde{G}_{d_i}$ of G_{d_i}. But the subalgebra $Vir_{[0]} \subset Vir_{[0,d_i-1]}$ integrates to a $\mathbb{C}^\times$. So the action of $\widetilde{G}_{d_i}$ on V_i descends to G_{d_i}.]

Instead of (38), we can then write:

$$Z_{\Sigma,\varphi_1(z_1;j_i),\ldots,\varphi_n(z_n;j_n)} := Z_{\Sigma^0}\big(g_1\xi_1 \otimes \ldots \otimes g_n\xi_n \otimes -\big), \tag{39}$$

where $g_i := f_i^{-1} \circ j_i \in G_{d_i}$ and, as before, $\Sigma^0 = \Sigma \setminus (f_1(\mathring{\mathbb{D}}) \sqcup \ldots \sqcup f_n(\mathbb{D}))$. Once again, we abbreviate things by writing $Z_{\Sigma,\varphi_1(z_1),\ldots,\varphi_n(z_n)}$ instead of $Z_{\Sigma,\varphi_1(z_1;j_i),\ldots,\varphi_n(z_n;j_n)}$.

In order to check that the map (39) is well defined (independent of the f_i), we proceed along the same lines as the previous proof. The analog of (37) (the most relevant part of the computation) is given by:

$$\begin{aligned}
\tilde{Z}_{\Sigma,\varphi_1(z_1),\ldots,\varphi_n(z_n)} &= Z_{\tilde{\Sigma}^0}\big(\tilde{g}_1\xi_1 \otimes \ldots \otimes \tilde{g}_n\xi_n \otimes -\big)\\
&= Z_{\Sigma^0}\big(Z_{A_1}(\tilde{g}_1\xi_1) \otimes \ldots \otimes Z_{A_n}(\tilde{g}_n\xi_n) \otimes -\big)\\
&= Z_{\Sigma^0}\big(\psi_1\tilde{g}_1(\xi_1) \otimes \ldots \otimes \psi_n\tilde{g}_n(\xi_n) \otimes -\big)\\
&= Z_{\Sigma^0}\big(g_1\xi_1 \otimes \ldots \otimes g_n\xi_n \otimes -\big)\\
&= Z_{\Sigma,\varphi_1(z_1),\ldots,\varphi_n(z_n)},
\end{aligned}$$

where $\psi_i = f_i^{-1} \circ \tilde{f}_i$ and $A_i = f_i(\mathbb{D}) \setminus \tilde{f}_i(\mathring{\mathbb{D}})$.

Lemma 26 *Let $\xi \in H_0$ be a finite energy vector, with corresponding field φ. Let $g \in G_d$ be a group element, and let $g\varphi$ be the field that corresponds to $g\xi$. Then we have:*

$$\varphi(z; j \circ g) = (g\varphi)(z; j).$$

Proof. $Z_{\Sigma,\varphi(z;j\circ g),\ldots} = Z_{\Sigma^0}((f^{-1}jg)\xi \otimes \ldots) = Z_{\Sigma^0}((f^{-1}j)(g\xi) \otimes \ldots) = Z_{\Sigma,(g\varphi)(z;j),\ldots}$ where, as before, $\Sigma^0 = \Sigma \setminus (f(\mathring{\mathbb{D}}) \sqcup \ldots)$ for some embeddings $f : \underset{0\,\mapsto\, z}{\mathbb{D} \to \Sigma}, \ldots$ □

When ξ is an eigenvector of L_0, one can also describe these more general types of fields axiomatically. In the definition of primary field, just replace the tangent vector v_i by a local coordinate $j_i : \underset{0\,\mapsto\, z_i}{\mathbb{C} \dashrightarrow \Sigma}$, and require the equation $\varphi(z; j \circ (z \mapsto az)) = a^\Delta \varphi(z; j)$ to hold:

$$Z_{\Sigma,\varphi(z_1;j_1),\ldots,\varphi(z_i;j_i\circ(z\mapsto az)),\ldots,\varphi(z_n;j_n)} = a^\Delta Z_{\Sigma,\varphi(z_1;j_1),\ldots,\varphi(z_n;j_n)} \qquad \forall a \in \mathbb{C}^\times.$$

We call such a thing a *field of conformal dimension* Δ. Similarly to the case of primary fields, we then have:

Theorem. (State-field correspondence) *There is a natural bijection:*

$$\Big\{\text{Fields of conformal dimension } \Delta\Big\} \longleftrightarrow \Big\{\xi \in H_0 \,\Big|\, L_0(\xi) = \Delta\xi\Big\}.$$

The proof goes along the same lines as the one in the previous section.

One of the defining properties of chiral CFT is that the fields $\varphi(z)$ 'depend holomorphically on z'. We formalize this in the following proposition:

Proposition. *Let φ be a field. Then the map*

$$\begin{array}{ccc} J^d\big(\Sigma \setminus \{\ldots\}\big) & \longrightarrow & \mathrm{Hom}\big(U(\lambda), U(F_\Sigma(\lambda))\big) \\ j \in J^d_z\Sigma & \mapsto & Z_{\Sigma,\varphi(z;j),\ldots} \end{array} \tag{40}$$

is holomorphic. (Here, the "$\setminus\{\ldots\}$" refers to the finitely many points where the other field insertions take place.)

Proof. Holomorphicity is a local condition. For every disc $D \subset \Sigma \setminus \{\ldots\}$, we have $Z_{\Sigma,\varphi(z;j),\ldots} = Z_{\Sigma\setminus D,\ldots} \circ Z_{D,\varphi(z;j)}$. So it's enough to prove the statement when Σ is a disc.

When $\Sigma = \mathbb{D}$, we can identify a jet $j \in J^d_z(\mathbb{D})$ with an element $g \in G_d$. By Lemma 26, for every $r < 1$, and every z of norm at most $1 - r$, we have

$$Z_{\mathbb{D},\varphi(z;j)} = Z_{\mathbb{D},(g\varphi)(z;w\mapsto w+z)} = Z_{\mathbb{D}\setminus(r\mathbb{D}+z)} \circ Z_{r\mathbb{D},(g\varphi)(0;\mathrm{id})}.$$

$Z_{\mathbb{D}\setminus(r\mathbb{D}+z)}$ depends holomorphically on z because $\mathbb{D} \setminus (r\mathbb{D} + z)$ does, and $Z_{r\mathbb{D},(g\varphi)(0;\mathrm{id})}$ depends holomorphically on j because $g\varphi$ does. So $Z_{\mathbb{D},\varphi(z;j)}$ depends holomorphically on z and on j. $\square$

Examples of fields

- If our conformal field theory is a chiral WZW model, or anything that admits affine Lie algebra symmetries, then, for every $X \in \mathfrak{g}$, we can consider the element $X_{-1}\Omega \in H_0$. The associated field is called a current and is denoted $J^X(z)$. This is a primary field of conformal dimension one:

$$L_1 X_{-1}\Omega \;=\; X_{-1}\underbrace{L_1\Omega}_{=\,0 \text{ because that's in degree } -1} + \underbrace{X_0\Omega}_{=\,0 \text{ because } \Omega \text{ is } \mathfrak{g}\text{-invariant}} \;=\; 0.$$

- If our conformal field theory is a chiral minimal model, or anything that contains Virasoro algebra symmetries (which is to say... any chiral CFT), then we can consider the element $\omega := L_{-2}\Omega \in H_0$, the so-called "conformal vector". The associated field is called the stress-energy tensor and is denoted $T(z)$. This field is also called the 'energy-momentum tensor' or 'Virasoro field', and it is *not* primary, unless $c = 0$:

$$
\begin{aligned}
L_1\omega &= L_1L_{-2}\Omega = 3\underbrace{L_{-1}\Omega} = 0\\
L_2\omega &= L_2L_{-2}\Omega = 4\underbrace{L_0\Omega}_{=0 \text{ because } \Omega \text{ is } PSU(1,1)\text{-invariant}} + \tfrac{c}{12}\cdot 6\cdot\Omega = \tfrac{c}{2}\cdot\Omega
\end{aligned}
$$

The action of $Vir_{\geq 0}$ on ω generates a two dimensional subspace $\mathrm{Span}\{\Omega,\omega\}\subset H_0$, on which the Lie algebra $Vir_{[0,2]}$ acts by

n	0	1	2
$L_n\Omega$	0	0	0
$L_n\omega$	2ω	0	$\frac{c}{2}\Omega$

(41)

Claim: At the Lie group level, this integrates to the action of $G_3 = \mathrm{Aut}(\mathbb{C}[z]/z^4)$ given by:

$$
\begin{cases}
g\cdot\Omega = \Omega\\
g\cdot\omega = g'(0)^2\omega + \frac{c}{12}\left(\frac{g'''(0)}{g'(0)} - \frac{3}{2}\left(\frac{g''(0)}{g'(0)}\right)^2\right)\Omega.
\end{cases}
\tag{42}
$$

In terms of the basis $\{\Omega,\omega\}$, that representation can be equivalently described as:

$$
g \mapsto \begin{pmatrix} 1 & \frac{c}{12}\left(\frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2\right)\\ 0 & g'^2\end{pmatrix}_{z=0}
$$

Here, the expression is called the *Schwarzian derivative*, and is often denoted by the symbol $\{g,z\}$.

Now, if someone hands you a representation of a Lie group sand says *"that's the representation which integrates the following Lie algebra rep"*, how do you check it? One way to proceed is as follows:

(1) You compute the formula to first order for an element close to the identity, and check that it agrees with the given Lie algebra representation. Computing modulo ε^2, we get:

$$
\begin{aligned}
&\text{If } g(z) = z+\varepsilon z, \text{ then } \{g,z\}=0 &&\rightsquigarrow && g\mapsto \begin{pmatrix}1 & 0\\ 0 & 1+2\varepsilon\end{pmatrix}. && \checkmark\\
&\text{If } g(z) = z+\varepsilon z^2, \text{ then } \{g,z\}=0 &&\rightsquigarrow && g\mapsto \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}. && \checkmark\\
{}^{31}&\text{If } g(z) = z+\varepsilon z^3, \text{ then } \{g,z\}=6\varepsilon &&\rightsquigarrow && g\mapsto \begin{pmatrix}1 & \frac{c}{2}\varepsilon\\ 0 & 1\end{pmatrix}. && \checkmark
\end{aligned}
$$

(2) You check that it's indeed a representation:

$$
\begin{pmatrix} 1 & \frac{c}{12}\left(\frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2\right)\\ 0 & g'^2\end{pmatrix}_{z=0}
\begin{pmatrix} 1 & \frac{c}{12}\left(\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2\right)\\ 0 & f'^2\end{pmatrix}_{z=0}
\overset{?}{=}
\begin{pmatrix} 1 & \frac{c}{12}\left(\frac{(g\circ f)'''}{(g\circ f)'} - \frac{3}{2}\left(\frac{(g\circ f)''}{(g\circ f)'}\right)^2\right)\\ 0 & (g\circ f)'^2\end{pmatrix}_{z=0}
$$

[31] We write $z+\varepsilon z^3$ as opposed to $z-\varepsilon z^3$ for the infinitesimal transformation corresponding to L_2 because of the issue pointed out in the remark on page 30.

This looks like an annoying computation, but it's actually not too bad. To begin with, we compute the first, second, and third derivatives of $g \circ f$ at zero:

$$\begin{aligned}(g \circ f)' &=_{z=0} g'f' \\ (g \circ f)'' &=_{z=0} g''f'^2 + g'f'' \\ (g \circ f)''' &=_{z=0} g'''f'^3 + 3g''f'f'' + g'f'''.\end{aligned}$$

Ignoring the $\frac{c}{12}$, the upper right corners of the two sides of the above equation are given by:

$$\frac{f'''}{f'} - \tfrac{3}{2}\Big(\frac{f''}{f'}\Big)^2 + \frac{g'''f'^2}{g'} - \tfrac{3}{2}\Big(\frac{g''f'}{g'}\Big)^2$$

and

$$\begin{aligned}&\frac{g'''f'^3 + 3g''f'f'' + g'f'''}{g'f'} - \tfrac{3}{2}\bigg(\frac{g''f'^2 + g'f''}{g'f'}\bigg)^2 \\ &= \frac{g'''f'^2}{g'} + \frac{3g''f''}{g'} + \frac{f'''}{f'} - \tfrac{3}{2}\Big(\frac{g''f'}{g'}\Big)^2 - 3\frac{g''f''}{g'} - \tfrac{3}{2}\Big(\frac{f''}{f'}\Big)^2.\end{aligned}$$

$\parallel$ ✓

Those two expressions are indeed equal.

This finishes the proof that the action (41) of $Vir_{[0,2]}$ integrates to the action (42) of the group G_3.

As an upshot of the above computation, we get the following special case of Lemma 26, known as the *'anomalous transformation of the stress-energy tensor'*: For $f : \mathbb{C} \dashrightarrow \mathbb{C}$, $f(0) = 0$, we have

$$T\big(w; j \circ f\big) \;=\; f'(0)^2 T\big(w; j\big) + \tfrac{c}{12}\{f, z\}_{z=0}\,\Omega.$$

For $w \in \mathbb{C}$, this becomes:

$$T\big(w; z \mapsto f(z)\big) \;=\; f'(0)^2 T\big(w; z \mapsto z + w\big) + \tfrac{c}{12}\{f, z\}_{z=0}\,\Omega.$$

In physics lingo, this is usually expressed in the following terms: "Under the map $z \to f(z)$, the stress-energy tensor transforms as $T(z) \to (\partial f)^2 T(f(z)) + \frac{c}{12}\{f, z\}$."

Remark 27 *The Schwarzian derivative also appears in the formulas which describe the action of $\varphi \in \mathrm{Diff}(S^1)$ on the universal central extensions of $\mathfrak{X}_{\mathbb{C}}(S^1)$. Namely, for $(f\frac{\partial}{\partial z}, a) \in {}^{\mathbb{C}}\mathfrak{X}_{\mathbb{C}}(S^1)$, we have $\varphi^*(f\frac{\partial}{\partial z}, a) = (\frac{f \circ \varphi}{\varphi'}\frac{\partial}{\partial z}, a + \frac{1}{12}\int_{S^1}\frac{f \circ \varphi(z)}{\varphi'(z)}\{\varphi, z\}\frac{dz}{2\pi i})$.*

Because the Schwarzian derivative vanishes on all infinitesimal coordinate transformations of the form $z \to z + \varepsilon$, $z \to z + \varepsilon z$ and $z \to z + \varepsilon z^2$, it vanishes on the group generated by them. Namely, on the group $PSL(2, \mathbb{C})$ of fractional linear transformations.

***Slogan*:** The Schwarzian derivative $\{f, z\}$ is a version of the third derivative which measures the failure of f being a fractional linear transformation (just like f''' measures the failure of f being a quadratic polynomial).

Let $\varphi(z)$ be a field of conformal dimension Δ. One way to say that $\varphi(z)$ is primary is to say that it satisfies

$$\varphi(z; j\circ f) = f'(0)^{\Delta}\varphi(z;j) \qquad \forall f : \mathbb{C} \underset{0\,\mapsto\,0}{\dashrightarrow} \mathbb{C}.$$

That's just a complicated way of saying that $\varphi(z)$ only depends on the vector $j'(0) \in T_z\Sigma$. An arbitrary field $\varphi(z)$ of conformal dimension Δ only satisfies

$$\varphi(z; j\circ(z\mapsto az)) = a^{\Delta}\varphi(z;j) \qquad \forall a \in \mathbb{C}^{\times}.$$

There's also an intermediate condition which is useful:

Definition: A field $\varphi(z)$ of conformal dimension Δ is called *quasi-primary* if

$$\varphi(z; j\circ f) = f'(0)^{\Delta}\varphi(z;j) \qquad \forall f \in PSL(2,\mathbb{C}),\ f(0)=0.$$

The stress-energy tensor $T(z)$ is a quasi-primary field of conformal dimension 2.

The state-field correspondence for quasi-primary fields reads as follows:

Theorem. (State-field correspondence) *There is a natural bijection*

$$\left\{\begin{array}{c}\text{Quasi-primary fields}\\ \text{of conformal dimension } \Delta\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{States } \xi\in H_0 \text{ s.t.}\\ L_0(\xi)=\Delta\xi \text{ and } L_1(\xi)=0\end{array}\right\}.$$

The Segal commutation relations

Let $\xi \in H_0$ be a finite energy vector, let φ be the associated field, let $V \subset H_0$ be the $Vir_{\geq 0}$-module generated by ξ, and let $d \in \mathbb{N}$ be such that the action of $Vir_{\geq 0}$ on V descends to an action of $Vir_{[0,d-1]}$ and hence to an action of the group G_d. Let Σ be a complex cobordism and let $\lambda \in \mathcal{C}(\partial_{in}\Sigma)$. By Lemma 26, we have a commutative diagram:

$$\begin{tikzcd}
J^d\mathring{\Sigma} \arrow[rr, "j\ \mapsto\ Z_{\Sigma,\varphi(j(0);j)}"] \arrow[d, "j \mapsto {[(j,\xi)]}"'] & & \mathrm{Hom}\Big(U(\lambda), U(F_\Sigma(\lambda))\Big) \\
V_{\mathring{\Sigma}} := J^d\mathring{\Sigma} \underset{G_d}{\times} V \arrow[urr, dashed, "\Phi_{\mathring{\Sigma}}"'] & &
\end{tikzcd} \tag{43}$$

The dotted map is given by $\Phi_{\mathring{\Sigma}} : [(j,\eta)] \mapsto Z_{\Sigma,\varphi_\eta(j(0);j)}$, where φ_η is the field associated to η. The map $\Phi_{\mathring{\Sigma}}$ is holomorphic, and linear on the fibers of the vector bundle $V_{\mathring{\Sigma}} \to \mathring{\Sigma}$.

Our next goal is to generalize the above picture to allow the insertion point $z = j(0)$ to be on the boundary $\partial\Sigma$. Assuming Σ is equipped with *collars* (see the picture (46) below), we'll upgrade (43) to a map

$$\Phi_\Sigma : V_\Sigma := J^d\Sigma \underset{G_d}{\times} V \to \mathrm{Hom}\Big(\widecheck{U(\lambda)}, U\big(\widehat{F_\Sigma(\lambda)}\big)\Big) \tag{44}$$

Here, the cech on $U(\lambda)$ means that we make the space a little bit 'thinner', in a way that we'll explain below, and the hat on $U(F_\Sigma(\lambda))$ means that we make the space a bit 'fatter'.

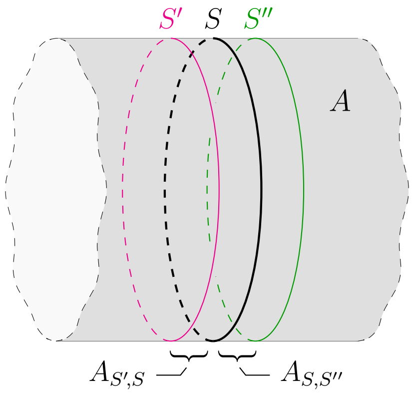
Remark: If φ is primary of conformal dimension Δ, then the vector space V is one dimensional, and $V_\Sigma = T^{\otimes\Delta}\Sigma$.

Let S be a circle. A <u>*collar*</u> is a piece of Riemann surface A in which S is analytically embedded. Two collars $S \hookrightarrow A$ and $S \hookrightarrow B$ are equivalent if there exist open subsets $U \subset A$ and $V \subset B$ containing S and an isomorphism $U \cong V$ that restricts to the identity on S (by analytic continuation, such an isomorphism is unique provided it exists.) An equivalence class of collars is the same thing as an *analytic structure* on S, i.e., a subsheaf $\mathcal{O}_S^{an} \subset \mathcal{O}_S$ of the sheaf of smooth functions on S which is locally isomorphic to the sheaf of analytic functions on $\mathbb{R}$.

Let $S \hookrightarrow A$ be a collar. Given two circles $S_1, S_2 \subset A$ that are isotopic to S and such that S_2 is 'in the future' of S_1, we write A_{S_1,S_2} for the part of A which lies between S_1 and S_2 (A_{S_1,S_2} is a thick annulus). For convenience, we abbreviate $F_{A_{S_1,S_2}}$ by F_{S_1,S_2} and $Z_{A_{S_1,S_2}}$ by Z_{S_1,S_2}. For any object $\lambda \in \mathcal{C}(S)$, we can then define

$$\check{H}_\lambda := \varinjlim_{\substack{S' \text{ in the}\\ \text{past of } S}} U\big(F^{-1}_{S',S}(\lambda)\big) \qquad \text{and} \qquad \widehat{H}_\lambda := \varprojlim_{\substack{S'' \text{ in the}\\ \text{future of } S}} U\big(F_{S,S''}(\lambda)\big)$$

where the maps used to define the limits are given by $Z_{S'_1,S'_2} : U(F^{-1}_{S'_1,S}(\lambda)) \to U(F^{-1}_{S'_2,S}(\lambda))$ and $Z_{S''_1,S''_2} : U(F_{S,S''_1}(\lambda)) \to U(F_{S,S''_2}(\lambda))$, respectively.



We then have dense inclusions $\check{H}_\lambda \subset H_\lambda \subset \widehat{H}_\lambda$. [32]

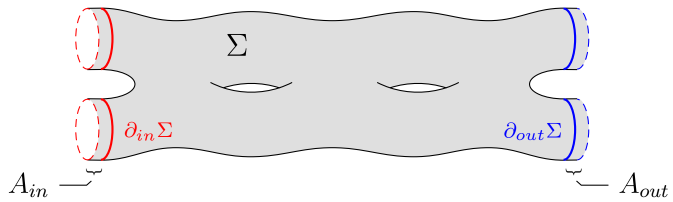
The advantage of working with the $\smallsmile$ and $\frown$ versions is that the map

$$Z_{\Sigma,\varphi_1(z_1),\dots,\varphi_n(z_n)} : \widecheck{U(\lambda)} \longrightarrow \widehat{U(F_\Sigma(\lambda))} \tag{45}$$

is now well defined for all $z_1, \dots, z_n$, including on the boundary of Σ. It is induced by the maps $Z_{\Sigma^+,\varphi_1(z_1),\dots,\varphi_n(z_n)} : U(F^{-1}_{A_{in}}(\lambda)) \to U(F_{A_{out}}(F_\Sigma(\lambda)))$, where $\Sigma^+ = A_{in} \cup \Sigma \cup A_{out}$,

[32] G. Segal suggests to add the axiom $H_\lambda = \widehat{H}_\lambda$ to the definition of a chiral CFT. Unfortunately, I think that this is incompatible with the requirement that the representations be smooth in the sense of Remark 14.

and A_{in} and A_{out} are thin collars on the outside of Σ:



(46)

We also have maps

$$\check{Z}_{\Sigma,\dots} : \widetilde{U(\lambda)} \to U(\widetilde{F_\Sigma(\lambda)}) \qquad \text{and} \qquad \widehat{Z}_{\Sigma,\dots} : \widehat{U(\lambda)} \to U(\widehat{F_\Sigma(\lambda)}),$$

defined in the obvious way.

As a particular case of (45), for every circle with collar S, and every point with local coordinate $z \in S$, we have a map

$$\boxed{\varphi(z) : \check{H}_\lambda \to \widehat{H}_\lambda}$$

given by $Z_{\mathrm{id}_S,\varphi(z)}$.

Let $V_S := J^d_{\mathbb{C}} S \times_{G_d} V$ be as in (43), where $J^d_{\mathbb{C}}$ denotes the complexified jet space of S, and let $\Phi_S : V_S \to \mathrm{Hom}(\check{H}_\lambda, \widehat{H}_\lambda)$ be as in (44). Given a smooth section $f \in \Gamma(\Omega^1_S \otimes V_S)$, we define the *smeared field* $\varphi[f] : \check{H}_\lambda \to \widehat{H}_\lambda$ to be the image of f under the map

$$\Gamma(\Omega^1_S \otimes V_S) \xrightarrow{\Phi_S} \Gamma\big(\Omega^1_S \otimes \mathrm{Hom}(\check{H}_\lambda, \widehat{H}_\lambda)\big) \xrightarrow{\int_S} \mathrm{Hom}(\check{H}_\lambda, \widehat{H}_\lambda).$$

Remark: When φ is primary of conformal dimension Δ, then f is just a section of $T^{\otimes(\Delta-1)}\Sigma$ (in particular, when φ is a current f is just a function). In that case, we can rewrite $\varphi[f]$ in the following more intuitive form:

$$\varphi[f] = \int_S f(z)\varphi(z)dz.$$

A priori, $\varphi[f]$ is only a map from $\check{H}_\lambda$ to $\widehat{H}_\lambda$. However, as far as I understand, when working with nuclear Fréchet spaces, this always extends by continuity to a map

$$\boxed{\varphi[f] : H_\lambda \to H_\lambda}$$

In that sense, quantum fields are *operator valued distributions*. They are things which take a test function f as input and produce an operator $H_\lambda \to H_\lambda$ as output.

Remark. When working with Hilbert spaces, a smeared field $\varphi[f]$ is typically not an operator $H_\lambda \to H_\lambda$. It is only a map $\check{H}_\lambda \to H_\lambda$, as well as a map $H_\lambda \to \widehat{H}_\lambda$. In other words, it is an *unbounded* operator on H_λ.

Proposition. (Segal commutation relations) *Let Σ be a complex cobordism, and let φ be a field. Then, for every holomorphic section $f \in \Gamma_{\mathrm{hol}}(\Omega^1_\Sigma \otimes V_\Sigma)$, letting $f_{in} := f|_{\partial_{in}\Sigma}$ and $f_{out} := f|_{\partial_{out}\Sigma}$, we have:*

$$\varphi[f_{out}] \circ \check{Z}_\Sigma \;=\; \widehat{Z}_\Sigma \circ \varphi[f_{in}]. \tag{47}$$

Proof. Consider the image of f under the map

$$\Gamma_{\mathrm{hol}}\big(\Omega^1_\Sigma \otimes V_\Sigma\big) \xrightarrow{\Phi_\Sigma} \Gamma_{\mathrm{hol}}\Big(\Omega^1 \otimes \mathrm{Hom}\big(\widetilde{U(\lambda)}, \widehat{U(F_\Sigma(\lambda))}\big)\Big).$$

Then $\Phi_\Sigma(f)$ is a $\mathrm{Hom}\big(\widetilde{U(\lambda)}, \widehat{U(F_\Sigma(\lambda))}\big)$-valued 1-form which is holomorphic on all of Σ. The integrals $\widehat{Z}_\Sigma \circ \varphi[f_{in}] = \displaystyle\int_{\partial_{in}\Sigma} \Phi_\Sigma(f)$ and $\varphi[f_{out}] \circ \check{Z}_\Sigma = \displaystyle\int_{\partial_{out}\Sigma} \Phi_\Sigma(f)$ are therefore equal by Cauchy's theorem. □

Assuming that all smeared fields extend to maps $H_\lambda \to H_\lambda$, the commutation relations (47) simplify to:

$$\boxed{\varphi[f_{out}] \circ Z_\Sigma \;=\; Z_\Sigma \circ \varphi[f_{in}]}$$

It is expected that, when $\partial\Sigma \neq \emptyset$, the functor $F_\Sigma : \mathcal{C}(\partial_{in}\Sigma) \to \mathcal{C}(\partial_{out}\Sigma)$ is universal with respect to the Segal commutation relations:

Conjecture 28 Fix a chiral Segal CFT. Let Σ be a complex cobordism with non-empty bondary, and let $\lambda \in \mathcal{C}(\partial_{in}\Sigma)$. Then, for every object $\mu \in \mathcal{C}(\partial_{out}\Sigma)$ and every linear map $\zeta : U(\lambda) \to U(\mu)$,

if for every field φ of the CFT and every holomorphic section $f \in \Gamma_{\mathrm{hol}}(\Omega^1_\Sigma \otimes V_\Sigma)$, the equation

$$\varphi[f_{out}] \circ \zeta \;=\; \zeta \circ \varphi[f_{in}]$$

holds,

then there exists a unique morphism $\kappa : F_\Sigma(\lambda) \to \mu$ such that

$$\zeta = U(\kappa) \circ Z_\Sigma.$$

2d chiral CFT as a boundary of 3d TQFT

We can generalize the moduli space (13) by replacing the disc $\mathbb{D}$ by an arbitrary complex cobordism. Given a complex cobordism Σ, let:

$$\mathcal{D}_\Sigma(n) := \left\{ \begin{array}{c} n \text{ holomorphic embeddings } \mathbb{D} \to \Sigma \\ \text{with non-overlapping images.} \end{array} \right\}$$

Then, for every $P \in \mathcal{D}_\Sigma(n)$, we get a functor

$$F_{\Sigma,P} : \mathcal{C}(\partial_{in}\Sigma) \otimes \mathcal{C}^n \longrightarrow \mathcal{C}(\partial_{out}\Sigma).$$

And for every path $\gamma : [0,1] \to \mathcal{D}_\Sigma(n)$ from P_1 to P_2 we get an isomorphism $T_\gamma : F_{\Sigma,P_1} \to F_{\Sigma,P_2}$. Moreover, the isomorphism T_γ only depends on the path γ up to homotopy.

If we fix objects $\mu_1, \ldots, \mu_n \in \mathcal{C}$ and interior points $z_1, \ldots, z_n \in \mathring{\Sigma}$ together with rays $\rho_i \subset T_{z_i}\Sigma$, we can always find embeddings $f_i : \mathbb{D} \to \Sigma$ satisfying $f_i(0) = z_i$ and $f_i'(0) \in \rho_i$. This yields a point $P \in \mathcal{D}_\Sigma(n)$, well defined up to contractible choice. The functor $F_{\Sigma,P}(- \otimes \mu_1 \otimes \ldots \otimes \mu_n)$ then agrees with what we had denoted

$$F_{\Sigma,\mu_1(z_1;\rho_1),\ldots,\mu_n(z_n;\rho_n)} : \mathcal{C}(\partial_{in}\Sigma) \to \mathcal{C}(\partial_{out}\Sigma).$$

Moreover, given finite energy vectors $\xi_i \in H_{\mu_i}$, the corresponding charged fields φ_i are maps

$$Z_{\Sigma,\varphi_1(z_1;j_1),\ldots,\varphi_n(z_n;j_n)} \,:\, U(\lambda) \;\to\; U\big(F_{\Sigma,\mu_1(z_1;\rho_1),\ldots,\mu_n(z_n;\rho_n)}(\lambda)\big), \tag{48}$$

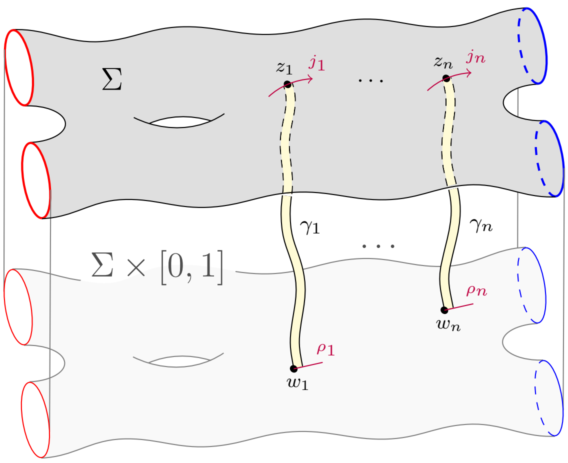
provided $j_i'(0) \in \rho_i$.

We'd like to say that $\varphi_i(z_i)$ depends holomorphically on z_i (and on j_i). This is a little bit tricky to formulate given that the place in which (48) takes its values depends on z_i (and on j_i). But it depends in a *flat* way, i.e., it's a vector bundle with flat connection over the moduli space of z_i's and ρ_i's. So we can locally trivialize the right hand side of (48) and pretend that all the $\varphi_i(z_i)$ take their values in the same space, at least locally in z_i and ρ_i.

Thinking more globally, we can make sense of the statement

$$Z_{\Sigma,\varphi_1(z_1;j_1),\ldots,\varphi_n(z_n;j_n)}(\eta) \;\in\; U\big(F_{\Sigma,\mu_1(w_1;\rho_1),\ldots,\mu_n(w_n;\rho_n)}(\lambda)\big) \quad \text{for} \quad \eta \in U(\lambda) \tag{49}$$

whenever we are provided with a (homotopy class of) path from $(z_i; j_1'(0)), \ldots, (z_n; j_n'(0))$ to $(w_1; \rho_1), \ldots, \mu_n(w_n; \rho_n)$. The picture which I wish to associate to (49) is the following:



Here, the path $\gamma : [0,1] \to \mathcal{D}(n)$ is interpreted as a ribbon braid $\gamma = (\gamma_1, \ldots, \gamma_n)$ inside $\Sigma \times [0,1]$, connecting the points $(w_1, \ldots, w_n) \subset \Sigma \times \{0\}$ to the points $(z_1, \ldots, z_n) \subset \Sigma \times \{1\}$. The fact that we could draw things as we did is a manifestation of the fact that *a chiral conformal field theory sits at the boundary of a 3d topological field theory*. The surface on which we drew the z_i's carries the chiral CFT, whereas the bulk, namely $\Sigma \times [0,1)$, is where the 3d TQFT lives.

In the special case $\partial\Sigma = \emptyset$, $\lambda = 1_\emptyset$, $\eta = 1$, one can rewrite (49) as follows:

$$\big\langle \varphi_1(z_1), \ldots, \varphi_n(z_n) \big\rangle_{\Sigma,\gamma_1,\ldots,\gamma_n} \in H_{\Sigma,\mu_1(w_1),\ldots,\mu_n(w_n)}. \tag{50}$$

Here, $H_{\Sigma,\mu_1(w_1),\ldots,\mu_n(w_n)}$ is just an equivalent name for the finite dimensional vector space $U(F_{\Sigma,\mu_1(w_1),\ldots,\mu_n(w_n)}(1_\emptyset))$. For charged fields, the correlator $\langle \varphi_1(z_1), \ldots, \varphi_n(z_n)\rangle$ doesn't just depend on the points z_i (and the local coordinates j_i), but also on the ribbon braid γ. (And if you were to try to think of it as a function of just the z_i and the j_i, then it would become a multivalued function.)

As before, a *conformal block* is a linear map $\mathcal{B} : H_{\Sigma,\mu_1(w_1),\ldots,\mu_n(w_n)} \to \mathbb{C}$, and the correlation function

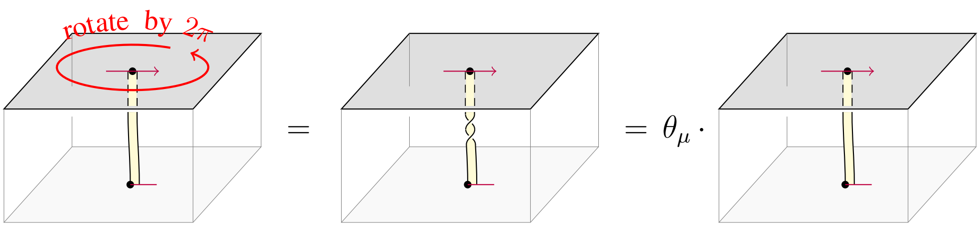
$$\big\langle \varphi_1(z_1), \ldots, \varphi_n(z_n) \big\rangle_{\Sigma,\mathcal{B},\gamma_1,\ldots,\gamma_n} \in \mathbb{C}.$$

is the image of (50) under that map.

Let $\varphi(z)$ be a field of charge μ which is primary. Then we have

$$\varphi(z; av) = a^\Delta \varphi(z; v) \qquad \text{for } v \in T_z\Sigma \tag{51}$$

as in (32). But the point z now has a ribbon attached to it, so a lives no longer in $\mathbb{C}^\times$ but instead in the universal cover of $\mathbb{C}^\times$. Correspondingly, the conformal dimension Δ is no longer an integer. It is an eigenvalue of the action of L_0 on H_μ. Namely, it is a number of the form $h_\mu + n$, where h_μ is the minimal energy of H_μ and $n \in \mathbb{N}$ is a natural number. For example, if $a = e^{2\pi i}$ then $a^\Delta = (e^{2\pi i})^\Delta = \theta_\mu$ is the conformal spin of μ (which is equal to the twist of μ, coming from the fact that $\mathcal{C}(S^1)$ is balanced). Equation (51) becomes:



Let $\mathcal{M}_{g,n}$ be the moduli space of closed Riemann surfaces Σ of genus g with n marked points $w_1, \ldots, w_n$ together with rays $\rho_i \subset T_{z_i}\Sigma$. The fiber of the map $\mathcal{M}_{g,n} \to \mathcal{M}_g := \mathcal{M}_{g,0}$ *[Note that $\mathcal{M}_g$ is not a space but something slightly more general, called a stack]* over a Riemann surface $\Sigma \in \mathcal{M}_g$ is the configuration space $\mathrm{Conf}_\Sigma(n)$. The latter is a space which is homotopy equivalent to the space $\mathcal{D}_\Sigma(n)$ defined above. So we have a fiber bundle

$$\begin{array}{ccc} \mathrm{Conf}_\Sigma(n) & \longrightarrow & \mathcal{M}_{g,n} \\ & & \downarrow \\ & & \mathcal{M}_g \end{array} \tag{52}$$

Fix objects $\mu_1, \ldots, \mu_n \in \mathcal{C}$. Then

$$(\Sigma, (w_1, \ldots, w_n)) \mapsto H_{\Sigma,\mu_1(w_1),\ldots,\mu_n(w_n)}$$

is a vector bundle of finite rank over $\mathcal{M}_{g,n}$ called (the dual of) the *bundle of conformal blocks*. Its fibers are called the (dual) *spaces of conformal blocks*.

The trivializations $T_{\tilde{A}}$ (see Table 1 on page 16) equip the bundle of conformal blocks with a flat projective connection. What this means is that the mapping class group $\pi_1(\mathcal{M}_{g,n})$ acts projectively on spaces of conformal blocks. Moreover, by the construction described in (14), that flat projective connection admits a lift to an honest (i.e. non-projective) flat connection in the direction of the fibers of (52). What this means is the above action of the mapping class group restricts to an honest action of the surface ribbon braid group $\pi_1(\mathrm{Conf}_\Sigma(n))$.