Course notes for *Conformal field theory* (math 290)

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Prologue: Let me spend a few words about what this class is not about. We will *not* be treating CFT in more than two dimensions. We will *not* be doing perturbation theory. More generally, I will *not* be assuming any prior knowledge of quantum field theory. Actually, those who happen to have some knowledge of quantum field theory should not expect it to help them much for this course.

Full and chiral CFT

Conformal manifolds. Recall that a manifold is called a *conformal manifold* if it is equipped with a conformal metric. Here, a conformal metric is an equivalence class of metric tensors (either Euclidian or Minkowskian) under the equivalence relation that declares g_1 and g_2 to be equivalent if $g_2 = f \cdot g_1$ for some $\mathbb{R}_{>0}$-valued function f.

In the case of 2-dimensional manifolds, a conformal metric can be re-expressed in terms of other, more familiar structures:

Euclidian signature: On a smooth surface, one has:

$$\text{conformal metric} \;+\; \text{orientation} \;=\; \text{complex structure.}$$

Indeed, given a conformal metric and a tangent vector $v \in T_xM$, one can define $iv \in T_xM$ to the be unique vector that is orthogonal to v, of the same length as v, and such that $\{v, iv\}$ forms an oriented basis of T_xM. Conversely, given a complex structure on a tangent space T_xM, then for every non-zero vector v there is a unique metric g such that $\{v, iv\}$ forms an oriented orthonormal basis. Changing the choice of vector v replaces the metric by a positive scalar multiple of it.

Minkowskian signature: In that case, we have

$$\text{conformal metric} \;=\; \begin{matrix}\text{two transverse}\\ \text{foliations}\end{matrix} \;+\; \begin{matrix}\text{information of what is}\\ \text{space and what is time,}\end{matrix}$$

from which it follows that:

$$\begin{matrix}\text{conformal metric}\\ +\text{ orientation}\end{matrix} \;=\; \begin{matrix}\text{two transverse}\\ \text{foliations}\end{matrix} \;+\; \begin{matrix}\text{information of what is}\\ \text{space and what is time}\end{matrix} \;+\; \text{orientation.}$$

Let us define a tangent vector v to be a *null-vector* if $g(v, v) = 0$. The notion of null-vector is invariant under the transformation $g \mapsto f \cdot g$, and is therefore an intrinsic notion to the conformal manifold M. Given a conformal metric, the two foliations are the ones defined by the null-vectors.

Conversely, given two transverse foliations, there exists a unique conformal metric that corresponds to them, up to sign. To see that, note that any pair of transverse foliations is locally diffeomorphic to the standard pair of foliations on $\mathbb{R}^2$ by lines parallel to the x- and y-axes. We may therefore restrict our attention to that special case. Let us write the components of the metric tensor g in matrix form:

$$g = \begin{pmatrix} g^{11} & g^{12} \\ g^{12} & g^{22} \end{pmatrix}.$$

Demanding that $(1, 0)$ be a null-vector implies $g^{11} = 0$ and, similarly, demanding that $(0, 1)$ be a null-vector implies $g^{22} = 0$. It follows that

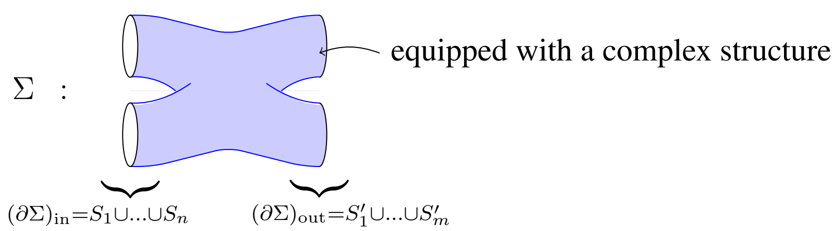
$$g = \begin{pmatrix} 0 & g^{12} \\ g^{12} & 0 \end{pmatrix} = g^{12} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \underset{\boxed{\text{if } g^{12} > 0}}{\sim} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If one furthermore demands that the vector $(1, 1)$ is space-like (and therefore that the vector $(1, -1)$ is time-like), then this forces g^{12} to be positive, and we see that $g \sim \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Therefore, the two transverse foliations along with the information of what space is and what time is completely determine the conformal metric.

There are, roughly speaking, two main approaches to conformal field theory: *Euclidian* and *Minkowskian*.

The Euclidian approach to ***CFT*** **:** In the Euclidian approach, a unitary CFT is a gadget that assigns to every connected compact oriented 1-manifold S a Hilbert space H_S (there is only one such manifold up to isomorphism). A diffeomorphism $f : S_1 \to S_2$ induces a map $H_{S_1} \to H_{S_2}$ that is unitary if f is orientation preserving and antiunitary if f is orientation reversing.

To every cobordism



there is a corresponding map $g_\Sigma : H_{\text{in}} \to H_{\text{out}}$,[1] where

$$\begin{aligned} & H_{\text{in}} := H_{S_1} \otimes H_{S_2} \otimes \ldots \otimes H_{S_n} \\ \text{and} \quad & H_{\text{out}} := H_{S_1'} \otimes H_{S_2'} \otimes \ldots \otimes H_{S_m'}. \end{aligned}$$

Here, the orientations on $S_1, \ldots, S_n$ are the ones induced by Σ, and the orientations on $S_1', \ldots, S_m'$ are the opposite of the ones induced by Σ.

The axioms are as follows:
(0) The map g_Σ depends smoothly on the choice of complex structure on Σ.
(0′) The image of a vector $\xi \in H_{S_1}$ under the map $H_{S_1} \to H_{S_2}$ depends continuously on the choice of diffeomorphism $S_1 \to S_2$. Similarly, if a sequence of cobordisms between two fixed 1-manifolds converges to a diffeomorphism, then the same pointwise convergence relation should hold.
(1) Given an orientation preserving diffeomorphism $\varphi : (\partial\Sigma)_{\text{out}} \cong (\partial\Sigma')_{\text{in}}$, then the map $g_{\Sigma \cup_\varphi \Sigma'}$ associated to the composite cobordism $\Sigma \cup_\varphi \Sigma'$ is equal to the composite

$$g_{\Sigma'} \circ g_\Sigma \quad : \quad H_{\text{in}} \longrightarrow H_{\text{out}} \cong H'_{\text{in}} \longrightarrow H'_{\text{out}}.$$

Here, the middle isomorphism $H_{\text{out}} \cong H'_{\text{in}}$ is induced by φ.
(2) For any two cobordisms Σ_1 and Σ_2, we have

$$g_{\Sigma_1 \sqcup \Sigma_2} = g_{\Sigma_1} \otimes g_{\Sigma_2}.$$

(3) Relabeling an "in" boundary component as "out" corresponds to taking a partial adjoint. We spell out this last condition in detail. Assume as before that $(\partial\Sigma)_{\text{in}} = S_1 \cup \ldots \cup S_n$ and $(\partial\Sigma)_{\text{out}} = S_1' \cup \ldots \cup S_m'$, and let us define $\tilde{\Sigma}$ to be the same manifold as Σ, but with S_1 relabeled as "out" and with its orientation reversed. Let us also write $\bar{S}_1$ for S_1 with the opposite orientation. Then for $\xi_i \in H_{S_i}$ and $\eta_i \in H_{S_i'}$ we have

$$\big\langle g_\Sigma(\xi_1 \otimes \ldots \otimes \xi_n), \eta_1 \otimes \ldots \otimes \eta_m \big\rangle = \big\langle g_{\tilde{\Sigma}}(\xi_2 \otimes \ldots \otimes \xi_n), \bar{\xi}_1 \otimes \eta_1 \otimes \ldots \otimes \eta_m \big\rangle.$$

Here, $\bar{\xi}_1 \in H_{\bar{S}_1}$ is the image of $\xi_1 \in H_{S_1}$ under the antiunitary $H_{S_1} \to H_{\bar{S}_1}$ induced by the orientation reversing diffeomorphism $\mathrm{Id} : S_1 \to \bar{S}_1$.

The Minkowskian approach to ***CFT*** **:** In the Minkowskian approach, we focus on the space-time manifold $M := S^1 \times \mathbb{R}$, equipped with its standard Minkowskian "metric" $ds^2 = dx^2 - dy^2$ given by the tensor

$$g((x_1, y_1), (x_2, y_2)) = x_1 x_2 - y_1 y_2.$$

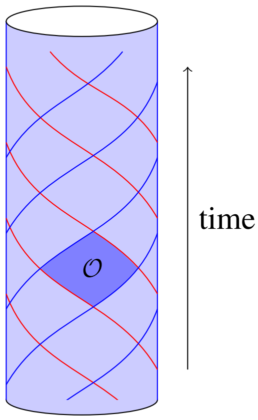
More precisely, we only care about the conformal equivalence class of the above metric on M. The null-vectors form two foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ on M, indicated in blue and red

[1] Really, the map g_Σ also depends on a Riemannian metric on Σ. Scaling the metric by e^f multiplies the map g_Σ by the constant e^{cS}, where $c \geq 0$ is an invariant of the CFT called the *central charge*, $S := \int_\Sigma (\frac{1}{2}\|df\|^2 + 2fK)$ is the so-called *Liouville action*, and K is the curvature of the metric.

in the picture below, and the equivalence class of g is entirely determined, up to sign, by those two foliations.

Definition: A *double cone* is an open simply connected subset $\mathcal{O} \subset M$ enclosed by four leaves of the above foliations. We also demand that $\mathcal{O}$ not be too big: the images of $\mathcal{O}$ in $M/\mathcal{F}_1$ and in $M/\mathcal{F}_2$ should not be dense. Here is an example of what a double cone looks like:

M :

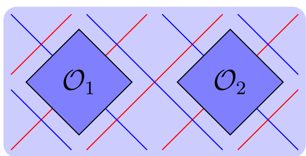


Let us call a tangent vector v *space-like* if $g(v, v) > 0$, and *time-like* if $g(v, v) < 0$.

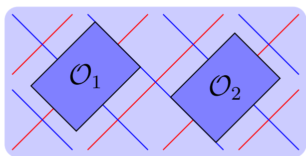
Definition: We call two double cones $\mathcal{O}_1$ and $\mathcal{O}_2$ *causally separated* if for any $x \in \mathcal{O}_1$ and $y \in \mathcal{O}_2$, every geodesic connecting x to y is space-like (you need to go faster than the speed of light to go from $\mathcal{O}_1$ to $\mathcal{O}_2$).

Similarly, we call $\mathcal{O}_1$ and $\mathcal{O}_2$ *causally well separated* if for any $x \in \overline{\mathcal{O}_1}$ and $y \in \overline{\mathcal{O}_2}$ in their closures, every geodesic connecting x to y is space-like.

Example:



The double cones $\mathcal{O}_1$ and $\mathcal{O}_2$ are causally well separated



The double cones are causally separated, but not well separated

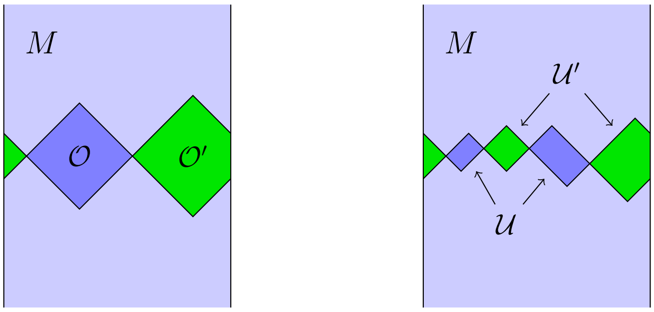
Similarly, we can talk about any two subsets of M being causally separated, or causally well separated.

Definition: If $\mathcal{U}$ is any subset of M, we define the *causal complement* of $\mathcal{U}$ to be the set

$$\mathcal{U}' := \big\{ x \in M \,\big|\, \{x\} \text{ is causally well separated from } \mathcal{U} \big\}.$$

Note that if $\mathcal{O}$ is a double cone, then its causal complement $\mathcal{O}'$ is also a double cone. Similarly, if $\mathcal{U} = \mathcal{O}_1 \cup \ldots \cup \mathcal{O}_n$ is a union of n causally well separated double cones, then $\mathcal{U}'$ is also a union of n causally well separated double cones.

Examples:



Roughly speaking, a CFT in the Minkowskian approach consists of a Hilbert space H with a projective action of the group of conformal transformations of M, and a map

$$\mathcal{A} : \mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$$

that sends a double cone $\mathcal{O} \subset M$ to a subalgebra of $B(H)$, the bounded operators on H.

However, before we can give the actual definition of CFT in the Minkowskian approach, we will need a couple of preliminaries on von Neumann algebras and on conformal transformations.

Von Neumann algebras. Fix a Hilbert space H and recall that $B(H)$ denotes the algebra of bounded operators of H.

Definition: If A is any sub-$*$-algebra of $B(H)$, we define the *commutant* of A to be $*$-algebra

$$A' := \big\{ b \in B(H) \,\big|\, ab = ba \text{ for every } a \in A \big\}.$$

The *double commutant* is then denoted A''; it is the commutant of the commutant of A.

It is easy to see that one always has $A \subset A''$. One can think of the double commutant operation as some kind of closure operation, and indeed there exists a topology on $B(H)$ such that A'' is exactly the closure of A in that topology.

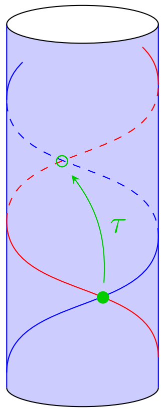
Definition: A sub-$*$-algebra $A \subset B(H)$ is called a *von Neumann algebra* if $A = A''$.

Conformal transformations. Let $\mathrm{Conf}(M)$ denote the group of orientation preserving conformal diffeomorphisms of M. This group has two connected components. The identity component $\mathrm{Conf}^{\uparrow}(M)$ is called the group of *orthochronous* transformations, and corresponds to those transformations that preserve the direction of time. The other component $\mathrm{Conf}^{\downarrow}(M)$ consists of transformations that reverse the direction of time.

The leaf spaces of M with respect to the two foliations are both circles. We therefore get two homomorphisms $\mathrm{Conf}^{\uparrow}(M) \to \mathrm{Diff}_+(S^1)$ to the group of orientation preserving diffeomorphisms of the circle. The kernel of the resulting homomorphism to $\mathrm{Diff}_+(S^1) \times \mathrm{Diff}_+(S^1)$ is still surjective and so one gets a short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{Conf}^{\uparrow}(M) \longrightarrow \mathrm{Diff}_+(S^1) \times \mathrm{Diff}_+(S^1) \longrightarrow 0$$

where the kernel $\mathbb{Z}$ is generated by the "twisted translation" τ:



(1)

There is also a subgroup $\mathrm{Möb}(M) = \mathrm{Möb}^{\uparrow}(M) \cup \mathrm{Möb}^{\downarrow}(M)$ of $\mathrm{Conf}(M)$, which we'll call the "Möbius transformations". It is defined as the preimage of the group

$$\mathrm{PGL}_2(\mathbb{R}) \times \mathrm{PGL}_2(\mathbb{R}) \subset \mathrm{Diff}(S^1) \times \mathrm{Diff}(S^1)$$

under the projection $\mathrm{Conf}(M) \to \mathrm{Diff}(S^1) \times \mathrm{Diff}(S^1)$. Here, $\mathrm{PGL}_2(\mathbb{R}) \subset \mathrm{Diff}_+(S^1)$ is the group of fractional linear transformations of the real projective line (both the orientation preserving and the orientation reversing ones), under the usual identification $S^1 \cong \mathbb{RP}^1$.

The identity component of $\mathrm{Möb}(M)$ is denoted $\mathrm{Möb}^{\uparrow}(M)$. It is the preimage $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$ in $\mathrm{Conf}^{\uparrow}(M)$, and fits into a short exact sequence that is similar to that for $\mathrm{Conf}^{\uparrow}(M)$:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{Möb}^{\uparrow}(M) \longrightarrow \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R}) \longrightarrow 0.$$

The definition of conformal field theory. Given all the above preliminaries, we can now write down the definition of a unitary CFT in the Minkowskian approach.

Definition: A CFT consists of the following data:

- A Hilbert spce H called the *state space* of the CFT.
- A continuous[2] projective action of $\mathrm{Conf}(M)$ on H that restricts to an honest action on the subgroup $\mathrm{Möb}(M)$. The orthochronous transformations act unitarily and the other ones act by antiunitaries.

[2]Here, continuity is with respect to the (quotient topology of the) strong operator topology, also known as the topology of pointwise convergence. By definition, $u_i \to u$ strongly if $u_i(\xi) \to u(\xi)$ for every $\xi \in H$.

- A unit vector $\Omega \in H$, called the *vacuum vector*.
- Finally and mot importantly, there is an assignment

$$\mathcal{A} : \mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$$

that sends each double cone $\mathcal{O} \subset M$ to a subalgebra of $B(H)$, the bounded operators on H. That algebra is called the *algebra of local observables*.

The above pieces of data are subject to the following axioms:

- If $\mathcal{O}_1 \subseteq \mathcal{O}_2$ then $\mathcal{A}(\mathcal{O}_1) \subseteq \mathcal{A}(\mathcal{O}_2)$.
- *Locality:* If $\mathcal{O}_1$ and $\mathcal{O}_2$ are causally separated, then the algebras $\mathcal{A}(\mathcal{O}_1)$ and $\mathcal{A}(\mathcal{O}_2)$ commute with each other.
- *Covariance:* For $g \in \mathrm{Conf}(M)$ with corresponding (anti)unitary u_g (well defined up to phase), we have

$$\mathcal{A}(g\mathcal{O}) = u_g \mathcal{A}(\mathcal{O}) u_g^*.$$

Moreover, if $g \in \mathrm{Conf}(M)$ fixes $\mathcal{O}$ pointwise, then $\mathrm{Ad}(u_g)$ fixes $\mathcal{A}(\mathcal{O})$ pointwise, in other words, u_g commutes with $\mathcal{A}(\mathcal{O})$.

- *Positive energy:* Let α_t be the unitary that corresponds to the transformation $(x, y) \mapsto (x + t, y + t)$ on $M = \mathbb{R}^2/\mathbb{Z} \oplus \{0\}$ (translation by t in the direction of one of the null-foliations). Similarly, let $\bar{\alpha}_t$ be the unitary that corresponds to the transformation $(x, y) \mapsto (x - t, y + t)$ (translation by t in the direction of the other the null-foliation). Then there exist unbounded *positive* operators L_0 and $\bar{L}_0$ on H such that

$$\alpha_t = e^{itL_0} \qquad \text{and} \qquad \bar{\alpha}_t = e^{it\bar{L}_0}.$$

- The vacuum vector Ω is fixed under the action of $\mathrm{M\ddot{o}b}(M)$, and it spans the subspace of $\mathrm{M\ddot{o}b}(M)$-fixed vectors of H.
- The vacuum vector Ω is cyclic for the joint actions of the algebras $\mathcal{A}(\mathcal{O})$. That is, the subspace generated by the action of those algebras on Ω is dense in H.
- *Anomaly cancellation:* The subgroup $\mathrm{Diff}(S^1) \subset \mathrm{Conf}(M)$ that fixes the Cauchy surface $S^1 \times \{0\} \subset S^1 \times \mathbb{R}$ acts honestly (as opposed to projectively) on H.
- *Strong Haag duality:* If $\mathcal{U} = \mathcal{O}_1 \cup \ldots \cup \mathcal{O}_n$ is a finite union of causally well separated double cones, let us define $\mathcal{A}(\mathcal{U}) := \mathcal{A}(\mathcal{O}_1) \vee \ldots \vee \mathcal{A}(\mathcal{O}_n)$ to be the von Neumann algebra generated by $\mathcal{A}(\mathcal{O}_1), \ldots, \mathcal{A}(\mathcal{O}_n)$. With the above notation in place, we demand that the algebra associated to the causal complement of $\mathcal{U}$ be equal to the commutant of the algebra associated to $\mathcal{U}$:

$$\mathcal{A}(\mathcal{U}') = \mathcal{A}(\mathcal{U})'.$$

Remark: Note that the locality axiom follows from the strong Haag duality axiom. We keep it in the list of axioms because we will soon be interested in weakenings of the above definition, where certain axioms are dropped.

At this point, one might ask why we have fixed our space-time manifold to be $M = S^1 \times \mathbb{R}$, as opposed to some other conformally Minkowskian manifold. The reason is that it is the above definition that is expected to line up with the correseponding Euclidian definition of CFT. At the moment, that is still a conjecture. For example, the <u>*WZW models*</u> associated to the group $SU(n)$ have been fully constructed in the Minkowskian approach, but not in the Euclidian approach.[3] (The WZW models form an important class of CFTs. There is one WZW model for every choice of compact, simple, simply connected Lie group G, and every "level" $k \in \mathbb{Z}_{\geq 1}$.)

The notion that we have been so far calling "CFT" is also called a <u>*full CFT*</u>.
Here are some related notions. We first present them in the **Mikowskian approach:**

Definition: A <u>*weak CFT*</u> is what one gets if one drops the 'anomaly cancellation' and 'strong Haag duality' axioms from the definition of a full CFT.

Definition: A <u>*chiral CFT*</u> is a weak CFT for which the algebra $\mathcal{A}(\mathcal{O})$ only depends on the image of $\mathcal{O}$ under the projection $M \to M/\mathcal{F}_1$, and such that the action of $\mathrm{Conf}(M)$ on H factors through its projection to $\mathrm{Diff}(M/\mathcal{F}_1) \cong \mathrm{Diff}(S^1)$.

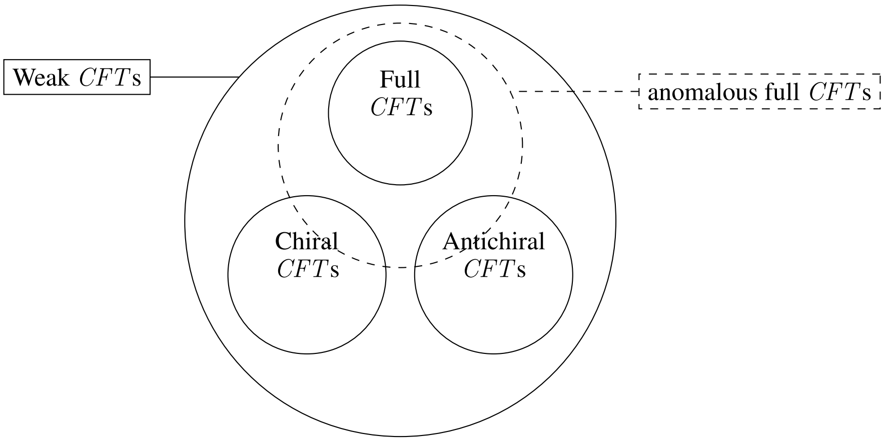
Definition: An <u>*antichiral CFT*</u> is a weak CFT for which the algebra $\mathcal{A}(\mathcal{O})$ only depends on the image of $\mathcal{O}$ under the projection $M \to M/\mathcal{F}_2$, and such that the action of $\mathrm{Conf}(M)$ on H factors through its projection to $\mathrm{Diff}(M/\mathcal{F}_2) \cong \mathrm{Diff}(S^1)$.

We also have the following intermediate notion between full and weak CFT. We won't really need that notion, but we include it for completeness:

Definition: If one only drops the 'anomaly cancellation' from the definition of full CFT, then one gets a notion that I'll call <u>*anomalous full CFT*</u>.

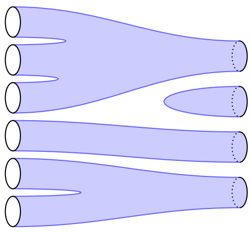
We should point out that there *do* exist models (=examples) of all the above notions. Below is a Venn diagram that indicates how they all fit with respect to each other. In particular, one sees that the classes of full, chiral, and antichiral CFTs are disjoint (with the exception of the trivial CFT, the one with $H = \mathbb{C}$, which is at the same time full, chiral, and antichiral; it is omitted from the Venn diagram below).

[3] Actually, there seems to be disagreement among experts in the field as to whether those models have been constructed or not...



These same concepts can also be formulated in the **Euclidian approach:**

Definition: A *weak CFT* is what one gets if one restricts our bordisms to be of the following form:



Namely, every connected components of Σ should have genus zero, and have exactly one outgoing boundary circle. Moreover, we should no longer require a map $S_1 \to S_2$ between circles to induce a map $H_{S_1} \to H_{S_2}$ between the corresponding Hilbert spaces; the latter should now only be well defined up to phase.[4]

Definition: An *(anti)chiral CFT* is a weak CFT for which the map $g_\Sigma : H_{\text{in}} \to H_{\text{out}}$ depends (anti)holomorphically on the choice of cobordism Σ.

For the latter definition to really make sense, we should also say what it means for a family of complex cobordisms between two fixed 'in' and 'out' 1-manifolds to be holomorphic (which only makes sense if the manifold that parametrises the family is itself complex). This would take us too far afield, and so we will not spell out the details here. We will also refrain from discussing the notion of anomalous full CFT in the Euclidian signature setup.

We will not have much further use of the Euclidian approach. We finish this chapter by mentioning an open problem:

[4]Moreover, the formula for the anomaly that appears in footnote [1] needs to be modified.

Conjecture: There are natural equivalences of categories

$$\begin{array}{ccc} \{\text{Minkowskian full } CFT\text{s}\} & \simeq & \{\text{Euclidian full } CFT\text{s}\} \\ \{\text{Minkowskian chiral } CFT\text{s}\} & \simeq & \{\text{Euclidian chiral } CFT\text{s}\} \\ \{\text{Minkowskian antichiral } CFT\text{s}\} & \simeq & \{\text{Euclidian antichiral } CFT\text{s}\} \end{array}$$

The chiral halves of a full *CFT*. If $(H, \Omega, \mathcal{A})$ is a full CFT then one can define its *associated chiral CFT* $(H^\chi, \Omega^\chi, \mathcal{A}^\chi)$ as follows — note that the procedure also makes sense for an arbitrary weak CFT. This is done by defining

$$\mathcal{A}^\chi(\mathcal{O}) := \bigcap_{\substack{\text{double cones } \tilde{\mathcal{O}} \text{ s.t.} \\ p_1(\tilde{\mathcal{O}}) = p_1(\mathcal{O})}} \mathcal{A}(\tilde{\mathcal{O}})$$

where $p_1 : M \to M/\mathcal{F}_1$ is the projection of M onto the leaf space of the foliation $\mathcal{F}_1$. The vacuum vector $\Omega^\chi := \Omega$ is unchanged, and one defines H^χ be the closure of the orbit of Ω under the action of the algebras $\mathcal{A}^\chi(\mathcal{O})$.

The action of the group $\mathrm{Conf}(M)$ on H^χ is somewhat tricky to define: the Hilbert space H^χ is typically *not* invariant under the action of $\mathrm{Conf}(M)$ and so one *cannot* define that action by restriction!

Here's something that doesn't quite work: Recall that we are looking for an action of $\mathrm{Diff}(M/\mathcal{F}_1)$ on H^χ. Given an element g in that group, pick a lift $\tilde{g} \in \mathrm{Conf}(M)$ whose image in $\mathrm{Diff}(M/\mathcal{F}_2)$ is trivial and define the action of g on H^χ to be that of $\tilde{g}$.

The reason this doesn't work is that the map

$$\mathrm{Conf}(M) \to \mathrm{Diff}(M/\mathcal{F}_1) \times \mathrm{Diff}(M/\mathcal{F}_2)$$

isn't surjective. Indeed, $\mathrm{Conf}(M)$ has only two connected components whereas the right hand side has four. So given g as above, there maybe *isn't* any lift $\tilde{g}$ with the property that its image in $\mathrm{Diff}(M/\mathcal{F}_2)$ is trivial.

The solution is to allow the lift $\tilde{g}$ to only satisfy a somewhat weaker condition than that of having trivial image in $\mathrm{Diff}(M/\mathcal{F}_2)$. Let

$$\begin{array}{ll} p_1 : M \to M/\mathcal{F}_1 & \qquad \pi_1 : \mathrm{Conf}(M) \to \mathrm{Diff}(M/\mathcal{F}_1) \\ p_2 : M \to M/\mathcal{F}_2 & \qquad \pi_2 : \mathrm{Conf}(M) \to \mathrm{Diff}(M/\mathcal{F}_2). \end{array}$$

be the projections. Let

$$\text{Möb}(S^1) := \mathrm{PGL}_2(\mathbb{R}) \subset \mathrm{Diff}(S^1)$$

be the subgroup of Möbius transformations (its identity component is $\mathrm{PSL}_2(\mathbb{R})$), and let us call $\text{Möb}(M/\mathcal{F}_1)$ and $\text{Möb}(M/\mathcal{F}_2)$ the corresponding subgroups of $\mathrm{Diff}(M/\mathcal{F}_1)$ and $\mathrm{Diff}(M/\mathcal{F}_2)$.

We are now ready to define the action of $\mathrm{Diff}(M/\mathcal{F}_1)$ on H^χ. Given an element g in that group, pick a lift $\tilde{g} \in \mathrm{Conf}(M)$ with the property that $\pi_2(\tilde{g}) \in \mathrm{M\ddot{o}b}(M/\mathcal{F}_2)$ and define the action of g on H^χ to be that of $\tilde{g}$. The fact that this procedure is well defined depends on the lemmas **3** and **5** below.

begin of technical lemmas →

Lemma 1 *The group* $\ker(\pi_1 : \mathrm{Conf}(M) \to \mathrm{Diff}(M/\mathcal{F}_1))$ *is the universal cover of* $\mathrm{Diff}_+(M/\mathcal{F}_2)$ *and we have a short exact sequence*

$$0 \longrightarrow \mathbb{Z} \longrightarrow \ker(\pi_1) \xrightarrow{\pi_2} \mathrm{Diff}_+(M/\mathcal{F}_2) \longrightarrow 0$$

whose kernel $\mathbb{Z} = \langle \tau \rangle$ *is generated by the 'twisted translation'* τ *from picture* (1).

Similarly, $\ker(\pi_2 : \mathrm{Conf}(M) \to \mathrm{Diff}(M/\mathcal{F}_2))$ *is the universal cover of* $\mathrm{Diff}_+(M/\mathcal{F}_1)$.

Proof: The map π_1 sends $\mathrm{Conf}^\downarrow(M)$ to orientation reversing diffeomorphisms of $M/\mathcal{F}_1$. In particular, $\ker(\pi_1 : \mathrm{Conf}(M) \to \mathrm{Diff}(M/\mathcal{F}_1)) = \ker(\pi_1 : \mathrm{Conf}^\uparrow(M) \to \mathrm{Diff}_+(M/\mathcal{F}_1))$.

Fix a leaf R of $\mathcal{F}_1$.

For every leaf L of $\mathcal{F}_2$, the map p_1 exhibits L as a universal cover of $M/\mathcal{F}_1$. Moreover, every element $g \in \ker(p_1)$ induces a morphism of universal covers:

$$\begin{tikzcd} L \arrow[rr, "g"] \arrow[dr, "p_1"'] & & g(L). \arrow[dl, "p_1"] \\ & M/\mathcal{F}_1 & \end{tikzcd}$$

Recall that, by the path lifting property, a morphism of universal covers is completely determined by what it does to a single point. Since $L \cap R \neq \emptyset$, the map $g|_L$ is completely determined by $g|_R$. This being true for every leaf L of $\mathcal{F}_2$, g is therefore completely determined by its restriction to R.

A map $f : R \to R$ is the restriction of an element of $\ker(p_1)$ iff it respects the equivalence relation induced by $\mathcal{F}_2$. Upon identifying R with $\mathbb{R}$, these are exactly the maps $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+1) = f(x) + 1$. Finally, the group of periodic diffeomorphisms

$$\mathrm{Diff}_{\mathrm{per}}(\mathbb{R}) := \left\{ f : \mathbb{R} \xrightarrow{\sim} \mathbb{R} \,\middle|\, f(x+1) = f(x) + 1 \right\}$$

is a universal cover of $\mathrm{Diff}_+(S^1)$, as can be seen from the fact that it is contractible (by straight-line homotopies to the identity map) and that its projection to $\mathrm{Diff}_+(S^1)$ induces a homeomorphism between small neighborhoods of the respective identity elements.

To finish the proof, we note that the kernel of $\pi_2 : \mathrm{Diff}_{\mathrm{per}}(\mathbb{R}) \to \mathrm{Diff}_+(S^1)$ is generated by the map $x \mapsto x + 1$, which is exactly the restriction of τ to R. $\square$

Lemma 2 *If* $g \in \mathrm{Conf}(M)$ *and* $\mathcal{O} \subset M$ *is a double cone, then*

$$u_g \mathcal{A}^\chi(\mathcal{O}) u_g^* = \mathcal{A}^\chi(g\mathcal{O}).$$

Proof: By the covariance axiom for $\mathcal{A}$, we have $u_g\mathcal{A}(\mathcal{O})u_g^* = \mathcal{A}(g\mathcal{O})$. It follows that:

$$\begin{aligned} u_g\mathcal{A}^\chi(\mathcal{O})u_g^* &= u_g\Bigg(\bigcap_{\substack{\text{double cones } \tilde{\mathcal{O}} \text{ s.t.}\\ p_1(\tilde{\mathcal{O}})=p_1(\mathcal{O})}} \mathcal{A}(\tilde{\mathcal{O}})\Bigg)u_g^* = \bigcap_{\substack{\text{double cones } \tilde{\mathcal{O}} \text{ s.t.}\\ p_1(\tilde{\mathcal{O}})=p_1(\mathcal{O})}} u_g\mathcal{A}(\tilde{\mathcal{O}})u_g^* \\ &= \bigcap_{\substack{\text{double cones } \tilde{\mathcal{O}} \text{ s.t.}\\ p_1(\tilde{\mathcal{O}})=p_1(\mathcal{O})}} \mathcal{A}(g\tilde{\mathcal{O}}) = \bigcap_{\substack{\text{double cones } \widehat{\mathcal{O}} \text{ s.t.}\\ p_1(\widehat{\mathcal{O}})=p_1(g\mathcal{O})}} \mathcal{A}(\widehat{\mathcal{O}}) = \mathcal{A}^\chi(g\mathcal{O}). \end{aligned}$$

□

Lemma 3 *If $g \in \mathrm{Conf}(M)$ is such that $\pi_2(g) \in \text{Möb}(M/\mathcal{F}_2)$, then*

$$u_g(H^\chi) = H^\chi.$$

(Note that it is enough to check that $u_g(H^\chi) \subseteq H^\chi$. Indeed, the statements $u_g(H^\chi) \subseteq H^\chi$ and $u_{g^{-1}}(H^\chi) \subseteq H^\chi$ together imply $u_g(H^\chi) = H^\chi$.)

Proof: Let H° be the dense subspace of H^χ spanned by the vectors $a_1a_2\ldots a_n\Omega$ with $a_i \in \mathcal{A}^\chi(\mathcal{O}_i)$. By continuity, it is enough to argue that $u_g(H^\circ) \subseteq H^\circ$. That is, we want to show that $u_g a_1a_2\ldots a_n\Omega$ lives in H°. That vector can be rewritten as

$$(u_ga_1u_g^*)(u_ga_2u_g^*)\ldots(u_ga_nu_g^*)u_g\Omega.$$

We have $u_ga_iu_g^* \in \mathcal{A}^\chi(g\mathcal{O}_i)$ by Lemma 2, and so it suffices to argue that $u_g\Omega \in H^\circ$.

Pick an element $h \in \text{Möb}(M)$ such that $\pi_2(h) = \pi_2(g)$, which is possible because the map $\text{Möb}(M) \to \text{Möb}(M/\mathcal{F}_2)$ is surjective. Letting $g_0 := gh^{-1}$ we have then written g as

$$g = g_0h \qquad \text{with} \qquad g_0 \in \ker(\pi_2),\ \ h \in \text{Möb}(M).$$

Since $u_g\Omega = u_{g_0}u_h\Omega = u_{g_0}\Omega$ (recall that Ω is Möbius-invariant), we have reduced our problem to showing that $u_{g_0}\Omega \in H^\circ$ for $g_0 \in \ker(\pi_2)$.

At this point, recall from Lemma 1 that π_1 exhibits $\ker(\pi_2)$ as the universal cover of $\mathrm{Diff}_+(M/\mathcal{F}_1)$. To simplify our notation, let us temporarily define

$$\begin{aligned} &S^1 := M/\mathcal{F}_1 \\ &\mathrm{Diff}_+(S^1) := \mathrm{Diff}_+(M/\mathcal{F}_1) \\ &\widetilde{\mathrm{Diff}}_+(S^1) := \ker\big(\pi_2 : \mathrm{Conf}(M) \to \mathrm{Diff}(M/\mathcal{F}_2)\big). \end{aligned}$$

For every interval $I \subset S^1$, let us write $\mathrm{Diff}_0(I)$ for the subgroup of $\mathrm{Diff}_+(S^1)$ that fixes the complement of I pointwise. The groups $\mathrm{Diff}_0(I)$ being contractible, we can lift them canonically to subgroups of $\widetilde{\mathrm{Diff}}_+(S^1)$. It is not too difficult to see that those subgroups generate a neighborhood of the identity in $\widetilde{\mathrm{Diff}}_+(S^1)$ and that they therefore generate the whole $\widetilde{\mathrm{Diff}}_+(S^1)$. We can then rewrite our element g_0 as the product

$$g_0 = g_1 \dots g_n$$

of elements $g_i \in \mathrm{Diff}_0(I_i)$.

Let $\mathcal{O}_i$ be a double cone such that $p_1(\mathcal{O}_i) = I_i$, and let $\mathcal{O}_i'$ be its causal complement. The map g_i fixes $\mathcal{O}_i'$ pointwise and so by the covariance axiom, u_{g_i} commutes with $\mathcal{A}(\mathcal{O}_i')$. By Haag duality, it then follows that $u_{g_i} \in \mathcal{A}(\mathcal{O}_i)$. The above reasoning holds for every $\tilde{\mathcal{O}}$ such that $p_1(\tilde{\mathcal{O}}) = I_i$, therefore

$$u_{g_i} \in \bigcap_{\substack{\text{double cones } \tilde{\mathcal{O}} \text{ s.t.} \\ p_1(\tilde{\mathcal{O}}) = p_1(\mathcal{O}_i)}} \mathcal{A}(\tilde{\mathcal{O}}) \;=\; \mathcal{A}^\chi(\mathcal{O}_i).$$

Finally, we get our desired equation $u_{g_0}\Omega = u_{g_1} \dots u_{g_n}\Omega \in H^\circ$. □

Lemma 4 *If $g \in \mathrm{Conf}(M)$ is in the kernel of π_1, then u_g commutes with $\mathcal{A}^\chi(\mathcal{O})$.*

Proof: Recall that $\ker(\pi_1)$ is the universal cover of $\mathrm{Diff}_+(M/\mathcal{F}_2)$ and that every element $g \in \ker(\pi_1)$ can be written as a product

$$g = g_1 \dots g_n$$

of elements g_i supported in intervals $I_i \subset M/\mathcal{F}_2$.

Let $\mathcal{O}_i$ be double cones with the property that $p_1(\mathcal{O}_i) = p_1(\mathcal{O})$ and $p_2(\mathcal{O}_i) \cap I_i = \emptyset$. Then g_i fixes $\mathcal{O}_i$ pointwise, and so u_{g_i} commutes with $\mathcal{A}^\chi(\mathcal{O}_i) = \mathcal{A}^\chi(\mathcal{O})$. The product $u_g = u_{g_1} \dots u_{g_n}$ therefore also commutes with $\mathcal{A}^\chi(\mathcal{O})$. □

Lemma 5 *If $g_1, g_2 \in \mathrm{Conf}(M)$ are in the preimage of $\mathrm{M\ddot{o}b}(M/\mathcal{F}_2)$ and are such that $\pi_1(g_1) = \pi_1(g_2)$, then*

$$u_{g_1}|_{H^\chi} \;=\; u_{g_2}|_{H^\chi}.$$

Proof: Let $g := g_1 g_2^{-1}$. We need to show that u_g acts trivially on H^χ. Clearly, it is enough to show that it acts trivially on vectors of the form $a_1 \dots a_n\Omega$, $a_i \in \mathcal{A}^\chi(\mathcal{O}_i)$. Since $g \in \mathrm{M\ddot{o}b}(M)$, u_g fixes Ω, and so we have

$$\begin{aligned} u_g a_1 \dots a_n \Omega &= (u_g a_1 u_g^*)(u_g a_2 u_g^*) \dots (u_g a_n u_g^*) u_g \Omega \\ &= (u_g a_1 u_g^*)(u_g a_2 u_g^*) \dots (u_g a_n u_g^*) \Omega. \end{aligned}$$

Since $g \in \ker(\pi_1)$, the latter is then equal to $a_1 \dots a_n\Omega$ by Lemma 4. □

→ end of technical lemmas

We have seen how, given a weak conformal field theory $(H, \Omega, \mathcal{A})$, one can define its *associated chiral conformal field theory* $(H^\chi, \Omega^\chi, \mathcal{A}^\chi)$. By exchanging the roles of $\mathcal{F}_1$ and $\mathcal{F}_2$, one can similarly define the *associated antichiral conformal field theory* $(H^{\bar\chi}, \Omega^{\bar\chi}, \mathcal{A}^{\bar\chi})$. These two sub-theories commute inside the big theory:

Lemma *For any double cones* $\mathcal{O}_1$ *and* $\mathcal{O}_2$*, the algebras* $\mathcal{A}^\chi(\mathcal{O}_1)$ *and* $\mathcal{A}^{\bar\chi}(\mathcal{O}_2)$ *commute inside* $B(H)$.

Proof: Recall that $p_1 : M \to M/\mathcal{F}_1$ and $p_2 : M \to M/\mathcal{F}_2$ denote the two projections.
Pick $\tilde{\mathcal{O}}_1$ so that

$$p_1(\tilde{\mathcal{O}}_1) = p_1(\mathcal{O}_1) \qquad \text{and} \qquad p_2(\tilde{\mathcal{O}}_1) \cap p_2(\mathcal{O}_2) = \emptyset.$$

Then $p_2(\mathcal{O}_2) \subset p_2(\tilde{\mathcal{O}}_1')$.
Pick $\tilde{\mathcal{O}}_2$ so that

$$p_2(\tilde{\mathcal{O}}_2) = p_2(\mathcal{O}_2) \qquad \text{and} \qquad \tilde{\mathcal{O}}_2 \subset \tilde{\mathcal{O}}_1'.$$

Then

$$\mathcal{A}^{\bar\chi}(\mathcal{O}_2) = \mathcal{A}^{\bar\chi}(\tilde{\mathcal{O}}_2) \subseteq \mathcal{A}(\tilde{\mathcal{O}}_2) \subseteq \mathcal{A}(\tilde{\mathcal{O}}_1') \subseteq \mathcal{A}(\tilde{\mathcal{O}}_1)' \subseteq \mathcal{A}^\chi(\tilde{\mathcal{O}}_1)' = \mathcal{A}^\chi(\mathcal{O}_1)'.$$

□

Let us now assume that $\mathcal{A} = (H, \Omega, \mathcal{A})$ is a full CFT. From the above lemma, we see that there is an inclusion

$$\mathcal{A}^\chi \otimes \mathcal{A}^{\bar\chi} \hookrightarrow \mathcal{A}$$

of the tensor product of the two chiral halves of $\mathcal{A}$ into the full theory (note that $\mathcal{A}^\chi \otimes \mathcal{A}^{\bar\chi}$ is typically only a weak CFT and that the above inclusion is typically not an isomorphism).

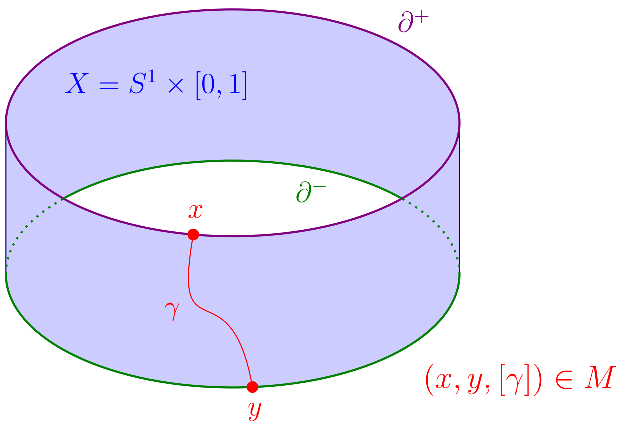
It is a ***basic problem of conformal field theory*** to classify all full CFTs with given chiral halves. Namely:

> Given a chiral CFT $\mathcal{A}_l$ and an antichiral CFT $\mathcal{A}_r$,
> classify all full CFTs $\mathcal{A}$ such that $\mathcal{A}^\chi \cong \mathcal{A}_l$ and $\mathcal{A}^{\bar\chi} \cong \mathcal{A}_r$.

***Digression*: Points in space-time as homotopy classes of paths.**

Recall that our space-time manifold M was defined to be $S^1 \times \mathbb{R}$. In this section, we provide an alternative, topological construction of M and of its foliations $\mathcal{F}_1$ and $\mathcal{F}_2$.

Consider the manifold $X := S^1 \times [0, 1]$. Here, the factor S^1 should be thought of as "geometric" and the factor $[0, 1]$ should be thought of as "topological". We then define M to be the following space: its elements are triples $(x, y, [\gamma])$ where x is a point on $S^1 \times \{1\}$, y is a point on $S^1 \times \{0\}$, and $[\gamma]$ an isotopy class (with fixed end points) of paths $\gamma : [0,1] \to S^1 \times [0,1]$ from x to y.



The fibers of the projection map $p_1 : (x, y, [\gamma]) \mapsto x$ define the foliation $\mathcal{F}_1$ on M, and the fibers of $p_2 : (x, y, [\gamma]) \mapsto y$ define the foliation $\mathcal{F}_2$. In that way, the upper boundary ∂^+ of X gets naturally identified with $M/\mathcal{F}_1$ and its lower boundary ∂^- gets naturally identified with $M/\mathcal{F}_2$.

The group of diffeomorphisms of X (not necessarily orientation preserving) that send ∂^+ to ∂^+ and ∂^- to ∂^-, modulo isotopy in the bulk, is canonically isomorphic to $\mathrm{Conf}(M)$. There is also a description of $\text{Möb}(M)$ along the same lines. It is the group of diffeomorphisms of the 3-manifold $\mathbb{D}^2 \times [0,1]$ that restrict to conformal automorphisms of $\mathbb{D}^2 \times \{1\}$ and of $\mathbb{D}^2 \times \{0\}$ (not necessarily orientation preserving), modulo isotopy in the bulk.

Conformal nets = chiral *CFT*s

So far, a chiral CFT has been defined as a theory on $M = S^1 \times \mathbb{R}$ satisfying certain conditions. But really, a chiral CFT is a theory that lives on S^1. Actually, people who work in that area don't call these things "chiral CFTs". Instead, they call them *conformal nets*. Recall that $\text{Möb}(S^1) := \mathrm{PGL}_2(\mathbb{R}) \subset \mathrm{Diff}(S^1)$ denotes the subgroup of Möbius transformation of S^1.

Let us agree that, from now on, an *interval* of S^1 will always mean an open non-empty interval with the property that its complement is not a single point (and of course, S^1 itself is not an interval).

Definition: A **Conformal net** consists of:

- A Hilbert space H, called the *state space* or *vacuum sector* of the chiral CFT.
- A continuous projective action of $\mathrm{Diff}(S^1)$ on H that restricts to an honest action of $\text{Möb}(S^1)$. The orientation preserving diffeomorphisms acts by projective unitaries while the orientation reversing diffeomorphisms acts by projective antiunitaries.
- For every interval $I \subset S^1$, a von Neumann algebra $\mathcal{A}(I)$.

- A vector $\Omega \in H$, called the *vacuum vector*.

The above pieces of data are subject to the following axioms:

- If $I_1 \subseteq I_2$, then $\mathcal{A}(I_1) \subseteq \mathcal{A}(I_2)$.
- *(locality)* If $I_1 \cap I_2 = \emptyset$, then $[\mathcal{A}(I_1), \mathcal{A}(I_2)] = 0$.
- $\mathcal{A}(gI) = u_g \mathcal{A}(I) u_g^*$ for every $g \in \mathrm{Diff}(S^1)$
- If g fixes I pointwise, then $[u_g, \mathcal{A}(I)] = 0$.
- *(positive energy)* Let $r_t : S^1 \to S^1$ denote rotation by angle t and let R_t be the corresponding unitary operator on H. Then there exists an unbounded *positive* operator L_0 such that $R_t = e^{itL_0}$.
- Ω is cyclic for the joint action of all the algebras $\mathcal{A}(I)$ and it spans the $\mathrm{M\ddot{o}b}(S^1)$ fixed points of H that is, $H^{\mathrm{M\ddot{o}b}(S^1)} = \mathbb{R}\Omega$.

The following is a slight weakening of the notion of conformal net:

Definition: A *Möbius covariant conformal net* is what one gets if one drops any reference to $\mathrm{Diff}(S^1)$ in the above definition, and only keeps the Möbius group action.

Warning: In the literature, the term "conformal net" is also commonly used for what I've been calling a "Möbius covariant conformal net", in which case the term "diffeomorphism covariant conformal net" will get used for the other notion.

In my opinion, the notion of a Möbius covariant conformal net is not a very important one: the interesting Möbius covariant conformal nets tend to also be diffeomorphism covariant, and the few that are known not to be can be considered to be 'pathological' (they are obtained by certain infinite tensor product constructions).

However, the notion of *morphism* of Möbius covariant conformal nets is important! There exist many important morphisms between conformal nets that are only Möbius covariant. For example, the inclusion of the trivial conformal net $(H = \mathbb{C}, \Omega = 1, \mathcal{A}(I) = \mathbb{C})$ into a non-trivial conformal net $(H, \Omega, \mathcal{A})$ is always Möbius covariant, but never diffeomorphism covariant. Another example of a morphism that is only Möbius covariant (now in the 2-dimensional context) is the inclusion $\mathcal{A}^\chi \hookrightarrow \mathcal{A}$ of the associated chiral CFT into a given full CFT.

The following result is our first general structure theorem about conformal nets:

Theorem (Reeh-Schlieder) *Let $(H, \Omega, \mathcal{A})$ be a Möbius covariant conformal net. Then for any interval $I \subset S^1$, the vacuum vector Ω is cyclic for the action of $\mathcal{A}(I)$ on H.*

Proof: We'll prove that $\mathcal{A}(I)\Omega$ is dense by showing that its orthogonal complement is trivial.

Let $J \subset S^1$ be an interval whose length is smaller than that of I. We claim that the following implication holds:

$$\xi \perp \mathcal{A}(I)\Omega \quad \Rightarrow \quad \xi \perp \mathcal{A}(J)\Omega. \tag{2}$$

Given a vector $\xi \in H$ that is orthogonal to $\mathcal{A}(I)\Omega$ and an element $a \in \mathcal{A}(J)$, we wish to show that $\langle \xi \mid a\Omega \rangle = 0$. Consider the following function:

$$f(t) := \big\langle \xi \,\big|\, \underbrace{R_t a R_{-t}}_{\in \mathcal{A}(r_t(J))} \Omega \big\rangle = \big\langle \xi \,\big|\, R_t a \Omega \big\rangle = \big\langle \xi \,\big|\, e^{itL_0} a \Omega \big\rangle.$$

Since L_0 is a positive operator, the last expression for f can be evaluated for any $t \in \mathbb{C}$ with nonnegative imaginary part. It is analytic for $\Im\mathrm{m}(t) > 0$ and continuous for $\Im\mathrm{m}(t) \geq 0$. For more details on this argument, see the lemmas (✠) after the proof. Moreover, it vanishes identically on a certain subinterval of $\mathbb{R}$ (namely, those values of t for which $r_t(J) \subset I$). By the Schwarz reflection principle and the fact that non-zero analytic functions cannot vanish identically on intervals, we conclude that $f \equiv 0$—see (✠) for a complete proof. Finally, evaluating f at $t = 0$ yields our desired relation $\langle \xi \mid a\Omega \rangle = 0$.

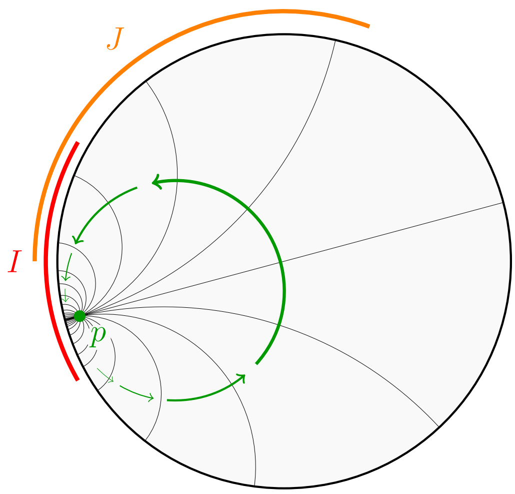
Let now J be any interval not containing I. We claim that, once again,

$$\xi \perp \mathcal{A}(I)\Omega \quad \Rightarrow \quad \xi \perp \mathcal{A}(J)\Omega \tag{3}$$

holds. The key step is to note that there exists a conjugate subgroup $\{g r_t g^{-1}\}_{t \in \mathbb{R}}$ of $\{r_t\} \subset \mathrm{M\ddot{o}b}(S^1)$ with which J can be "rotated" into I. Namely,

$$\exists\, g \in \mathrm{M\ddot{o}b}(S^1) \quad \text{such that} \quad g r_t g^{-1}(J) \subset I \quad \text{for some } t \in \mathbb{R}.$$

To construct such a subgroup of $\mathrm{M\ddot{o}b}(S^1)$, one proceeds as follows. Identify S^1 with the boundary of the Poincaré disc (which is a model of the hyperbolic plane). Pick a point $x \in I$ not in J, and let p be a point in the interior of the disc, very close to x. Then the group of hyperbolic rotations fixing p has all the desired properties:



The argument that was used for (2) can now be easily adapted to prove (3): just replace L_0 everywhere by $u_g L_0 u_g^*$.

We are now ready for the main argument of this proof. Recall that we are trying to show that

$$\xi \perp \mathcal{A}(I)\Omega \quad \Rightarrow \quad \xi = 0.$$

Since Ω is cyclic for the joint actions of all the algebras, it is enough to show that

$$\langle \xi \,|\, a_1 a_2 \dots a_n \Omega \rangle = 0$$

for any choice of elements $a_i \in \mathcal{A}(I_i)$ and of intervals I_i. Fix an interval $J \subset S^1$ that is not contained in I, and whose length is bigger than that of all the I_i's. By (3), we know that $\xi \perp \mathcal{A}(J)\Omega$. Now consider the expression

$$\begin{aligned} f := \big\langle \xi \,\big|\, (R_{t_1} a_1 R_{-t_1}) \dots (R_{t_n} a_n R_{-t_n})\Omega \big\rangle &= \big\langle \xi \,\big|\, e^{it_1 L_0} a_1 e^{i(t_2 - t_1)L_0} a_2 \dots e^{i(t_n - t_{n-1})L_0} a_n \Omega \big\rangle \\ &= \big\langle \xi \,\big|\, e^{iz_1 L_0} a_1 e^{iz_2 L_0} a_2 \dots e^{iz_n L_0} a_n \Omega \big\rangle \end{aligned}$$

as a function of $z_i := t_i - t_{i-1}$. By construction, there exist small intervals $K_i \subset \mathbb{R}$ such that

$$(z_i \in K_i,\ \forall i) \quad \Rightarrow \quad f(z_1, \dots, z_n) = 0.$$

Namely, that happens when the rotation angles t_i are such that $r_{t_i}(I_i) \subset J$ for every i. Applying the same analytic continuation argument used in (2) to f viewed as a function of just z_1, we learn that

$$(z_2 \in K_2, \dots, z_n \in K_n) \quad \Rightarrow \quad f(z_1, \dots, z_n) = 0.$$

By the same argument applied to f as a function of just z_2, we get

$$(z_3 \in K_3, \dots, z_n \in K_n) \quad \Rightarrow \quad f(z_1, \dots, z_n) = 0.$$

Eventually, after n steps of the above procedure, we learn that $f \equiv 0$. Finally, plugging in $z_1 = z_2 = \dots = 0$ into the definition of f, we see that $\langle \xi \,|\, a_1 a_2 \dots a_n \Omega \rangle = 0$. □

The main argument of the above proof was a functional analytical version of the notion of analytic continuation. Here is some background material that we didn't spell out above:

(✠) **Lemma** *Let L be an unbounded positive self-adjoint operator on some Hilbert space H. Then the function $t \mapsto e^{itL}$ is analytic for $\Im\mathrm{m}(t) > 0$ with respect to the norm topology on $B(H)$.*

Proof: By the spectral theorem, L is unitarily equivalent to the operator $m_f : L^2(X) \to L^2(X)$ of multiplication by some (unbounded) function $f : X \to \mathbb{R}_{\geq 0}$ on some measure space X. The Banach space $L^\infty(X)$ embeds isometrically into $B(H)$, and so the question reduces to the analyticity of the function

$$\begin{array}{rcl} \mathbb{C}_{\Im\mathrm{m}(z)>0} & \to & L^\infty(X) \\ t & \mapsto & e^{itf} \end{array}$$

with respect to the sup-norm on $L^\infty(X)$.

Thus, we have to show that $\lim\limits_{h\to 0}\frac{e^{i(t+h)f}-e^{itf}}{h}$ exists in the sup-norm for any $t\in\mathbb{C}_{\Im\mathrm{m}(z)>0}$, and that ife^{itf} is sup-norm continuous in t. This is indeed the case because the limits

$$\frac{e^{i(t+h)x}-e^{itx}}{h} \xrightarrow{h\to 0} ixe^{itx} \qquad \text{and} \qquad ixe^{i(t+h)x} \xrightarrow{h\to 0} ixe^{itx}$$

are uniform on $[0,\infty)$. □

(✠) **Lemma** *Let L be an unbounded positive operator on H, and let $\eta\in H$ be a vector in the Hilbert space. Then the function $t\mapsto e^{itL}\eta$ is continuous for $\Im\mathrm{m}(t)\geq 0$ with respect to the norm topology on H.*

Warning: The function $t\mapsto e^{itL}$ is typically not norm-continuous on $\mathbb{C}_{\Im\mathrm{m}(z)\geq 0}$!

Proof: Once again, by the spectral theorem, we can replace H by $L^2(X)$, and L by some multiplication operator m_f. The question then reduces to the continuity of

$$\begin{array}{rcl}\mathbb{C}_{\Im\mathrm{m}(z)\geq 0} & \to & L^2(X)\\ t & \mapsto & e^{itf}\eta\end{array}$$

with respect to the L^2-norm. We will show the continuity of the above function by arguing that for any $\epsilon>0$, it is ϵ-close to a continuous function.

We assume without loss of generality that $\|\eta\|=1$. The sequence of subspaces $X_n:=\{x\in X\,|\,f(x)<n\}$ exhausts X, so we may chose $n\in\mathbb{N}$ so that $\|\eta-\eta|_{X_n}\|<\epsilon$. Here, $\eta|_{X_n}$ denotes the function given by η on X_n and zero on the rest. Since f is bounded on X_n, it is easy to see that

$$\begin{array}{rcccl}\mathbb{C}_{\Im\mathrm{m}(z)\geq 0} & \to & L^2(X_n) & \hookrightarrow & L^2(X)\\ t & \mapsto & e^{itf}\eta & \mapsto & e^{itf}\eta|_{X_n}\end{array}$$

is analytic, and in particular continuous. To finish the argument, we note that $\|e^{itf}\|\leq 1$, and so $e^{itf}\eta|_{X_n}$ is indeed ϵ-close $e^{itf}\eta$. □

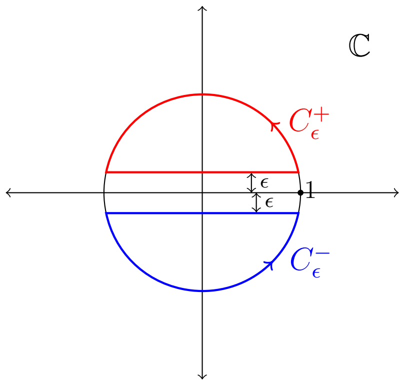
The following consequence of the Schwarz reflection principle was also used in the proof of the Reeh-Schlieder theorem:

(✠) **Lemma** *Let $f:\mathbb{C}_{\Im\mathrm{m}(z)\geq 0}\to\mathbb{C}$ be a continuous function whose restriction to $\mathbb{C}_{\Im\mathrm{m}(z)>0}$ is analytic, and whose restriction to $[-1,1]$ is identically zero. Then $f\equiv 0$.*

Proof: Consider the function

$$\hat{f}\,:\,\mathbb{C}\setminus\big((-\infty,-1)\cup(1,\infty)\big)\ \longrightarrow\ \mathbb{C}$$

given by $f(z)$ on the upper half plane and $\overline{f(\bar{z})}$ on the lower half plane. It is clearly continuous and holomorphic away from $[-1,1]$. We claim that $\hat{f}$ is also holomorphic on the interval $(-1, 1)$. To see that, consider the following contours in $\mathbb{C}$:



By Cauchy's residue formula, we have

$$\hat{f}(z) = \frac{1}{2\pi i} \oint_{C_\epsilon^+ \cup C_\epsilon^-} \frac{\hat{f}(w)}{w - z} dw$$

for any $z \in \mathbb{D}$ such that $|\Im\mathrm{m}(z)| > \epsilon$. Taking the limit as $\epsilon \to 0$ and noting that the contributions along the real axis cancel each other, we get

$$\hat{f}(z) = \frac{1}{2\pi i} \oint_{\partial \mathbb{D}} \frac{\hat{f}(w)}{w - z} dw$$

for any $z \in \mathbb{D}$ such that $\Im\mathrm{m}(z) \neq 0$. By continuity, the last formula then also holds for z on the real axis. Finally, the right hand side is analytic for every $z \in \mathbb{D}$ (including $\Im\mathrm{m}(z) = 0$) because each of the functions $\frac{\hat{f}(w)}{w-z}$ is.

The function $\hat{f}$ is analytic and vanishes on a whole interval: it is therefore identically zero, and the same must then also hold for f (by continuity). $\square$

The WZW models

The WZW models form an important family of full CFTs. They are parametrized by pairs (G, k), where G is a <u>c</u>ompact, <u>s</u>imple, <u>c</u>onnected, <u>s</u>imply <u>c</u>onnected (abbreviated hereafter 'cscsc') Lie group, and k is a positive integer, commonly referred to as the "level" (one can also go beyond simply connected Lie group, but we shall not discuss those models here). The *chiral WZW models* are the chiral CFTs associated to the above full CFTs. To avoid possible confusions, we shall call the latter the *full WZW models*.

Our general approach will be as follows: we'll first construct the chiral WZW models, prove various theorems about them, and only at the very end will we then use them to construct the full WZW models.

Throughout these notes, we'll emphasize the case $G = SU(2)$. We'll formulate things in full generality when it doesn't cost much to do so... but *only* then. Therefore, quite often, we'll present the case $SU(2)$ first, and only quickly mention what needs to be modified to take care of general cscsc groups G, but without going too much in depth.

Let:

$$\begin{aligned} \mathfrak{g} &:= \text{ the complexified Lie algebra of } G \\ L\mathfrak{g} &:= \mathcal{C}^\infty(S^1,\mathfrak{g}) \quad \text{with bracket defined pointwise} \\ \widetilde{L\mathfrak{g}}_k &:= L\mathfrak{g}\oplus\mathbb{C} \quad \text{with bracket given by} \end{aligned}$$

$$\Big[(f,a),(g,a')\Big]_k := \Big([f,g],\ \tfrac{k}{2\pi i}\cdot\int_{S^1}\langle f,dg\rangle\Big)$$

Here, $\langle\ ,\ \rangle : \mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ is the inner product on $\mathfrak{g}$ given as follows. For $G = SU(2)$, we have $\mathfrak{g} = \mathfrak{sl}(2)$ and $\langle\ ,\ \rangle$ is given by the formula $\langle X,Y\rangle := -tr(XY)$, equivalently, $\big\langle \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right), \left(\begin{smallmatrix} a' & b' \\ c' & d'\end{smallmatrix}\right)\big\rangle = -(aa' + bc' + cb' + dd')$. The minus in the formula ensures that this inner product is positive definite when restricted to $\mathfrak{su}(2)\subset\mathfrak{sl}(2)$.

For the case of general cscsc Lie group G, one defines $\langle\ ,\ \rangle$ to be the smallest $\mathfrak{g}$-invariant inner product on $\mathfrak{g}$ whose restriction to any $\mathfrak{sl}(2)\subset\mathfrak{g}$ is a positive integer multiple of $(X,Y)\mapsto -tr(XY)$. This inner product is the negative of what is traditionally called the "basic inner product" (the latter is negative definite on the Lie algebra of G, which is why we don't like it and prefer to work with its opposite). Note that we have used the word "inner product" despite the fact that it is bilinear as opposed to sesquilinear.

Here, a bilinear inner product $\langle\ ,\ \rangle : \mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ is called *$\mathfrak{g}$-invariant* if it satisfies

$$\langle [X,Y],Z\rangle = \langle X,[Y,Z]\rangle$$

for all $X,Y,Z\in\mathfrak{g}$. The terminology attached to that strange looking formula is justified by the following fact, which we leave as an exercise:

Exercise: Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ (complexified or not). Prove that a bilinear inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ satisfies $\langle [X,Y],Z\rangle = \langle X,[Y,Z]\rangle$ for all $X,Y,Z\in\mathfrak{g}$ iff it satisfies $\langle \mathrm{Ad}(g)X,\mathrm{Ad}(g)Y\rangle = \langle X,Y\rangle$ for all $X,Y\in\mathfrak{g}$ and $g\in G$.
Hints: To show $\Leftarrow$, replace g by e^{tZ} and differentiate with respect to t. For the other implication $\Rightarrow$, show that $\langle \mathrm{Ad}(e^{tZ})X,\mathrm{Ad}(e^{tZ})Y\rangle$ is constant by arguing that its derivative is zero. Then use the fact that G is connected to write any element as a product of exponentials.

Note: Our inner product on $\mathfrak{g}$ is defined completely invariantly. As a consequence, it is invariant under *all* automorphisms of $\mathfrak{g}$, not just the inner ones:

$$\big\langle \alpha(X),\alpha(Y)\big\rangle = \langle X,Y\rangle \qquad \forall\alpha\in\mathrm{Aut}(\mathfrak{g}).$$

Before going on, let us check that $[\ ,\ ]_k$ is indeed a Lie bracket. Antisymmetry boils down to

$$\int_{S^1}\langle f,dg\rangle = -\int_{S^1}\langle df,g\rangle,$$

which is just integration by parts. In order to verify the Jacobi identity, we have to check that the last term in the expression

$$\sum_{\substack{\swarrow\, 3\, \nwarrow \\ 1 \,\to\, 2}} \big[\big[(f_1,a_1),(f_2,a_2)\big]_k,(f_3,a_3)\big]_k = \Bigg(\sum_{\substack{\swarrow\, 3\, \nwarrow \\ 1 \,\to\, 2}}[[f_1,f_2],f_3],\ \tfrac{k}{2\pi i}\cdot\sum_{\substack{\swarrow\, 3\, \nwarrow \\ 1 \,\to\, 2}}\int_{S^1}\langle [f_1,f_2],df_3\rangle\Bigg)$$

vanishes. This is indeed the case since:

$$\underbrace{\int_{S^1}\langle[f_1,f_2],df_3\rangle}_{} + \int_{S^1}\langle[f_2,f_3],df_1\rangle + \int_{S^1}\langle[f_3,f_1],df_2\rangle$$

$$\underset{\text{by parts}}{=} -\Big(\int_{S^1}\langle[df_1,f_2],f_3\rangle + \int_{S^1}\langle[f_1,df_2],f_3\rangle\Big).$$

by $\mathfrak{g}$-invariance of $\langle\ ,\ \rangle$

A few notions of Lie algebra cohomology. Given a Lie algebra L, a bilinear function $c : L\times L\to\mathbb{C}$ is called a *2-cocycle* if the operation

$$\big[(f,a),(g,a')\big]_c := \big([f,g],c(f,g)\big) \tag{4}$$

on $L\oplus\mathbb{C}$ satisfies the axioms of a Lie bracket. Concretely, this translates into these two conditions for c:

$$c(f,g) = -c(g,f) \qquad \text{and} \qquad c([f,g],h)+c([g,h],f)+c([h,f],g)=0$$

for every $f,g,h\in L$.

The resulting Lie algebra $\widetilde{L}_c := (L\oplus\mathbb{C},[\ ,\ ]_c)$ is called a *central extension* of L by $\mathbb{C}$, and fits into the following short exact sequence of Lie algebras:

$$0\longrightarrow\mathbb{C}\longrightarrow\widetilde{L}_c\longrightarrow L\longrightarrow 0.$$

The fact that a and a' do not appear in the right hand side of (4) reflects the fact that $\mathbb{C}$ is in the center of $\widetilde{L}_c$, hence the name *central* extension.

Two central exentsions $\widetilde{L}_c$ and $\widetilde{L}_{c'}$ corresponding to cocycles c and c' are said to be isomorphic if there exists a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{C} & \longrightarrow & \widetilde{L}_c & \longrightarrow & L & \longrightarrow & 0\\
 & & {\scriptstyle \mathrm{id}_{\mathbb{C}}}\downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}_L} & & \\
0 & \longrightarrow & \mathbb{C} & \longrightarrow & \widetilde{L}_{c'} & \longrightarrow & L & \longrightarrow & 0.
\end{array}$$

where the vertical arrows on the left and on the right are identity maps. Note that this is much stronger than just requiring an isomorphism of short exact sequences (where the left and right vertical maps are allowed to be anything), which is itself much stronger than requiring a mere isomorphism $\widetilde{L}_c\cong\widetilde{L}_{c'}$.

An isomorphism of central extensions $(L\oplus\mathbb{C},[\ ,\ ]_c)\to(L\oplus\mathbb{C},[\ ,\ ]_{c'})$ is always of the form $(f,a)\mapsto(f,a+b(f))$ for some linear functional $b : L\to\mathbb{C}$. The two cocylces c and c' are then related by

$$c'(f,g) = c(f,g)+b([f,g]).$$

Two 2-cocycles that are related by the above relation are called *cohomologous*, and the quotient vector space

$$H^2(L) := \{2\text{-cocylces}\}/f \sim g \text{ if } f \text{ and } g \text{ are cohomologous}$$

is called the *second cohomology group* of L (with coefficients in $\mathbb{C}$). By construction, there is a natural bijection between elements of $H^2(L)$ and isomorphism classes of central extensions of L by $\mathbb{C}$.

In the case when our Lie algebra L is equipped ith a natual topology (such as our example of interest $L = L\mathfrak{g}$), it is natural to restrict attention to *continuous* 2-cocycles, and to the equivalence relation generated by *continuous* isomorphisms of central extensions. The resulting cohomology groups are then called the second continuous cohomology groups of L with coefficients in $\mathbb{C}$, and is denoted $H^2_{cts}(L)$. The following result will not be proved (and also not used) in this class:

Theorem *If $\mathfrak{g}$ is a simple Lie algebra, then its second continuous cohomology group is one dimensional:*

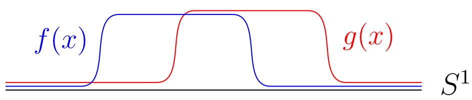
$$H^2_{cts}(L\mathfrak{g}) \cong \mathbb{C},$$

and it is generated by the (equivalence class of the) 2-cocycle $(f, g) \mapsto \frac{1}{2\pi i}\int_{S^1}\langle f, dg\rangle$.

We'll only prove the following watered down version of this theorem:

Lemma *For every $k \neq 0$, the 2-cocycle $\frac{k}{2\pi i}\int_{S^1}\langle f, dg\rangle$ generates a non-trivial central extension of $L\mathfrak{g}$.*

Proof: Pick $X \in \mathfrak{g}$ with $\langle X, X\rangle \neq 0$ and consider two functions $f, g : S^1 \to \mathfrak{g}$ whose graphs look roughly as follows:



They are $\mathbb{R}$-valued as drawn, so multiply them by X to make them $\mathfrak{g}$-valued. The commutator $[(f, 0), (g, 0)]_k = \frac{k}{2\pi i}\int_{S^1}\langle f, dg\rangle$ is non-zero and lies in the center $\mathbb{C}$ of $\widetilde{L\mathfrak{g}}_k$. Now, in a trivial central extension, it is not possible to express a non-zero element of the center as a commutator: Contradiction $\lightning$. Hence $\widetilde{L\mathfrak{g}}_k$ is non-trivial. $\square$

To help us formulate things most cleanly, I'll introduce the following non-standard terminology (here "non-standard terminology" mean that I just invented a name for this):

Definition: A *Lie algebra with unit* is a pair $(L, 1_L)$ consisting of a Lie algebra L, and a distinguished central element $1_L \in Z(L)$.

A *representation of a Lie algebra with unit* $(L, 1_L)$ on a vector space V is a representation of L on V in which 1_L acts as 1. Equivalently, it is a homomorphism of Lie algebras with unit $(L, 1_L) \to (\mathfrak{gl}(V), 1_V)$.

Example: The pair

$$\widetilde{L\mathfrak{g}}_k := \Big((L\mathfrak{g} \oplus \mathbb{C}, [\ ,\ ]_k),\ 1_{\widetilde{L\mathfrak{g}}_k} := (0,1) \Big)$$

is a Lie algebra with unit. Note that $\widetilde{L\mathfrak{g}}_k \to \widetilde{L\mathfrak{g}}_{k'} : (f,a) \mapsto (f, \frac{k'}{k}a)$ is an isomorphism of Lie algebras, but not of Lie algebras *with unit* as it does not send 1 to 1. As it turns out, $\widetilde{L\mathfrak{g}}_k$ and $\widetilde{L\mathfrak{g}}_{k'}$ are not isomorphic as Lie algebras with unit unless $k = \pm k'$.

Lemma *If $k \neq \pm k'$, then $\widetilde{L\mathfrak{g}}_k$ and $\widetilde{L\mathfrak{g}}_{k'}$ are not isomorphic as Lie algebras with unit.*

Note: A unital isomorphism $\widetilde{L\mathfrak{g}}_k \simeq \widetilde{L\mathfrak{g}}_{-k}$ is provided by the map $(f,a) \mapsto (f \circ \bar{\cdot}, a)$, where $\bar{\cdot}$ denotes complex conjugation on S^1.

Proof: An isomorphism $\alpha : \widetilde{L\mathfrak{g}}_k \to \widetilde{L\mathfrak{g}}_{k'}$ of Lie algebras with unit induces an isomorphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \widetilde{L\mathfrak{g}}_k & \longrightarrow & L\mathfrak{g} & \longrightarrow & 0 \\ & & {\scriptstyle \mathrm{id}_{\mathbb{C}}}\downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \bar{\alpha}} & & \\ 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \widetilde{L\mathfrak{g}}_{k'} & \longrightarrow & L\mathfrak{g} & \longrightarrow & 0 \end{array} \tag{5}$$

The manifold S^1 can be identified with the set of maximal ideals of $L\mathfrak{g}$: to a point x corresponds the ideal $\mathfrak{m}_x \subset L\mathfrak{g}$ of functions that vanish at x. Let $\phi : S^1 \to S^1$ be the diffeomorphism uniquely given by $\bar{\alpha}(\mathfrak{m}_x) = \mathfrak{m}_{\phi^{-1}(x)}$. For every point $x \in S^1$, let us also define $\alpha_x \in \mathrm{Aut}(\mathfrak{g})$ by

$$\alpha_x : \mathfrak{g} \simeq L\mathfrak{g}/\mathfrak{m}_{\phi(x)} \xrightarrow{\bar{\alpha}} L\mathfrak{g}/\bar{\alpha}(\mathfrak{m}_{\phi(x)}) = L\mathfrak{g}/\mathfrak{m}_x \simeq \mathfrak{g}.$$

Then the automorphism $\bar{\alpha}$ can then be recovered as $\bar{\alpha}(f)(x) = \alpha_x(f \circ \phi(x))$.

The following calculation now shows that our standard cocycle $c(f,g) = \frac{1}{2\pi i}\int_{S^1}\langle f, dg\rangle$ pulls back to plus or minus itself under the action of $\bar{\alpha}$:

$$\tfrac{1}{2\pi i}\int_{S^1} \big\langle \bar{\alpha}(f), d\bar{\alpha}(g) \big\rangle = \tfrac{1}{2\pi i}\int_{S^1} \langle \alpha_x(f \circ \phi(x)), \alpha_x(dg \circ \phi(x)) \rangle = \tfrac{1}{2\pi i}\int_{S^1} \langle f \circ \phi, dg \circ \phi \rangle = \pm \tfrac{1}{2\pi i}\int_{S^1} \langle f, dg \rangle.$$

More precisely, c pulls back to itself if ϕ is orientation preserving and to minus itself if ϕ is orientation reversing. Let us now define

$$k'' := \begin{cases} k' & \text{if } \phi \text{ is orientation preserving} \\ -k' & \text{if } \phi \text{ is orientation reversing.} \end{cases}$$

By the above result about cocycles, the following is an isomorphism of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \widetilde{L\mathfrak{g}}_{k''} & \longrightarrow & L\mathfrak{g} & \longrightarrow & 0 \\ & & {\scriptstyle \mathrm{id}_{\mathbb{C}}}\downarrow & & \downarrow{\scriptstyle \bar{\alpha}\oplus 1} & & \downarrow{\scriptstyle \bar{\alpha}} & & \\ 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \widetilde{L\mathfrak{g}}_{k'} & \longrightarrow & L\mathfrak{g} & \longrightarrow & 0 \end{array} \tag{6}$$

Composing (5) with the inverse of (6), we see that $(\bar{\alpha}^{-1} \oplus 1) \circ \alpha$ induces an isomorphism of central extensions between $\widetilde{L\mathfrak{g}}_k$ and $\widetilde{L\mathfrak{g}}_{k''}$ (that is, an isomorphism between short exact sequences where both the left and right vertical maps are identity maps). The 2-cocycles kc and $k''c$ are therefore cohomologous. But we have seen in the previous lemma that the map $\lambda \mapsto [\lambda c] : \mathbb{C} \to H^2(L\mathfrak{g})$ is injective. Hence $k = k''$. $\square$

Next step towards the construction of the χWZW model: pick a Representation.

Our next goal is to construct a suitable representation of $\widetilde{L\mathfrak{g}}_k$, the so-called *vacuum representation*. As a first step towards that goal, one defines the subalgebra

$$L\mathfrak{g}_{\geq 0} \subset L\mathfrak{g}$$

as follows. By definition, it is the set all of functions $S^1 \to \mathfrak{g}$ that are boundary values of holomorphic functions on the unit disk $\mathbb{D}$. Equivalently, these are the functions whose Fourier series (= Laurent series) only involves non-negative terms (non-negative powers of z).

By the residue theorem, the 2-cocycle $(f, g) \mapsto \frac{k}{2\pi i} \int_{S^1} \langle f, dg \rangle$ vanishes identically on $L\mathfrak{g}_{\geq 0}$. The subalgebra

$$\big(L\mathfrak{g}_{\geq 0} \oplus \mathbb{C}, [\ ,\ ]_k\big)$$

of $\widetilde{L\mathfrak{g}}_k$ therefore splits as a direct sum of Lie algebras (and in particular, it doesn't depend on k).

Let $\mathbb{C}_0$ denote the trivial 1-dimensional representation of $L\mathfrak{g}_{\geq 0} \oplus \mathbb{C}$, where the action of $L\mathfrak{g}_{\geq 0}$ is identically zero, and $1 \in \mathbb{C}$ acts by 1. We will denote by Ω the standard basis vector of $\mathbb{C}_0$. We then define

$$W_{0,k} := \mathrm{Ind}_{L\mathfrak{g}_{\geq 0} \oplus \mathbb{C}}^{\widetilde{L\mathfrak{g}}_k} \mathbb{C}_0.$$

to be the corresponding *induced representation*. This space can be described as the tensor product of the universal enveloping algebra of $\widetilde{L\mathfrak{g}}_k$ with the representation $\mathbb{C}_0$ over the universal enveloping algebra of $L\mathfrak{g}_{\geq 0} \oplus \mathbb{C}$. More down to earth, $W_{0,k}$ is the set of formal linear combinations of expressions of the form $f_1 f_2 \ldots f_n \Omega$ (physicists would use the notation $|f_1 f_2 \ldots f_n \Omega\rangle$) with $f_i \in \widetilde{L\mathfrak{g}}_k$, modulo the equivalence relation generated by:

The symbol $f_1 f_2 \ldots f_n \Omega$ depends linearly on f_i

- $f_1 f_2 \ldots (\lambda f_i) \ldots f_n \Omega = \lambda \cdot f_1 \ldots f_n \Omega \quad \text{for } \lambda \in \mathbb{C}$
- $f_1 \ldots (f_i' + f_i'') \ldots f_n \Omega = f_1 \ldots f_i' \ldots f_n \Omega + f_1 \ldots f_i'' \ldots f_n \Omega$

The action of $\widetilde{L\mathfrak{g}}_k$ on $W_{0,k}$ is a Lie algebra action

- $f_1 \ldots f_i f_{i+1} \ldots f_n \Omega - f_1 \ldots f_{i+1} f_i \ldots f_n \Omega = f_1 \ldots [f_i, f_{i+1}] \ldots f_n \Omega$

The action of $L\mathfrak{g}_{\geq 0} \oplus \mathbb{C}$ on Ω is prescribed

- $f_1 f_2 \ldots f_n \Omega = 0 \quad \text{if } f_n \in L\mathfrak{g}_{\geq 0}$
- $f_1 f_2 \ldots 1\,\Omega = f_1 f_2 \ldots f_{n-1} \Omega$

Note: By the third and fifth relations, it is sufficient to use $f_i \in L\mathfrak{g} \subset \widetilde{L\mathfrak{g}}_k$ in order to write any element of $W_{0,k}$.

We will now present a couple of lemmas that we'll prove only much later (the proofs are actually rather difficult), and only for the case $G = SU(n)$.

The Lie algebra $\widetilde{L\mathfrak{g}}_k$ is equipped with the following natural $*$-operation:

$$(f, a)^* := (f^*, \bar{a}) \qquad\qquad {\scriptstyle (f \in L\mathfrak{g},\, a \in \mathbb{C})}$$

Here, $f^*(x) = f(x)^*$ for $x \in S^1$, where $*$ is the unique antilinear operation on $\mathfrak{g}$ with the property that $X = -X^*$ iff X is in the Lie algbera of G. (For $G = SU(n)$, this is the usual $*$-operation on $n \times n$ matrices.) Let us call an action ρ of a $*$-Lie algebra L on a vector space W ***unitary*** with respect to some inner product $\langle\ ,\ \rangle$ on W if

$$\langle \rho(X)v, w\rangle = \langle v, \rho(X^*)w\rangle$$

for all $X \in L$, and all $v, w \in W$.

Lemma *The vector space $W_{0,k}$ carries a unique positive semi-definite inner product with respect to which $\langle \Omega, \Omega\rangle = 1$, and the action of $\widetilde{L\mathfrak{g}}_k$ is unitary.*

The existence and positive semi-definiteness are difficult, but the uniqueness part of the statement is not too difficult to prove:

Proof of uniqueness: It is enough to determine the value of expressions of the form $\langle f_1 f_2 \dots f_n \Omega, \Omega\rangle$, $f_i \in L\mathfrak{g}$, since the more general ones $\langle f_1 f_2 \dots f_n \Omega, g_1 g_2 \dots g_m \Omega\rangle$ can easily be reduced to those by bringing the g_i's to the other side.

Let us suppose that, by induction, we have determined the values of $\langle f_1 f_2 \dots f_{n-1}\Omega, \Omega\rangle$ for every $f_1, \dots, f_{n-1} \in L\mathfrak{g}$. We then want to determine

$$\langle f_1 f_2 \dots f_n \Omega, \Omega\rangle.$$

Decompose f_1 as $f_1^+ + f_1^-$ with $f_1^+ \in L\mathfrak{g}_{\geq 0}$ and $f_1^- \in L\mathfrak{g}_{\leq 0}$. (Here, $L\mathfrak{g}_{\leq 0} \subset L\mathfrak{g}$ denotes the functions that extend to holomorphically to $|z| \geq 1$, including ∞.)

Mini-lemma: *If $f \in L\mathfrak{g}_{\leq 0}$, then $f^* \in L\mathfrak{g}_{\geq 0}$.*

Proof: If $F(z)$ is a holomorphic extension of f defined for $|z| \geq 1$, then $F(1/\bar{z})^*$ is a holomorphic extension of f^* defined for $|z| \leq 1$. $\square$

We can then write

$$\begin{aligned}
\big\langle f_1 f_2 \dots f_n \Omega, \Omega\big\rangle &= \big\langle f_1^+ f_2 \dots f_n \Omega, \Omega\big\rangle + \big\langle f_1^- f_2 \dots f_n \Omega, \Omega\big\rangle \\
&= \big\langle f_1^+ f_2 \dots f_n \Omega, \Omega\big\rangle + \big\langle f_2 \dots f_n \Omega, \underbrace{(f_1^-)^* \Omega}_{=0}\big\rangle \\
&= \quad \big\langle [f_1^+, f_2] f_3 f_4 \dots f_n \Omega, \Omega\big\rangle \\
&\quad + \big\langle f_2 [f_1^+, f_3] f_4 \dots f_n \Omega, \Omega\big\rangle \\
&\quad + \big\langle f_2 f_3 [f_1^+, f_4] \dots f_n \Omega, \Omega\big\rangle \\
&\quad + \qquad \cdots\cdots \\
&\quad + \big\langle f_2 f_3 f_4 \dots [f_1^+, f_n] \Omega, \Omega\big\rangle \\
&\quad + \big\langle f_2 f_3 f_4 \ \dots\ f_n \underbrace{f_1^+ \Omega}_{=0}, \Omega\big\rangle
\end{aligned} \tag{7}$$

Those terms have only $n-1$ things acting on Ω and so their value is already known by induction.

$\square$

Let $H_{0,k}$ be the Hilbert space completion of $W_{0,k}/(\text{null-vectors})$.

Lemma *The action of $\widetilde{L\mathfrak{g}}_k$ on $H_{0,k}$ exponentiates to a projective unitary action of the loop group $LG := C^\infty(S^1, G)$.*

(Once again, the proof of this lemma is difficult.)

We are now ready to define the chiral WZW conformal nets (but, unfortunately, only as a Möbius covariant conformal net—we'll need to do more work to construct the projective action of $\mathrm{Diff}(S^1)$):

Definition: The **Chiral WZW Conformal Net** for G at level k is given by:

- The Hilbert space is $H_{0,k} = \left[\text{Hilbert space completion of } \mathrm{Ind}_{L\mathfrak{g}_{\geq 0}\oplus\mathbb{C}}^{\widetilde{L\mathfrak{g}}_k}\mathbb{C}_0/(\text{null-vectors})\right]$
- The action
$$u_g\big(f_1 f_2 \dots f_n \Omega\big) := (g\cdot f_1)(g\cdot f_2)\dots(g\cdot f_n)\Omega$$
of $g \in \mathrm{Möb}(S^1)$ on $f_1 f_2 \dots f_n\Omega \in H_{0,k}$ is expressed in terms of the action
$$\big(g\cdot f\big)(z) := \begin{cases} f(g^{-1}(z)) & \text{if } g \in \mathrm{Möb}_+(S^1) \\ -f(g^{-1}(z))^* & \text{if } g \in \mathrm{Möb}_-(S^1). \end{cases}$$
on elements $f \in L\mathfrak{g}$. In the formula for $g\cdot f$ when $g \in \mathrm{Möb}_-(S^1)$, the $*$ ensures that the action is complex antilinear and that $f \in L\mathfrak{g}_{\geq 0} \Rightarrow g\cdot f \in L\mathfrak{g}_{\geq 0}$. The minus sign is then needed so that $\begin{cases} L\mathfrak{g} \to L\mathfrak{g} \\ f \mapsto g\cdot f \end{cases}$ and $\begin{cases} \widetilde{L\mathfrak{g}}_k \to \widetilde{L\mathfrak{g}}_k \\ (f,a) \mapsto (g\cdot f, \bar{a}) \end{cases}$ be compatible with the Lie brackets.
- The vacuum vector is Ω.
- The local algebras $\mathcal{A}_{G,k}(I)$ are given by
$$\mathcal{A}_{G,k}(I) := \big\{u_\gamma \,\big|\, \gamma \in L_I G\big\}''$$
where $L_I G = \{\gamma \in LG \,|\, \mathrm{supp}(\gamma) \subset I\}$ is the *local loop group*, consisting of all loops supported in I. Here, u_γ is the operator (well defined up to phase) that corresponds to a loop $\gamma \in LG$.

We recall that, in the definition of $\mathcal{A}_{G,k}(I)$, the double prime indicates the operation of taking the von Neumann algebra generated by.

The Lie algebra $L\mathfrak{g}$ comes with the following four subspaces, of which we have already seen some:

$$L\mathfrak{g}_{\geq 0} = \big\{\textstyle\sum_{n\geq 0} X_n z^n \,|\, X_n \in \mathfrak{g}\big\} \qquad L\mathfrak{g}_{> 0} = \big\{\textstyle\sum_{n> 0} X_n z^n \,|\, X_n \in \mathfrak{g}\big\}$$
$$L\mathfrak{g}_{\leq 0} = \big\{\textstyle\sum_{n\leq 0} X_n z^n \,|\, X_n \in \mathfrak{g}\big\} \qquad L\mathfrak{g}_{< 0} = \big\{\textstyle\sum_{n< 0} X_n z^n \,|\, X_n \in \mathfrak{g}\big\}$$

Recall that the vacuum vector Ω is annihilated by the elements of $L\mathfrak{g}_{\geq 0}$. The following lemma should sound therefore feel intuitive:

Lemma *The vectors of the form $f_1 f_2 \dots f_n\Omega$ with $f_i \in L\mathfrak{g}_{<0}$ span a dense subspace of the vacuum sector $H_{0,k}$.*

Proof: By definition, vectors of the form $f_1 f_2 \dots f_n\Omega$ with $f_i \in L\mathfrak{g}$ span a dense subspace of $H_{0,k}$. We argue by induction on n that every such element can be rewritten as a linear combination of vectors with $f_i \in L\mathfrak{g}_{<0}$.

• Case $n = 0$: vacuously satisfied.

• Induction step: We assume that all expressions of length $n-1$ have been dealt with, and consider $f_1 f_2 \dots f_n\Omega$, $f_i \in L\mathfrak{g}$. By the induction hypothesis, we may assume that $f_2, \dots, f_n \in L\mathfrak{g}_{<0}$. Writing f_1 as $f_1^+ + f_1^-$ with $f_1^+ \in L\mathfrak{g}_{\geq 0}$ and $f_1^- \in L\mathfrak{g}_{<0}$, we have

$$\begin{aligned} f_1 f_2 \dots f_n\Omega &= \qquad f_1^+ f_2 \dots f_n\Omega \quad + \quad f_1^- f_2 \dots f_n\Omega \\ &= \begin{bmatrix} \quad [f_1^+, f_2] f_3 f_4 \dots f_n\Omega \\ + f_2 [f_1^+, f_3] f_4 \dots f_n\Omega \\ + f_2 f_3 [f_1^+, f_4] \dots f_n\Omega \\ + \qquad \cdots\cdots \\ + f_2 f_3 f_4 \dots [f_1^+, f_n]\Omega \\ + f_2 f_3 f_4 \ \dots \ f_n \underbrace{f_1^+\,\Omega}_{=0} \end{bmatrix} + f_1^- f_2 \dots f_n\Omega \end{aligned}$$

and so we're done by induction. $\square$

Corollary *The vacuum sector $H_{0,k}$ is topologically spanned (i.e., they span a dense subspace) by vectors*

$$(X_1 z^{-a_1})(X_2 z^{-a_2}) \dots (X_n z^{-a_n})\Omega, \tag{8}$$

with $X_i \in \mathfrak{g}$ and $a_i > 0$.

Proof: By the previous lemma, it is enough to argue that every $f_1 f_2 \dots f_n\Omega$ with $f_i \in L\mathfrak{g}_{<0}$ can be approximated by linear combinations of vectors of the form (8).

Let us assume for the moment that $\lim\limits_{N\to\infty} f_i^N = f_i$ in the $\mathcal{C}^\infty$ topology implies

$$\lim_{N\to\infty} f_1^N f_2^N \dots f_n^N\Omega \;=\; f_1 f_2 \dots f_n\Omega \tag{9}$$

in the Hilbert space. The argument then goes as follows. Approximate each function $f_i = \sum\limits_{n<0} X_{i,n} z^n$ by a finite Fourier series $f_i^N := \sum\limits_{-N<n<0} X_{i,n} z^n$. Since $f_i^N \to f_i$ in the $\mathcal{C}^\infty$ topology, we have $f_1^N f_2^N \dots f_n^N\Omega \;\to\; f_1 f_2 \dots f_n\Omega$ by our assumption (9). Finally, we note that $f_1^N f_2^N \dots f_n^N\Omega$ is a linear combination of vectors of the form (8).

We now turn to the proof of (9). By carefully examining (7), we note that the only operations that get used in the computation of $\langle g_1 g_2 \dots g_n\Omega, h_1 h_2 \dots h_m\Omega\rangle$ are those of differentiation and of projection onto $L\mathfrak{g}_{\geq 0}$. These operations preserve smoothness (recall that a function is smooth iff its Fourier coefficients decay rapidly—it is quite obvious that the operator of projection onto $L\mathfrak{g}_{\geq 0}$ preserves that property) and are continuous in the $\mathcal{C}^\infty$ topology. Therefore, the above inner product depends continuously on the functions g_i and h_i (in the $\mathcal{C}^\infty$ topology). In particular, taking $g_i = h_i = f_i^N$, we get

$$\|f_1^N f_2^N \dots f_n^N \Omega\|^2 \to \|f_1 f_2 \dots f_n \Omega\|^2$$

and taking $g_i = f_i^N$, $h_i = f_i$, we get

$$\langle f_1^N f_2^N \dots f_n^N \Omega, f_1 f_2 \dots f_n \Omega\rangle \to \|f_1 f_2 \dots f_n \Omega\|^2.$$

For convenience, let us write $\xi_N = f_1^N f_2^N \dots f_n^N \Omega$ and $\xi = f_1 f_2 \dots f_n \Omega$. We then have

$$\lim_{N\to\infty} \|\xi_N - \xi\|^2 = \lim_{N\to\infty} \Big(\langle \xi_N, \xi_N\rangle - \langle \xi_N, \xi\rangle - \langle \xi, \xi_N\rangle + \langle \xi, \xi\rangle\Big) = 0.$$

□

The strange minus signs in the above corollary might make one doubt that the positive energy condition is satisfied... but everything is ok:

Lemma *The operator L_0 that generates the action of rotations on $H_{0,k}$ has positive spectrum.*

Proof: Recall that if we let $R_t : H_{0,k} \to H_{0,k}$ be the operator that corresponds to the rotation $r_t : z \mapsto e^{it}z$ then, by definition, we have $R_t = e^{itL_0}$, and thus

$$L_0 = -i\frac{d}{dt}\Big|_{t=0} R_t.$$

Let us also recall that the action of $g \in \text{Möb}_+(S^1)$ is given by

$$u_g\big(f_1 f_2 \dots f_n \Omega\big) := (f_1 \circ g^{-1})(f_2 \circ g^{-1}) \dots (f_n \circ g^{-1})\Omega.$$

We now argue that the spanning set $\{(X_1 z^{-a_1})(X_2 z^{-a_2}) \dots (X_n z^{-a_n})\Omega\}$ constructed in the previous corollary consists of eigenvectors of L_0 with positive eigenvalues. Indeed, we have

$$-i\frac{d}{dt}\Big|_{t=0}\big(z^{-a} \circ r_{-t}\big) = -i\frac{d}{dt}\Big|_{t=0}\big(e^{ita} z^{-a}\big) = a z^{-a},$$

and so

$$\begin{aligned}
&L_0\ (X_1 z^{-a_1})(X_2 z^{-a_2}) \dots (X_n z^{-a_n})\Omega \\
&= -i\tfrac{d}{dt}\big|_{t=0}\ (X_1 z^{-a_1} \circ r_{-t})(X_2 z^{-a_2} \circ r_{-t}) \dots (X_n z^{-a_n} \circ r_{-t})\Omega \\
&= -\,i\tfrac{d}{dt}\big|_{t=0}\ e^{it\sum a_i}(X_1 z^{-a_1})(X_2 z^{-a_2}) \dots (X_n z^{-a_n})\Omega \\
&= \Big[\sum_{i=1}^n a_i\Big](X_1 z^{-a_1})(X_2 z^{-a_2}) \dots (X_n z^{-a_n})\Omega.
\end{aligned}$$

Finally, we note that $\sum a_i \geq 0$. $\square$

Representations of conformal nets.

Our next goal is to construct the irreducible representations of the chiral WZW conformal nets $\mathcal{A}_{G,k}$. But first, some definitions:

Definition: A **representation** (or **sector**) of a conformal net $(H_0, \Omega, \mathcal{A})$ consists of:

- A Hilbert space H
- For every interval $I \subset S^1$, an action $\rho_I : \mathcal{A}(I) \to B(H)$

such that $\rho_I = \rho_J|_{\mathcal{A}(I)}$ whenever $I \subset J$.

The following is a consequence of the above definition:

Lemma *If H is a representation of $\mathcal{A}$, and I_1 and I_2 are two disjoint intervals, then the algebras $\rho_{I_1}(\mathcal{A}(I_1))$ and $\rho_{I_2}(\mathcal{A}(I_2))$ commute inside $B(H)$.*

Proof: If the union of I_1 and I_2 is not dense in S^1, then we may pick an interval $K \subset S^1$ that contains them both. The maps $\rho_{I_i} : \mathcal{A}(I_i) \to B(H)$ factor through $\mathcal{A}(K)$ and because $\mathcal{A}(I_1)$ and $\mathcal{A}(I_2)$ commute in $\mathcal{A}(K)$, so do their images $\rho_K(\mathcal{A}(I_1))$ and $\rho_K(\mathcal{A}(I_2))$ inside $\rho_K(\mathcal{A}(K)) \subset B(H)$.

Let us now assume that $I_1 \cup I_2$ is dense in S^1, namely that $I_2 = I_1'$. We claim that

$$\bigcup_{J \subsetneq I_2} \mathcal{A}(J)$$

is dense in $\mathcal{A}(I_2)$ in the strong operator topology.[5]

Sub-lemma: *The subalgebra $\bigcup_{J \subsetneq I} \mathcal{A}(J)$ is dense in $\mathcal{A}(I)$ in the strong operator topology.*

Proof: The denseness of $\bigcup_{J \subsetneq I} \mathcal{A}(J)$ inside $\mathcal{A}(I)$ is a general fact, but we'll only prove it under the additional assumption (easily verified in all examples of interest) that H_0 is separable. If H_0 is separable, then the unit ball of $B(H_0)$ is separable and metrizable. **Exercise:** If $(\xi_n)_{n\in\mathbb{N}}$ is a sequence of vectors that is dense in the unit ball of H_0, then the metric $d(a,b) := \sum \frac{1}{2^n}\|a(\xi_n) - b(\xi_n)\|$ recovers the strong operator topology on the unit ball of $B(H_0)$.

Let us identify I with $[-1,1]$. Assuming by contradiction that $\bigcup_{r<1} \mathcal{A}[-r,r]$ is not dense in $\mathcal{A}([-1,1])$, there exists an element a_1 in the unit ball of $\mathcal{A}([-1,1])$ whose distance to $\bigcup_{r<1} \mathcal{A}[-r,r]$ is $\epsilon > 0$. Similarly, for every real number $x \in (0,1)$, there is an element a_x in the unit ball of $\mathcal{A}([-x,x])$ that is at the same distance $\epsilon > 0$ from $\bigcup_{r<x} \mathcal{A}[-r,r]$ (use Möbius covariance). This yields an uncountable family of elements $(a_x)_{x\in(0,1)}$ that form a discrete subspace of the unit ball of $B(H_0)$, contradicting the fact that it is separable. $\square$

[5] Recall that, by definition, $a_i \to a$ in the strong operator topology if $a_i(\xi) \to a(\xi)$ for every $\xi \in H$.

The algebra $\mathcal{A}(I_1)$ commutes with $\bigcup \mathcal{A}(J)$ by the first argument and so by continuity it commutes with all of $\mathcal{A}(I_2)$ (this uses the fact that the commutator map $a \mapsto [a, b]$ is continuous for the strong operator topology). $\square$

Construction of representations of $\mathcal{A}_{G,k}$. Let V_λ be an irreducible unitary representation (necessarily finite dimensional) of the Lie algebra $\mathfrak{g}$. That is, we have a Lie algebra homomorphism $\rho_\lambda : \mathfrak{g} \to \mathfrak{gl}(V_\lambda)$ satisfying $\rho_\lambda(X^*) = \rho_\lambda(X)^*$. Here, λ denotes the highest weight of V_λ which, in the case $\mathfrak{g} = \mathfrak{sl}(2)$, is just a element of $\mathbb{Z}_{\geq 0}$.

Let us spend some time to recall the theory of finite dimensional unitary representations of $\mathfrak{sl}(2)$. The Lie algebra $\mathfrak{sl}(2)$ has a basis given by

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and commutation relations

$$[X, Y] = H \qquad [H, X] = 2X \qquad [H, Y] = -2Y.$$

The irreducible unitary representations of $\mathfrak{sl}(2)$ are as indicated in the following figure:

physics name:	"spin 0"	"spin $\frac{1}{2}$"	"spin 1"	"spin $\frac{3}{2}$"	"spin 2"
Highest weight λ:	0	1	2	3	4 ...
Irrep V_λ:					

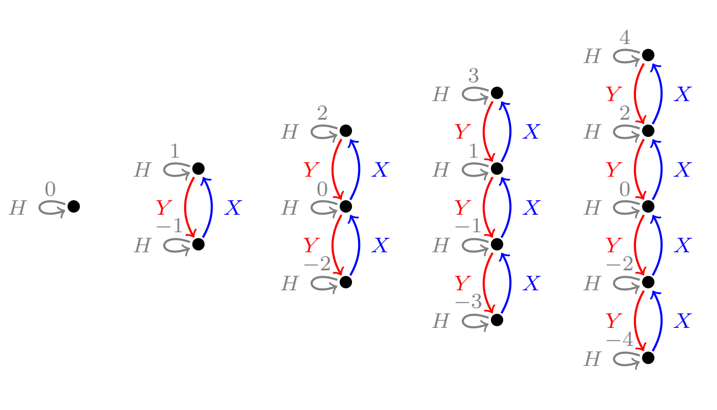


Figure: The irreducible unitary representations of $\mathfrak{sl}(2)$

We have $\dim(V_\lambda) = \lambda + 1$ and each bullet represents a basis element of V_λ. The action of the generators X, Y, H of $\mathfrak{sl}(2)$ are indicated by the arrows between the bullets. If we call v_i the basis element that corresponds to the ith bullet (the top one being v_0) then, for example, a red arrow between the ith and the $(i+1)$st bullet indicates the relation $Y(v_i) = av_{i+1}$ for some non-zero scalar a. If we normalize the basis vectors v_i so that

$\|v_i\| = 1$, and so that the structure constants for X and Y are positive, then the action is given by:

$$X(v_{i+1}) = \sqrt{(i+1)(\lambda - i)}v_i \qquad Y(v_i) = \sqrt{(i+1)(\lambda - i)}v_{i+1}$$
$$H(v_i) = (\lambda - 2i)v_i.$$

The last equation is usually referred to by saying that "v_i has weight $\lambda - 2i$" (the weights are indicated in gray in the picture).

Note that we could get rid of the square roots if we dropped the condition $\|v_i\| = 1$. The price to pay would be that the relation $X^* = Y$ would no longer be visible from the presentation of the action.

The next step in order to construct a representation of $\mathcal{A}_{G,k}$ is to endow V_λ with an action of $L\mathfrak{g}_{\geq 0} \oplus \mathbb{C}$. We do so by letting an element $f : S^1 \to \mathfrak{g}$ of $L\mathfrak{g}_{\geq 0}$ act by $F(0)$, and $1 \in \mathbb{C}$ act by 1.

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d, phantom, "\cap"] & \mathfrak{g} \\ \mathbb{D} \arrow[ur, dashed, "F"'] \end{tikzcd} \qquad \boxed{\text{holomorphic extension of } f}$$

As before, we then consider the induced representation

$$W_{\lambda,k} := \mathrm{Ind}_{L\mathfrak{g}_{\geq 0} \oplus \mathbb{C}}^{\widetilde{L\mathfrak{g}}_k} V_\lambda.$$

We now present a couple of lemmas without proofs:

• **Lemma** *The representation $W_{\lambda,k}$ admits a unique $\widetilde{L\mathfrak{g}}_k$-invariant inner product that agrees given to us on V_λ. Moreover, if $k \in \mathbb{Z}_{\geq 0}$ and if λ is subject to a suitable condition, then this inner product is positive semi-definite.*

If $\mathfrak{g} = \mathfrak{sl}(2)$, then the "suitable condition" is simply the statement that $\lambda \in \{0, 1, \ldots, k\}$. [For general simple Lie algbera $\mathfrak{g}$, the condition is that the inner product $\langle \lambda, \alpha_{max} \rangle$ of λ with the highest root α_{max} should be at most k, where the inner product on $\mathfrak{h}^*$ (the dual of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$) is the dual of the basic inner product. Note that for any given $k \in \mathbb{Z}_{\geq 0}$, there are only finitely many λ's that satisfy that condition.]

Let $H_{\lambda,k}$ denote the Hilbert space completion of $W_{\lambda,k}/(\text{null-vectors})$. [6]

• **Lemma** *The action of $\widetilde{L\mathfrak{g}}_k$ on $W_{\lambda,k}/$*(null-vectors) *exponentiates to a projective unitary action of the loop group LG on the Hilbert space $H_{\lambda,k}$.*

Recall that L_IG denotes the group of loops $S^1 \to G$ whose support is contained in I.

• **Lemma** *The von Neumann algbera generated by L_IG inside $B(H_{\lambda,k})$ is canonically isomorphic to the von Neumann algbera generated by L_IG inside $B(H_{0,k})$. The Hilbert space $H_{\lambda,k}$ is therefore a representation of the chiral WZW conformal net $\mathcal{A}_{G,k}$.*

[6]The case $k = 0$, $\lambda = 0$ is allowed, but the resulting Hilbert space $H_{0,0}$ is not very interesting: it is one dimensional, spanned by Ω. For that reason, one usually restricts to $k \in \mathbb{Z}_{\geq 1}$.

We will now argue that, in the case $\mathfrak{g} = \mathfrak{sl}(2)$, the conditions $k \in \mathbb{Z}_{\geq 0}$ and $\lambda \in \{0, 1, \ldots, k\}$ are *necessary*[7] for the inner product on $W_{\lambda,k}$ to be positive semidefinite. Let us first introduce some useful notation. The elements

$$X(n) := \begin{pmatrix} 0 & z^n \\ 0 & 0 \end{pmatrix} \qquad Y(n) := \begin{pmatrix} 0 & 0 \\ z^n & 0 \end{pmatrix} \qquad H(n) = \begin{pmatrix} z^n & 0 \\ 0 & -z^n \end{pmatrix}$$

span $L\mathfrak{sl}(2)$ topologically, their commutation relations in $\widetilde{L\mathfrak{sl}(2)}_k$ are given by

$$\begin{aligned} [X(n), Y(m)] &= H(n+m) + nk\,\delta_{n+m,0} \\ [H(n), X(m)] &= 2X(n+m) \\ [H(n), Y(m)] &= -2Y(n+m) \\ [H(n), H(m)] &= 2nk\,\delta_{n+m,0} \qquad \text{(all other brackets are zero)} \end{aligned}$$

and the $*$-structure is given by $X(n)^* = Y(-n)$ and $H(n)^* = H(-n)$.
The operators $X(n)$, $Y(n)$, $H(n)$ act as *creation operators* for $n < 0$, and as *annihilation operators* for $n > 0$.

Exercise: Check that the above commutation relations are correct (check also the signs!).

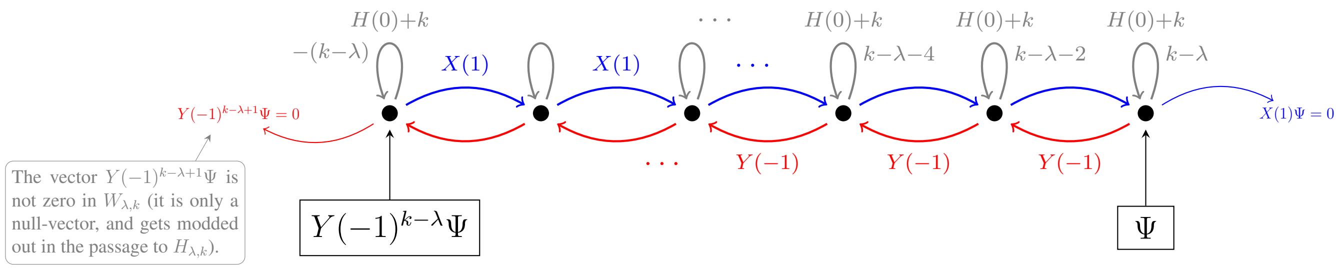
The main observation is that there is an interesting copy of $\mathfrak{sl}(2)$ as a sub-Lie-$*$-algebra inside $\widetilde{L\mathfrak{sl}(2)}_k$, given by

$$X \mapsto X(1) \qquad Y \mapsto Y(-1) \qquad H \mapsto H(0) + k$$

(such a gadget is called an "$\mathfrak{sl}(2)$-triple"). Let us consider the action of that $\mathfrak{sl}(2)$-triple on the lowest weight vector $\Psi \in V_\lambda$ (of weight $-\lambda$). The vector Ψ satisfies

$$(H(0) + k)\Psi = (k - \lambda)\Psi \qquad X(1)\Psi = 0.$$

Given the above information and the classification of unitary representations of $\mathfrak{sl}(2)$ (on vector spaces with *positive definite* inner products), the only possibility for the $\mathfrak{sl}(2)$-subrepresentation generated by Ψ is:



In particular, we see that $k - \lambda$ must be in $\mathbb{Z}_{\geq 0}$. Given that λ was itself in $\mathbb{Z}_{\geq 0}$, the only possibilities this leaves are:

$$\boxed{k \in \mathbb{Z}_{\geq 0}, \qquad \lambda \in \{0, 1, \ldots, k\}}$$

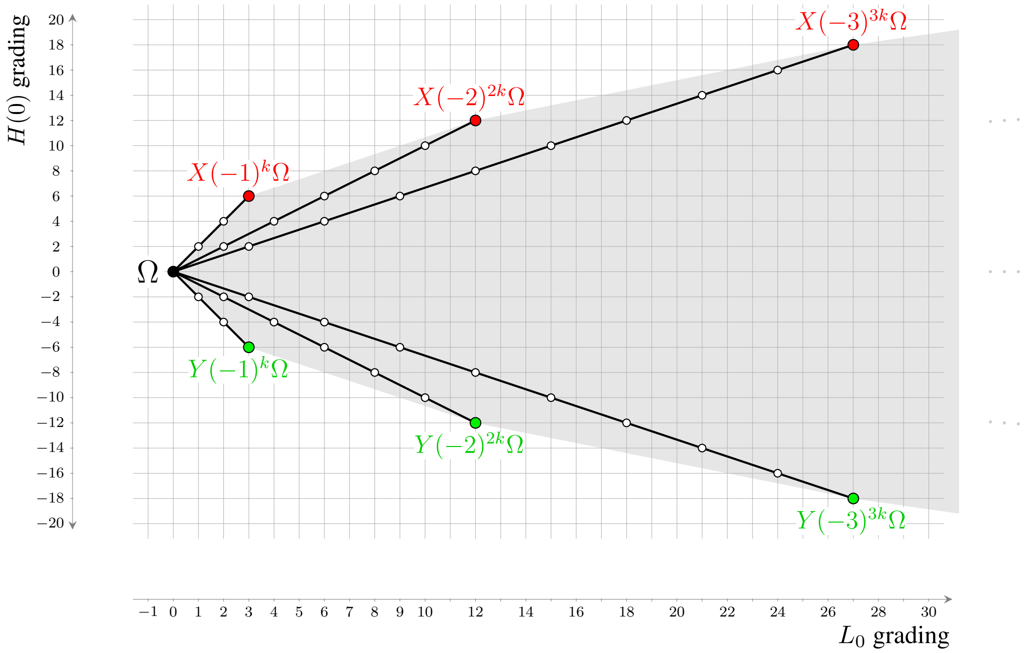
[7]The proof that these conditions are also *sufficient* will appear much later, on page 87.

Using the same sort of analysis with the $\mathfrak{sl}(2)$-triples

$$\big(X(n), Y(-n), H(0) + nk\big),$$

one can get a rather detailed understanding of the Hilbert space $H_{\lambda,k}$. Here is how things look like in the case of the vacuum sector (the case $\lambda = 0$).

The weight of Ω with respect $H(0)+nk$ is nk and therefore Ω supports an $\mathfrak{sl}(2)$-chain of length $|nk|$. There are two relevant gradings on $H_{0,k}$, coming from the two commuting actions of L_0 and of $H(0)$. The operator $X(n)$ raises bidegree by $(-n, 2)$, and its adjoint $Y(-n)$ raises bidegree by $(n, -2)$. The vacuum Hilbert space $H_{0,k}$ then looks as follows. Here, we take $k = 3$:



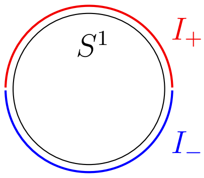
The shaded area is the so-called *weight polytope* of $H_{0,k}$. It indicates those bidegrees (= weights) whose corresponding graded pieces (= weight spaces) are non-zero.

Haag duality

Recall that if $I \subset S^1$ is an interval then I' denotes the interior of its complement and that if $A \subset B(H)$ is a von Neumann algebra then A' denotes its commutant. Our next goal is the following theorem:

> **Theorem (Haag duality)** *Let $(H_0, \Omega, \mathcal{A})$ be a Möbius covariant conformal net and let $I \subset S^1$ be an interval. Then:*
>
> $$\boxed{\mathcal{A}(I') = \mathcal{A}(I)'}$$

Note that by Möbius covariance, we may assume without loss of generality that I is the upper half of S^1 and that I' is its lower half:



The statement then becomes $\mathcal{A}(I_-)' = \mathcal{A}(I_+)$.

The above theorem is known to be true but we'll only prove it for the chiral WZW models. However, the proof that we'll present can also be adapted to other models.

The easiest way to prove Haag duality uses a big result about von Neumann algebras, due to Tomita and Takesaki. We will start by stating Tomita-Takesaki's result in its full generality (without proof). On page 44, we will show how this result can be used to prove Haag duality. Later on, we will show how one can be sneaky and prove Haag duality without needing to rely on Tomita-Takesaki's result.

A primer on Tomita-Takesaki theory:

The basic setup of the theory is as follows:

A : a von Neumann algebra
H : a Hilbert space with an action of A
$\Omega \in H$: a vector that is cyclic for both A and A'

(In our example of interest, we will take $A = \mathcal{A}(I_-)$ and $H = H_0$. By the Reeh-Schlieder theorem, the vacuum vector $\Omega \in H_0$ is cyclic for both $\mathcal{A}(I_-)$ and its commutant.)

Lemma *If $\Omega \in H$ is cyclic for A', then the map $A \to H : x \mapsto x\Omega$ is injective. (When the latter condition is satisfied, one says that Ω is <u>separating</u> for A.)*

Proof:

$$(x\,\Omega = 0) \;\Rightarrow\; (x\,\underbrace{\Omega\,y}_{\text{dense in } H} = 0 \quad \forall y \in A') \;\Rightarrow\; (x = 0),$$

where we wrote A' as acting on the right. □

The next step in the setup of Tomita-Takesaki theory is to consider the following densely defined operator:

$$S_0 : x\,\Omega \mapsto x^*\Omega. \tag{10}$$

By the above lemma, that operator is well defined. Namely, if a vector $\xi \in H$ is of the form $x\Omega$, then it is so in a unique way, and so its value $x^*\Omega$ under S_0 is unambiguously defined. Let S be the closure of S_0 (I'll talk about closures of unbounded operators in the next section—see Lemma 6 below). One then defines

$$\Delta := S^*S \qquad \text{and} \qquad J := S\Delta^{-\frac{1}{2}}$$

(strictly speaking, J is the closure of $S\Delta^{-\frac{1}{2}}$) so that

$$S = J\Delta^{\frac{1}{2}}$$

is the polar decomposition of S. We then have:

Theorem (Tomita-Takesaki) *Under the above assumptions, and with the above notations, we have:*

$$A' = JAJ.$$

Here, the operators S and Δ are *unbounded operators*. We should therefore step back and give some basic definitions before going on too fast...

Unbounded operators on Hilbert spaces:

Definition: A (possibly) *unbounded operator* $a : H \dashrightarrow H$ consists of a domain $\mathcal{D}_a \subset H$ and a linear map $a : \mathcal{D}_a \to H$. We will always assume that $\mathcal{D}_a$ is dense in H.

An unbounded operator $a : H \dashrightarrow H$ is *closed* if its graph

$$\Gamma_a := \big\{(\xi, a(\xi)) \,\big|\, \xi \in \mathcal{D}_a\big\}$$

is a closed subspace of $H \oplus H$.

Really, the good notion is that of a closed operator. But not all unbounded operators come to us as closed operators. Therefore, one introduces the following definition:

An unbounded operator $a : H \dashrightarrow H$ is *closeable* if $\overline{\Gamma_a}$, the closure of its graph, is the graph of something. In other words, a is closeable if the first projection $p_1 : H \oplus H \to H$ restricts to an injective map $p_1 : \overline{\Gamma_a} \to H$. Equivalently, a is closeable if

$$\left.\begin{array}{r} \xi_i \to 0 \\ a(\xi_i) \to \eta \end{array}\right\} \Rightarrow \eta = 0$$

for every sequence $\xi_i \in \mathcal{D}_a$.

If a is a closed operator, then a subspace $\mathcal{D}_0 \subset \mathcal{D}_a$ is called a *core* of a if a is the closure of $a|_{\mathcal{D}_0}$. Equivalently, $\mathcal{D}_0$ is a core of a if it is dense in $\mathcal{D}_a$ in the *graph norm*

$$\|\cdot\|_a := \|\cdot\| + \|a(\,\cdot\,)\|.$$

(We could also have defined the graph norm to be $\sqrt{\|\cdot\|^2 + \|a(\,\cdot\,)\|^2}$. That's an equivalent norm and so it doesn't matter which one of those two norms one uses.)

Finally, if $\mathcal{D}_a \subset \mathcal{D}_b$ and $a = b|_{\mathcal{D}_b}$, then we write $a \subset b$.

With the above definitions in place, we can now go back and prove a claim that was used implicitly in our earlier discussion:

Lemma 6 *The operator S_0 defined in* (10) *is closeable.*

Proof: Assume that we have a sequence $x_i \in A$ with $x_i\Omega \to 0$ and $x_i^*\Omega \to \eta$. We need to show that $\eta = 0$. It is enough to show that $\langle \xi, \eta \rangle = 0$ for every $\xi \in H$. Actually, since Ω is cyclic for A', it is enough to show $\langle \xi, \eta \rangle = 0$ for every ξ of form Ωy with $y \in A'$ (once again, we write the action of A' on the right). And indeed:

$$\langle \Omega y, \eta \rangle = \langle \Omega y, \lim x_i^*\Omega \rangle = \lim \langle \Omega y, x_i^*\Omega \rangle = \lim \langle x_i \Omega y, \Omega \rangle = \langle \underbrace{\lim x_i \Omega}_{=0}\, y, \Omega \rangle = 0$$

□

Note that bounded operators are a special case of closed operators:

Exercise: Show that a densely defined operator $a : H \dashrightarrow H$ satisfies $\sup_{\xi \neq 0} \frac{\|a\xi\|}{\|\xi\|} < \infty$ if and only if $\overline{\Gamma_a}$ is the graph of a continuous everywhere defined map.

Definition: The *adjoint* of an unbounded operator $a : H \dashrightarrow H$ is given by...
Let's slow down and first think a bit about what we want:

$$\langle a\xi, \eta \rangle = \langle \xi, \underbrace{a^*\eta}_{} \rangle \tag{11}$$

We expect this to be not always defined...

...so we should first start by describing the domain of a^*:

$$\mathcal{D}_{a^*} := \left\{ \eta \in H \,\middle|\, \begin{smallmatrix} \mathcal{D}_a \to \mathbb{C} \\ \xi \mapsto \langle a\xi, \eta \rangle \end{smallmatrix} \text{ is bounded} \right\}$$

(Indeed, if (11) is to hold, then the map $\xi \mapsto \langle a\xi, \eta \rangle$ should definitely be bounded.) Given $\eta \in \mathcal{D}_{a^*}$, one then defines $a^*\eta$ to be the unique vector for which (11) holds.

Note: If a is an *antilinear* operator, then (11) should be replaced by $\langle a\xi, \eta \rangle = \langle a^*\eta, \xi \rangle$. Adjoints actually make sense for any real linear maps (and they recover the usual notion of adjoint in the case of a $\mathbb{C}$-linear or $\mathbb{C}$-antilinear operator). To define them, just replace equation (11) by the corresponding formula involving only $\mathbb{R}$-valued inner products: $\Re\mathrm{e}\langle a\xi, \eta \rangle = \Re\mathrm{e}\langle \xi, a^*\eta \rangle$.

The graph of a^* can also be described directly in terms of the graph of a:

$$\Gamma_{a^*} = \big\{ (\eta, \text{“}a^*\eta\text{”}) \in H \oplus H \,\big|\, \langle a\xi, \eta \rangle = \langle \xi, \text{“}a^*\eta\text{”} \rangle \;\; \forall (\xi, a\xi) \in \Gamma_a \big\} \tag{12}$$

Here, we put quotes around $a^*\eta$ to remind ourselves that the correct thing to write should have been $\{(\eta, X) \in H \oplus H \,|\, \langle a\xi, \eta \rangle = \langle \xi, X \rangle \;\; \forall (\xi, a\xi) \in \Gamma_a\}$. An alternative description of Γ_{a^*} which is easily seen to be equivalent to (12) is given by:

$$\Gamma_{a^*} = \begin{pmatrix} 0 & \mathbf{1}_H \\ -\mathbf{1}_H & 0 \end{pmatrix} \Gamma_a^\perp.$$

From this last characterization, it follows that a^* is always closed, and that a^{**} is the closure of a.

Note: In our definition of unbounded operator, we could have dropped the condition that $\mathcal{D}_a$ be dense in H, and used (12) to define a^*. In that case, we would have

$$(\mathcal{D}_a \text{ is dense}) \Leftrightarrow (a^* \text{ is single-valued}).$$

By replacing a by a^* in the above equivalence and using that a^{**} is the closure of a, we then get:

$$(a \text{ is closeable}) \Leftrightarrow (\mathcal{D}_{a^*} \text{ is dense}).$$

Definition: An unbounded operator $a : H \dashrightarrow H$ is called ***self-adjoint*** if $a = a^*$.

Note that this is much stronger than simply asking

$$\langle a\xi, \eta\rangle = \langle \xi, a\eta\rangle \qquad \forall\, a \in \mathcal{D}_a.$$

The latter only implies (and is actually equivalent to) $a \subset a^*$, but it does not entail $\mathcal{D}_a = \mathcal{D}_{a^*}$ even if a is closed.

Example: Let X be a measure space and let $H := L^2(X)$ be the Hilbert space of square integrable functions on X. Then for every measurable function $f : X \to \mathbb{R}$, we have the multiplication operator

$$m_f : L^2(X) \dashrightarrow L^2(X) : \xi \mapsto f\xi$$

with domain $\mathcal{D}_{m_f} := \{\xi \in L^2(X) \,|\, f\xi \in L^2(X)\}$.

Aside: my preferred way of stating the spectral theorem is to say that every self-adjoint operator is unitarily equivalent to one of the above form.

Lemma *Let $a : H \dashrightarrow H$ be a closed operator. Then a^*a with domain*

$$\mathcal{D}_{a^*a} := \{\xi \in \mathcal{D}_a \,|\, a\xi \in \mathcal{D}_{a^*}\}$$

is self-adjoint (and in particular closed). Moreover, $\mathcal{D}_{a^*a}$ is a core of a.
Furthermore, we have $\mathcal{D}_{\sqrt{a^*a}} = \mathcal{D}_a$.

For a proof, see for example Proposition 3.18 and Lemma 7.1 in Konrad Schmüdgen's book *Unbounded self-adjoint operators on Hilbert space*.

We now turn to the subject of the ***polar decomposition*** of a closed operator $a : H \dashrightarrow H$. Let us assume for simplicity that $\ker(a) = 0$ and that $\operatorname{im}(a)$ is dense.

Claim: Any such operator can be written *uniquely* as

$$\boxed{a = up}$$

with u unitary and p positive (note: the requirement that $\mathcal{D}_a = \mathcal{D}_p$ is implicit in the statement $a = up$).

Given a, let us first define $p := \sqrt{a^*a}$ (clearly, if we want $a = up$ to hold, then p has to be equal to $\sqrt{a^*a}$). Now, one might be tempted to write $u := ap^{-1}$, but that is not quite right because ap^{-1} is not everywhere defined. Instead, we first define

$$u_0 := ap^{-1} : \ \mathrm{im}(p) \xrightarrow{p^{-1}} \mathcal{D}_p = \mathcal{D}_a \xrightarrow{a} H.$$

That operator is a densely defined because $\mathrm{im}(p)^\perp = \ker(p) = \ker(a) = 0$, and is well defined (i.e. single-valued) because $\ker(p) = 0$. Moreover, its range is dense in H by our assumption on a. We then define u to be the closure of u_0. The fact that u_0 is closeable and that its closure is unitary is the content of the following lemma:

Lemma *The map $u_0 : p\xi \mapsto a\xi$ is isometric.*

Proof: For any $\xi \in \mathcal{D}_{a^*a}$ and $\eta \in \mathcal{D}_a = \mathcal{D}_p$, we have

$$\langle a\xi, a\eta\rangle = \langle a^*a\xi, \eta\rangle = \langle p^2\xi, \eta\rangle = \langle p\xi, p\eta\rangle.$$

In particular, $\|a\xi\| = \|p\xi\|$ for every $\xi \in \mathcal{D}_{a^*a}$.

By the previous lemma, $\mathcal{D}_{a^*a} = \mathcal{D}_{p^2}$ is a core of p. For every $\xi \in \mathcal{D}_p$, we may therefore pick a sequence $\xi_i \in \mathcal{D}_{a^*a}$ that converges to ξ in the graph norm $\|\cdot\|_p$. By the above computation, the two graph norms $\|\cdot\|_a$ and $\|\cdot\|_p$ agree on $\mathcal{D}_{a^*a}$. The sequence ξ_i is therefore also Cauchy for $\|\cdot\|_a$ (with necessarily the same limit ξ since every $\|\cdot\|_a$-limit is also a $\|\cdot\|$-limit). We then have $\|a\xi\| = \lim \|a\xi_i\| = \lim \|p\xi_i\| = \|p\xi\|$. $\square$

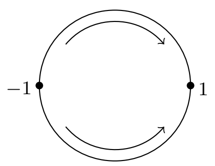
Note: The polar decomposition of antilinear operators works in exactly the same way as the one for linear operators. The only difference is that u is then antiunitary instead of unitary.

We now go back to the proof of Haag duality: $\mathcal{A}(I_-)' = \mathcal{A}(I_+)$.

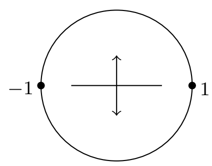
Our strategy will be to apply Tomita-Takesaki's theorem to the algebra $\mathcal{A}(I_-)$ with respect to the vacuum vector $\Omega \in H_0$. We will compute the polar decomposition $S = J\Delta^{\frac{1}{2}}$ by the method of 'lucky guess' and then observe that $J\mathcal{A}(I_-)J = \mathcal{A}(I_+)$.

We will need the following elements of $\mathrm{Möb}(S^1)$

v_t :



ϑ :



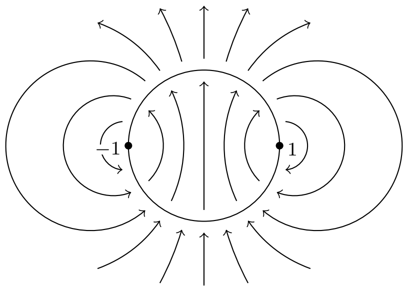
and the corresponding operators on H_0

$$V_t : H_0 \to H_0 \qquad\qquad\qquad \Theta : H_0 \to H_0.$$

Here, $\{v_t\}_{t\in\mathbb{R}}$ is the unique one-parameter subgroup of $\text{Möb}(S^1)$ that fixes the points $\{1,-1\}$, normalized so that the derivative at -1 of v_t is e^t. The map ϑ also fixes $\{1,-1\}$, and satisfies $\vartheta^2 = 1$.

The homomorphism $\mathbb{R} \to \text{Möb}(S^1) : t \mapsto v_t$ can be analytically continued to a homomorphism $\mathbb{C} \to \text{Aut}(\mathbb{CP}^1) : z \mapsto v_z$, which again fixes the points $\{-1,1\}$. For example, the map $v_{i\pi}$ exchanges I_- and I_+:

$v_{i\pi}$:



Let L be the infinitesimal generator of $\{V_t\}$, so that $V_t = e^{itL}$ for $t \in \mathbb{R}$, and let us define V_z for $z \in \mathbb{C}$ by

$$V_z := e^{izL} : H_0 \dashrightarrow H_0.$$

A vector ξ is in the domain of V_z if and only if the map $\mathbb{R} \to H_0 : t \mapsto e^{itL}\xi$ analytically continues to the strip $\{\zeta \in \mathbb{C} \,|\, 0 \le \Im\mathrm{m}(\zeta) \le \Im\mathrm{m}(z)\}$. More precisely, the extension should be analytic for $0 < \Im\mathrm{m}(\zeta) < \Im\mathrm{m}(z)$ and continuous for $0 \le \Im\mathrm{m}(\zeta) \le \Im\mathrm{m}(z)$ (reverse all inequalities if $\Im\mathrm{m}(z)$ is negative). To see that, use the spectral theorem and the explicit description of domains of multiplication operators given two pages above.

Proposition 1 *We have* $\boxed{\Theta V_{i\pi} a\Omega = a^*\Omega}$ *for every* $a \in \mathcal{A}(I_-)$*. Equivalently,*

$$V_{i\pi} a\Omega = \Theta a^*\Omega. \tag{13}$$

(Implicitly, we are also making the claim that $a\Omega \in \mathcal{D}_{V_{i\pi}}$*.)*

We will only present the proof of the above proposition for the case of the chiral WZW models, but our proof generalizes to other models too.

Proof: In order to attack the problem, we first need to understand the action of $V_{i\pi}$ on H_0.

Claim 1: if $f_i : S^1 \to \mathfrak{g}$ is supported in I_- and if $f_i|_{I_-}$ analytically continues to a function $F_i(z)$ defined for $|z| \ge 1$, including ∞, smooth on the boundary, then the function

$$t \mapsto V_t f_1 \dots f_n \Omega = (f_1 \circ v_{-t}) \dots (f_n \circ v_{-t})\Omega \qquad \text{for } t \in \mathbb{R}$$

analytically continues to

$$z \mapsto (F_1 \circ v_{-z})|_{I_-} \dots (F_n \circ v_{-z})|_{I_-}\Omega$$

for $z \in \mathbb{C}$ with imaginary part between 0 and π (analytic on the interior of the strip and continuous on the closed strip). It follows that $f_1 \dots f_n\Omega$ is in the domain of V_z and that

$$V_z\, f_1 \dots f_n\Omega = (F_1 \circ v_{-z})|_{I_-} \dots (F_n \circ v_{-z})|_{I_-}\Omega.$$

technical stuff →

Proof: Recall from (9) that $f_1 \dots f_n\Omega$ depends continuously on $f_1, \dots, f_n$ in the $\mathcal{C}^\infty$ topology. If now $f_i = f_i^{(z)}$ depends holomorphically on some parameter $z \in \mathbb{C}$, then the map $z \mapsto f_1^{(z)} \dots f_n^{(z)}\Omega$ is complex differentiable:

$$\begin{aligned}
&\lim_{h\to 0} \frac{f_1^{(z+h)} \dots f_n^{(z+h)}\Omega - f_1^{(z)} \dots f_n^{(z)}\Omega}{h} \\
&= \lim_{h\to 0} \sum_{i=1}^{n} f_1^{(z+h)} \dots f_{i-1}^{(z+h)} \big(\tfrac{f_i^{(z+h)} - f_i^{(z)}}{h}\big) f_{i+1}^{(z)} \dots f_n^{(z)}\Omega \\
&= \sum_{i=1}^{n} f_1^{(z)} \dots f_{i-1}^{(z)} (\tfrac{d}{dz} f_i^{(z)}) f_{i+1}^{(z)} \dots f_n^{(z)}\Omega.
\end{aligned}$$

Here, the last equality holds because the function $h \mapsto \frac{f_i^{(z+h)} - f_i^{(z)}}{h}$ admits a continuous extension to $h = 0$.

Since $(F_i \circ v_{-z})|_{I_-}$ depends holomorphically on z, the above argument applies. The vector $(F_1 \circ v_{-z})|_{I_-} \dots (F_n \circ v_{-z})|_{I_-}\Omega$ depends holomorphically on z for $z \in \mathbb{C}$ with imaginary part between 0 and π (analytic on the interior of the strip and continuous on the closed strip). □

Claim 2: Let $\mathcal{D}_0$ be the linear span of vectors of the form $f_1 \dots f_n\Omega$ with $\operatorname{supp}(f_i) \subset I_-$ and with the property that $f_i|_{I_-}$ that analytically continues to $|z| \geq 1$, including ∞. Then $\mathcal{D}_0$ is dense in H_0.

Proof: We first show that if $\operatorname{supp}(f_i) \subset I_-$, then $f_1 \dots f_n\Omega$ is in the closure of $\mathcal{D}_0$. For every function f_i, define the 'convolution'

$$f_i^N := \tfrac{N}{\sqrt{2\pi}} \int_{t\in\mathbb{R}} e^{-\frac{(Nt)^2}{2}} \cdot f_i \circ v_{-t}.$$

Then $\lim_N f_i^N = f_i$ in the $\mathcal{C}^\infty$ topology and so $\lim_N f_1^N \dots f_n^N\Omega = f_1 \dots f_n\Omega$ by (9). Moreover, each f_i^N extends to an analytic function

$$F_i^N(z) := \tfrac{N}{\sqrt{2\pi}} \int_{\mathbb{R}+\alpha(z)} e^{-\frac{(Nt)^2}{2}} \cdot f_i(v_{-t}(z))\, dt,$$

where $\alpha(z)$ is chosen so that $t \mapsto v_{-t}(z)$ maps $\alpha(z)$ to some fixed point in I_- (say $-i$), and thus maps $\mathbb{R} + \alpha(z)$ to the whole of I_-. To finish the proof, we need to argue that vectors of the form $f_1 \dots f_n\Omega$ with $\operatorname{supp}(f_i) \subset I_-$ are dense in H_0. This is very similar to the statement of the Reeh-Schlieder theorem, and the proof that we presented in pages 16–20 applies word for word. $\square$

Given $f_1, \ldots, f_n \in \mathcal{C}^\infty(S^1, \mathfrak{g})$ with support in I_- and analytic continuations $F_1, \ldots, F_n$ as above, we define

$$g_i := (F_i \circ v_{-i\pi})|_{I_-} \qquad\qquad h_i := (F_i \circ v_{-i\pi})|_{I_+}$$

so that

$$V_{i\pi}\, f_1 \ldots f_n \Omega = g_1 \ldots g_n \Omega \qquad\qquad \Theta\, f_1 \ldots f_n \Omega = (-1)^n h_1^* \ldots h_n^* \Omega.$$

(The action of $\mathrm{M\ddot{o}b}(S^1)$ on vectors $f_1 \ldots f_n \Omega$ was described on page 27.)

Recall that we are trying to prove equation (13). By Claim 2, it is enough to test it against vectors of the form $f_1 \ldots f_n \Omega$, with f_i with support in I_- and analytic continuations as above. Ignoring for the moment the question of whether $a\Omega$ is in the domain of $V_{i\pi}$, we have to show that

$$\big\langle V_{i\pi}\, a\Omega, f_1 \ldots f_n \Omega \big\rangle \;=\; \big\langle \Theta\, a^* \Omega, f_1 \ldots f_n \Omega \big\rangle \tag{14}$$

for all $a \in \mathcal{A}(I_-)$. Recall that both $V_{i\pi}$ and Θ are self-adjoint (and Θ is antilinear). We can then compute:

$$\langle V_{i\pi} a\Omega, f_1 \ldots f_n \Omega \rangle = \langle a\Omega, V_{i\pi}\, f_1 \ldots f_n \Omega \rangle = \langle a\Omega, g_1 \ldots g_n \Omega \rangle$$

and

$$\begin{aligned}
\langle \Theta a^* \Omega, f_1 \ldots f_n \Omega \rangle &= \langle \Theta f_1 \ldots f_n \Omega, a^* \Omega \rangle \\
&= \langle (-1)^n h_1^* \ldots h_n^* \Omega, a^* \Omega \rangle^{①} \\
&= \langle \Omega, (-1)^n h_n \ldots h_1 a^* \Omega \rangle \\
&= \langle \Omega, (-1)^n a^* h_n \ldots h_1 \Omega \rangle = \langle a\Omega, (-1)^n h_n \ldots h_1 \Omega \rangle.^{②}
\end{aligned}$$

I'm not allowed to write this because I don't know whether $a^*\Omega$ is in the domain of $h_n \ldots h_1$. So I'll need to find another more roundabout way of going from ① to ②.

We now address the technical problem raised above. We assume WLOG that the h_i are skew adjoint (otherwise, write $h_i = \frac{1}{2}[(h_i - h_i^*) - i((ih_i) - (ih_i)^*)]$ and use linearity) so that they exponentiate to elements in $\mathcal{A}(I_+)$. Since $\mathcal{A}(I_+)$ and $\mathcal{A}(I_-)$ commute, we have:

$$\begin{aligned}
\big\langle h_1^* \ldots h_n^* \Omega, a^* \Omega \big\rangle &= \big\langle (\tfrac{d}{dt_1}\big|_{t_1=0} e^{t_1 h_1^*}) \ldots (\tfrac{d}{dt_n}\big|_{t_n=0} e^{t_n h_1^*})\, \Omega, a^* \Omega \big\rangle \\
&= \tfrac{d^n}{dt_1 \ldots dt_n}\big|_{\vec{t}=0} \big\langle (e^{t_1 h_1^*}) \ldots (e^{t_n h_n^*})\, \Omega, a^* \Omega \big\rangle \\
&= \tfrac{d^n}{dt_1 \ldots dt_n}\big|_{\vec{t}=0} \big\langle a\Omega, (e^{t_n h_n}) \ldots (e^{t_1 h_1})\, \Omega \big\rangle \\
&= \big\langle a\Omega, (\tfrac{d}{dt_n}\big|_{t_n=0} e^{t_n h_n}) \ldots (\tfrac{d}{dt_1}\big|_{t_1=0} e^{t_1 h_1})\, \Omega \big\rangle = \big\langle a\Omega, h_n \ldots h_1 \Omega \big\rangle.
\end{aligned}$$

Note: For this argument to work, we need to know that Ω is a smooth vector for the action of the appropriate central extension $\widetilde{LG}_k$ of LG (that is, the map $\widetilde{LG}_k \to H_0 : \gamma \mapsto u_\gamma \Omega$ is smooth, at least when restricted to finite dimensional submanifolds of $\widetilde{LG}_k$).

To finish the argument, we need to check that $g_1 \ldots g_n \Omega = (-1)^n h_n \ldots h_1 \Omega$. Since $g_i + h_i$ admits a holomorphic extension to the unit disc (namely $F_i \circ v_{-i\pi}$), it annihilates the vacuum vector:

$$g_i \Omega + h_i \Omega = 0.$$

Moreover, g_i and h_j commute in $\widetilde{L\mathfrak{g}}_k$ because they have disjoint supports. So we get:

$$\begin{aligned}
g_1 \dots g_n\Omega &= -g_1 \dots g_{n-1}h_n\Omega \\
&= -h_n g_1 \dots g_{n-1}\Omega \\
&= h_n g_1 \dots g_{n-2}h_{n-1}\Omega \\
&= h_n h_{n-1} g_1 \dots g_{n-2}\Omega \\
&= -h_n h_{n-1} g_1 \dots g_{n-3}h_{n-2}\Omega \\
&= -h_n h_{n-1} h_{n-2} g_1 \dots g_{n-3}\Omega \\
&= h_n h_{n-1} h_{n-2} g_1 \dots g_{n-4}h_{n-3}\Omega \\
&= h_n \dots h_{n-3} g_1 \dots g_{n-4}\Omega = \dots = (-1)^n h_n \dots h_1\Omega.
\end{aligned}$$

This finishes the proof of Proposition 1 modulo the claim that $a\Omega$ is in the domain of $V_{i\pi}$.

Now let us now return to the question of whether $a\Omega$ is in the domain of $V_{i\pi}$. If you look carefully at the above computation, you'll see that the only thing I've really shown is

$$\langle a\Omega, V_{i\pi}\, f_1 \dots f_n\Omega\rangle = \langle \Theta a^*\Omega, f_1 \dots f_n\Omega\rangle \tag{15}$$

because I haven't yet argued that the left hand side of (14) is well defined. However, we can already see from (15) that the map $f_1 \dots f_n\Omega \mapsto \langle a\Omega, V_{i\pi}\, f_1 \dots f_n\Omega\rangle$ is bounded on

$$\mathcal{D}_0 = \mathrm{Span}\big\{ f_1 \dots f_n\Omega \,\big|\, \mathrm{supp}(f_i) \subset I_-,\ \ f_i|_{I_-} \text{ analytically continues to } |z| \geq 1 \big\}. \tag{16}$$

If we new that $\mathcal{D}_0$ is a core of $V_{i\pi}$, then we'd be able to conclude that $a\Omega \in \mathcal{D}_{V_{i\pi}^*} = \mathcal{D}_{V_{i\pi}}$.

The following general result will help us finish the argument:

(($\star$)) **Proposition** *Let a be a self-adjoint operator. If $\mathcal{D}_0 \subset \mathcal{D}_a$ is dense and invariant under $\{e^{ita}\}_{t\in\mathbb{R}}$, then $\mathcal{D}_0$ is a core of a.*

Proof: Given $\xi \in \mathcal{D}_a$, we need to show that it is in the $\|\ \|_a$-closure of $\mathcal{D}_0$. We begin with two observations:

(1) Given $\eta \in \mathcal{D}_a$, the map $t \mapsto e^{ita}\eta$ is $\|\ \|_a$-continuous.
Indeed both $t \mapsto e^{ita}\eta$ and $t \mapsto a(e^{ita}\eta) = e^{ita}(a\eta)$ are $\|\ \|$-continuous.

(2) If $\eta \in \mathcal{D}_0$, then $\int_0^T e^{ita}\eta$ is in the $\|\ \|_a$-closure of $\mathcal{D}_0$.
Indeed, the integral of a continuous function can be approximated by Riemann sums, each one of which is in $\mathcal{D}_0$ by our assumption $e^{ita}\mathcal{D}_0 = \mathcal{D}_0$ (where both 'continuous' and 'approximated' are with respect to $\|\ \|_a$).

Now, given $\xi \in \mathcal{D}_a$, we may pick $\eta_n \in \mathcal{D}_0$ such that $\eta_n \xrightarrow{\|\ \|} \xi$. We then have:

$$\tfrac{1}{T}\int_0^T e^{ita}\eta_n\, dt \xrightarrow[n\to\infty]{\|\ \|_a} \tfrac{1}{T}\int_0^T e^{ita}\xi\, dt \xrightarrow[T\to 0]{\|\ \|_a} \xi$$

Uses $a\big(\int_0^T e^{ita}\eta\big)\, dt = \int_0^T ae^{ita}\eta\, dt = -ie^{ita}\eta\big|_0^T$ and our assumption $\eta_n \xrightarrow{\|\ \|} \xi$.

Because $t \mapsto e^{ita}\eta$ is $\|\ \|_a$-continuous.

Therefore ξ is in the $\| \ \|_a$-closure of $\mathcal{D}_0$. $\square$

The domain $\mathcal{D}_0$ defined in (16) is invariant under the action of $\{V_t\}_{t\in\mathbb{R}}$. By $((\star))$, it is therefore a core for $V_{i\pi}$. From (15), we then get:

<u>*Corollary*</u>: For every $a \in \mathcal{A}(I_-)$, the vector $a\Omega$ is in the domain of $V_{i\pi}$ and equation (14) holds:

$$\big\langle V_{i\pi}\, a\Omega, \xi\big\rangle \;=\; \big\langle \Theta\, a^*\Omega, \xi\big\rangle, \qquad \forall \xi \in \mathcal{D}_0.$$

Since $\mathcal{D}_0$ is dense in H, we can then conclude that $V_{i\pi}\, a\Omega = \Theta\, a^*\Omega$, equivalently:

$$\boxed{\Theta V_{i\pi} a\Omega = a^*\Omega} \tag{17}$$

This finishes the proof of Proposition 1. $\square$

At this point, we should admit that (17) was really the wrong thing to ask. What we really want to know is $S = \Theta V_{i\pi}$, whereas the above equation only gives us $S \subset \Theta V_{i\pi}$. Once again, Proposition $((\star))$ comes to our rescue:

Theorem *Let S be the closure of $S_0 : a\Omega \mapsto a^*\Omega$ with domain $\mathcal{D}_{S_0} := \mathcal{A}(I_-)\Omega$. Then:*

$$\boxed{S = \Theta V_{i\pi} \;\; \textit{is the polar decomposition of } S}$$

Proof: The domain $\mathcal{A}(I_-)\Omega$ on which we have checked (17) is invariant under the action of $\{V_t\}_{t\in\mathbb{R}}$. By $((\star))$, it is a core for $V_{i\pi}$, and therefore also a core for $\Theta V_{i\pi}$. We've checked that the operators S and $\Theta V_{i\pi}$ agree on a common core. They are therefore equal. $\square$

At this point, we could get Haag duality by simply quoting the result of Tomita-Takesaki theory, according to which $A' = JAJ$:

$$\mathcal{A}(I_-)' \;\underset{\text{By Tomita-Takesaki theory}}{=}\; \Theta\mathcal{A}(I_-)\Theta \;\underset{\text{By Möbius covariance}}{=}\; \mathcal{A}(I_+).$$

However, this would be philosophically unsatisfactory, as both Haag duality and the Tomita-Takesaki result are of the form:

"Under such and such assumptions, these two algebras are each other's commutants"

Let's step back and think about the general setup of Tomita-Takesaki theory.

It turns out that the operator S can be entirely described in terms of the real subspace

$$K := \text{closure of } \big\{a\Omega \,\big|\, a \in A,\ a = a^*\big\}$$

Note that $K \cap iK = \{0\}$. Indeed, $\xi \in K \cap iK$ means that ξ is both of the form $a\Omega$ and of the form $ib\Omega$, for self-adjoint elements a and b. Since Ω is separating, this implies $a = ib$, which can only happen if $a = b = 0$.

Lemma *The operator with domain $K + iK$ given by (the linear extension of)*

$$\begin{cases} +1 & \text{on } K \\ -1 & \text{on } iK \end{cases}$$

is S.

Proof: The graph $\Gamma := \big\{(\xi,\xi) + (\eta,-\eta) \,\big|\, \xi \in K, \eta \in iK\big\}$ of that operator is the orthogonal direct sum of $\{(\xi,\xi) \,|\, \xi \in K\}$ and of $\{(\eta,-\eta) \,|\, \eta \in iK\}$. Both of them being closed, so is Γ (unlike $K + iK$, which is typically not closed).

If K_0 is a dense subspace of K, then

$$\Gamma_0 := \big\{(\xi,\xi) + (\eta,-\eta) \,\big|\, \xi \in K_0, \eta \in iK_0\big\}$$

is dense in Γ. Therefore Γ is the closure of Γ_0. To finish the argument, note that if we let $K_0 := \{a\Omega \,|\, a \in A, a = a^*\}$, then Γ_0 is exactly the graph of S_0. $\square$

Let e denote that orthogonal projection onto K (the operator e is only $\mathbb{R}$-linear!). Our Hilbert space then becomes a representation of the real $*$-algebra

$$\mathcal{K} := \big\langle e, i \,\big|\, e^2 = e^* = e, i^2 = -1, i^* = -i \big\rangle$$

Since S is entirely determined by K and i, it is entirely determined by H as a representation of $\mathcal{K}$. Now a miracle happens:

The algebra $\mathcal{K}$ has a very simple representation theory!

Lemma *The element $z := (e+iei)^2 = e+ieie+eiei-iei$ is self-adjoint and contained in the center of $\mathcal{K}$.*

Proof: direct computation. $\square$

Lemma *In any representation on a Hilbert space, the spectrum of z is contained in $[0,1]$.*

Proof: Let $K := eH$ and let $f := -iei$ be the orthogonal projection onto iK. Then:
$\mathrm{Spec}(e + (1-f)) \subset [0,2] \;\Rightarrow\; \mathrm{Spec}(e-f) \subset [-1,1] \;\Rightarrow\; \mathrm{Spec}(e-f)^2 \subset [0,1]$. $\square$

By the above results, we can desintegrate any representation of $\mathcal{K}$ on a Hilbert space according to the action of z:

$$H = \int_{\lambda\in[0,1]}^{\oplus} H_\lambda.$$

(Given a bundle Hilbert spaces $\{H_\lambda\}_{\lambda\in X}$ parametrized by some measure space X, the *direct integral* $\int_{\lambda\in X}^{\oplus} H_\lambda$ is the space of L^2 sections of that bundle.) Each fiber H_λ then carries an action of

$$\mathcal{K}_\lambda := \big\langle e, i \,\big|\, e^2 = e^* = e, i^2 = -1, i^* = -i, (e+iei)^2 = \lambda \big\rangle.$$

Lemma *The algebra $\mathcal{K}_\lambda$ is spanned by the elements* $\{1, e, i, ei, ie, eie, iei, eiei\}$.
In particular, $\dim_{\mathbb{R}}(\mathcal{K}_\lambda) \leq 8$.

Proof: Any word in the letters e and i that is not in the above list can be simplified using the rules $ee = e$, $ii = -1$ and $ieie = \lambda - e - eiei + iei$. $\square$

We now analyze a couple special things that happen when $\lambda = 0$ or 1.

Lemma (1) *In any representation of $\mathcal{K}_0$ on a Hilbert space, the extra relation $e+iei = 0$ holds. The quotient algebra $\tilde{\mathcal{K}}_0 := \mathcal{K}_0/(e + iei = 0)$ is spanned by $\{1, e, i, ei\}$ and is therefore at most four dimensional.*

(2) *In any representation of $\mathcal{K}_1$ on a Hilbert space, the extra relation $1+iei = e$ holds. The quotient algebra $\tilde{\mathcal{K}}_1 := \mathcal{K}_1/(1 + iei = e)$ is spanned by $\{1, e, i, ei\}$ and is therefore at most four dimensional.*

Proof: (1) On a Hilbert space, $(e + iei)^2 = 0$ implies $e + iei = 0$ because that element is self-adjoint. Given that relation, we can then rewrite $ie = -ieii = ei$, $eie = eei = ei$, and so on.

(2) $1-(e+iei)^2 = (1+iei-e)^2$. On a Hilbert space, because $1+iei-e$ is self-adjoint, the relation $(e + iei)^2 = 1$ therefore implies $1 + iei - e = 0$. Given that relation, we can then rewrite $ie = -ieii = (1 - e)i = i - ei$, $eie = e(i - ei) = 0$, etc. $\square$

Lemma *Let $\theta \in [0, \frac{\pi}{4}]$ be the unique solution of the equation $\lambda = 4\cos^2(\theta)\sin^2(\theta)$, and let us abbreviate $c := \cos(\theta)$ and $s := \sin(\theta)$. Then the representation $\mathcal{K}_\lambda \to M_2(\mathbb{C})$ given by*

$$e \mapsto \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix} \qquad i \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

induces an isomorphism $\mathcal{K}_\lambda \cong M_2(\mathbb{C})$ for $0 < \lambda < 1$.

For $\lambda = 0$ or 1, that representation descends to a faithful representation of $\tilde{\mathcal{K}}_0$ or $\tilde{\mathcal{K}}_1$. The image is $\cong \mathbb{C} \oplus \mathbb{C}$ for $\lambda = 0$, and $\cong M_2(\mathbb{R})$ for $\lambda = 1$.

Proof: If $0 < \lambda < 1$, we have $0 < \theta < \frac{\pi}{4}$, and the matrices

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e = \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix}, \quad iei = \begin{pmatrix} -c^2 & cs \\ cs & -s^2 \end{pmatrix},$$
$$\text{and} \quad eiei = \begin{pmatrix} -c^4 + c^2s^2 & c^3s - cs^3 \\ -c^3s + cs^3 & c^2s^2 - s^4 \end{pmatrix} = (c^2 - s^2)\begin{pmatrix} -c^2 & cs \\ -cs & s^2 \end{pmatrix}$$

are easily seen to span all reall two-by-two matrices. Similarly, by multiplying the above matrices by $\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$ on the right, we see that the elements i, ei, ie and eie span all two-by-two matrices with purely imaginary entires. It follows that $\dim_{\mathbb{R}}(\mathcal{K}_\lambda) \geq 8$. Combined with the previous lemma, we conclude that $\dim_{\mathbb{R}}(\mathcal{K}_\lambda) = 8$ and that the above representation is an isomorphism.

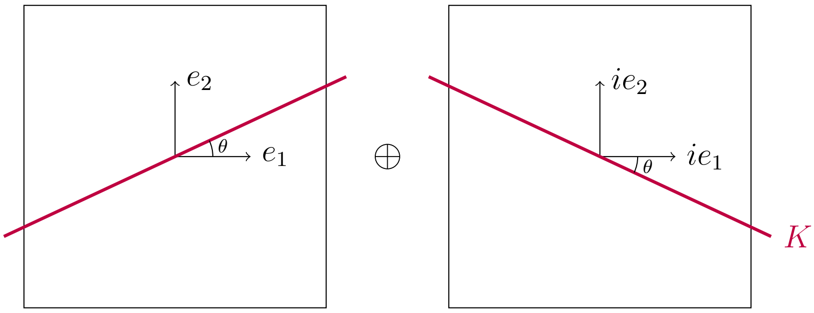
If $\lambda = 0$, the image of that representation is $\{\left(\begin{smallmatrix} z & 0 \\ 0 & w \end{smallmatrix}\right) \mid z, w \in \mathbb{C}\}$, which is isomorphic to $\mathbb{C} \oplus \mathbb{C}$. If $\lambda = 1$, the image is $\{\left(\begin{smallmatrix} z & w \\ \bar{w} & \bar{z} \end{smallmatrix}\right) \mid z, w \in \mathbb{C}\}$ and is equal to the image of $M_2(\mathbb{R})$

under the map $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} 1 & i \\ i & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & i \\ i & 1 \end{smallmatrix}\right)^{-1}$.

The existence of those representations implies $\dim_{\mathbb{R}}(\tilde{\mathcal{K}}_0) \geq 4$ and $\dim_{\mathbb{R}}(\tilde{\mathcal{K}}_1) \geq 4$. Combined with the result of the previous lemma, we conclude that $\dim_{\mathbb{R}}(\tilde{\mathcal{K}}_0) = \dim_{\mathbb{R}}(\tilde{\mathcal{K}}_1) = 4$. $\square$

The upshot of the above lemmas is that the irreducible representations of $\mathcal{K}$ are very easy to understand. Generically they are four real dimensional, and in some limit cases they can also be two dimensional.

This is what a generic irrep of $\mathcal{K}$ looks like:



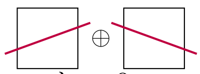
(i is indicated on the basis vectors, and e is the orthogonal projection onto the 2-dimensional subspace K).

In the non-generic cases (i.e., when θ equals 0 or $\frac{\pi}{4}$), those representations are no longer irreducible: they split as a direct sums of two 2-dimensional irreps. The above picture therefore contains all the information about the irreps of $\mathcal{K}$, including the special boundary cases $\lambda = 0, 1$ (i.e., $\theta = 0, \frac{\pi}{4}$).

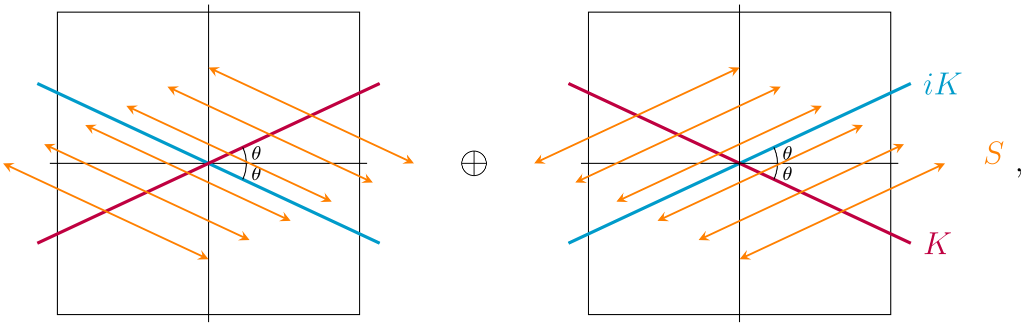
Equipped with the above complete understanding of the representation theory of $\mathcal{K}$, we can now get some results. Recall that S is the closure of the operator $a\Omega \mapsto a^*\Omega$, and that it can also be described as $\left\{\begin{smallmatrix} +1 & \text{on } K \\ -1 & \text{on } iK \end{smallmatrix}\right.$. Recall also that $\Delta := S^*S$ and that J is the closure of $S\Delta^{-\frac{1}{2}}$, so that $S = J\Delta^{\frac{1}{2}}$ is the polar decomposition of S.

Our Hilbert space H decomposes as a direct integral of the above representations. Note that because Ω is cyclic and separating, we have $(K + iK)^{\perp} = K \cap iK = 0$, so the 2-dimensional irreps that correspond to $\lambda = 0$ cannot occur as direct summands of H.

Lemma $J(iK^{\perp}) = K$.

Proof: Our Hilbert space decomposes as a $\int^{\oplus}$ of copies of  or direct summands thereof in the boundary case $\lambda = 1$ (recall that the case $\lambda = 0$ cannot occur here). It is therefore enough to show that the relation $J(iK^{\perp}) = K$ holds in every such representation.

Geometrically, the operator S looks as follows:



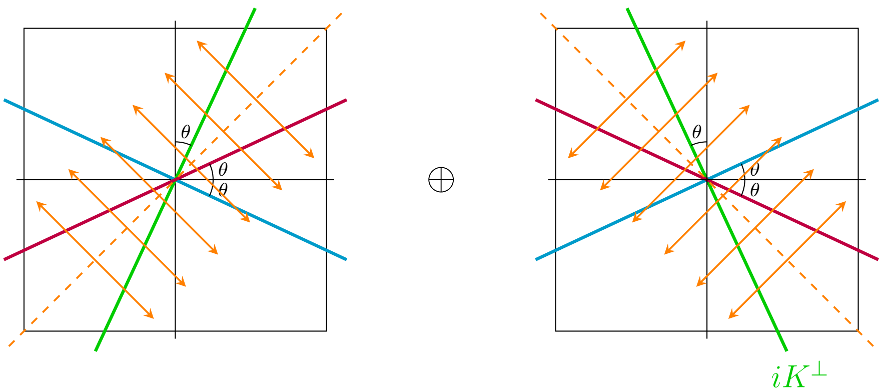
which translates algebraically into:

$$S = \begin{pmatrix} 0 & \cot(\theta) \\ \tan(\theta) & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -\cot(\theta) \\ -\tan(\theta) & 0 \end{pmatrix}.$$

It is then easy to compute

$$\begin{aligned} J :&= S(S^*S)^{-\frac{1}{2}} \\ &= \begin{pmatrix} 0 & \cot(\theta) \\ \tan(\theta) & 0 \end{pmatrix} \begin{pmatrix} \tan^2(\theta) & 0 \\ 0 & \cot^2(\theta) \end{pmatrix}^{-\frac{1}{2}} \oplus \begin{pmatrix} 0 & -\cot(\theta) \\ -\tan(\theta) & 0 \end{pmatrix} \begin{pmatrix} \tan^2(\theta) & 0 \\ 0 & \cot^2(\theta) \end{pmatrix}^{-\frac{1}{2}} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \end{aligned}$$

Putting this all together, it is now a simple geometric exercise to check that the relation $J(iK^\perp) = K$ holds:



□

Lemma *If $\xi, \eta \in iK^\perp$, then $\langle J\xi, \eta\rangle \in \mathbb{R}$.*

Proof: By the previous lemma, $J\xi$ sits in K and is therefore orthogonal to $i\eta$. We then have $\Re\mathrm{e}\langle J\xi, i\eta\rangle = 0 \Rightarrow \langle J\xi, i\eta\rangle \in i\mathbb{R} \Rightarrow \langle J\xi, \eta\rangle \in \mathbb{R}$. □

Recall that A is our von Neumann algebra and that it acts on a Hilbert space H with cyclic and separating vector $\Omega \in H$.

Lemma *If $x \in A'$ is self-adjoint, then $x\Omega \in iK^\perp$.*

Proof: Let x be as above. If $a \in A$ is self-adjoint, then $\langle a\Omega, x\Omega\rangle = \langle x\Omega, a\Omega\rangle$ and so $\Im\mathrm{m}\langle x\Omega, a\Omega\rangle = 0$. It follows that $\Re\mathrm{e}\langle x\Omega, ia\Omega\rangle = 0$ and so $x\Omega$ is orthogonal to $ia\Omega$. This being true for every a, we have shown that $x\Omega \in iK^\perp$. $\square$

Theorem (Haag duality) *In our case of interest*
(namely, the case $A = \mathcal{A}(I_-)$, $J = \Theta$, $JAJ = \mathcal{A}(I_+)$),
we have $A' = JAJ$.

Proof: The crucial extra piece of information that we have due to the specifics of our situation is the relation $JAJ \subseteq A'$ (that's the locality axiom of our conformal net).

If $x, y \in A'$ are self-adjoint, then by the last two lemmas, we have

$$\langle Jx\Omega, y\Omega\rangle = \langle y\Omega, Jx\Omega\rangle.$$

Using $J\Omega = \Omega$ and $J^* = J$, we can rewrite this as

$$\langle yJxJ\Omega, \Omega\rangle = \langle JxJy\Omega, \Omega\rangle. \tag{18}$$

By linearity in y and antilinearity in x, equation (18) then holds for every x and y in A'. One way to think of that last equation is as saying that "y and JxJ commute as far as Ω can see". We'd like to know that y and JxJ actually commute. Since Ω is cyclic for A, it is enough to check

$$\langle yJxJa\Omega, b\Omega\rangle \stackrel{?}{=} \langle JxJya\Omega, b\Omega\rangle$$

for every $a, b \in A$. Let us rewrite that last equation in the following way:

$$\langle yJ(JbJxJaJ)J\Omega, \Omega\rangle \stackrel{?}{=} \langle J(JbJxJaJ)Jy\Omega, \Omega\rangle. \tag{19}$$

As pointed out at the beginning of the proof, we do know that JaJ and JbJ are in A'. Therefore so is $JbJxJaJ$, and equation (19) is a special case of (18). Conclusion:

$$yJxJ = JxJy.$$

The latter holds for every $y \in A'$, and so we have shown that $JxJ \in A'' = A$. It follows that $JA'J \subseteq A$. In other words, we have proven that the inclusion $A' \subseteq JAJ$ holds. $\square$

The Free Fermion

In this section we will construct a certain conformal net called the *Majorana Free Fermion*. There are actually two chiral CFTs that go by the name *Free Fermion*: the *'Majorana Free Fermion'* and the *'Dirac Free Fermion'* (warning: there are also some full CFTs that go by the same names), related by:

$$(\text{Dirac Free Fermion}) = (\text{Majorana Free Fermion})^{\otimes 2}.$$

Actually, the Majorana Free Fermion is not a conformal net in the sense we introduced so far. It's a *super*-conformal net, that is, it's a conformal net where everything is $\mathbb{Z}/2$-graded and where some of the axioms are modified. In order to set up the Majorana Free Fermion (or any other super-conformal net), we'll need one more piece of structure on our standard circle: a *spinor bundle*.

The spinor bundle is a complex line bundle $\mathbb{S}$ over the unit disc $\mathbb{D}$, equipped with an isomorphism between its tensor square $\mathbb{S}^{\otimes 2}$ and the cotangent bundle $T^*\mathbb{D}$ (the isomorphism is part of the data). As a bundle, $T^*\mathbb{D}$ is just the trivial bundle, $\mathbb{S}$ is also trivial, and the isomorphism $\mathbb{S}^{\otimes 2} \cong T^*\mathbb{D}$ is also trivial. Only later will we see the real meaning of introducing the spinor bundle, when we'll want to consider the action of the Möbius group or when we'll want to extend it to $\mathbb{CP}^1$.

For the moment, we'll just write $f(z)\sqrt{dz}$ for sections of $\mathbb{S}$, where "$\sqrt{dz}$" is formal symbol. Under the isomorphism $\mathbb{S}^{\otimes 2} \cong T^*\mathbb{D}$, the tensor product $\big(f(z)\sqrt{dz}\big)\circ\big(g(z)\sqrt{dz}\big)$ goes to the 1-form $f(z)g(z)dz$.

The CAR algebra. Let $\Gamma(\mathbb{S}) := \Gamma(S^1, \mathbb{S})$ denote the space of sections of $\mathbb{S}$ over S^1. For our purposes, it doesn't matter what kind of sections one takes (we could take $\mathcal{C}^\infty$ sections, or $\mathcal{C}^0$ sections, or L^2 sections; let's say that we take $\mathcal{C}^0$ sections).

Definition: The algebra $CAR(S^1)$ of **C**anonical **A**nticommutation **R**elations is given by:

Generators: There is one generator $c(f)$ for every section $f \in \Gamma(\mathbb{S})$ and the symbol $c(f)$ depends linearly on f, namely, $c(f+g) = c(f) + c(g)$ and $c(\lambda f) = \lambda c(f)$ for $\lambda \in \mathbb{C}$.

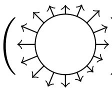
Relations: For any sections $f, g \in \Gamma(\mathbb{S})$, we have:
$$[c(f), c(g)]_+ = \frac{1}{2\pi i}\int_{S^1} fg$$
where $[\,,\,]_+$ is the anticommutator $[A, B]_+ := AB + BA$. Here, fg is viewed as a 1-form via the isomorphism $\mathbb{S}^{\otimes 2} \cong T^*\mathbb{D}$.

$*$-structure: If we let $f \mapsto \bar{f}$ be the antilinear involution on $\Gamma(\mathbb{S})$ given by $\overline{z^n\sqrt{dz}} := z^{-n-1}\sqrt{dz}$, then we set $c(f)^* := c(\bar{f})$
Exercise: Check that the $*$-structure is compatible with the relations: $[c(g)^*, c(f)^*]_+ = (\frac{1}{2\pi i}\int_{S^1} fg)^*$.

The way to remember the formula for $f \mapsto \bar{f}$ is to view $\sqrt{dz}$ as some kind of substitute for $z^{\frac{1}{2}}$. The formula $z^n\sqrt{dz} \mapsto z^{-n-1}\sqrt{dz}$ then becomes $z^{n+\frac{1}{2}} \mapsto z^{-(n+\frac{1}{2})}$, which agrees with our intuition about bar.

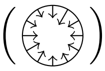
The operation $f \mapsto \bar{f}$ on sections of $\mathbb{S}$ also admits a geometric description:

Lemma *The sections $f \in \Gamma(\mathbb{S})$ that satisfy $\bar{f} = f$ are those whose square pairs positively with every normal outgoing vectors field; the sections $f \in \Gamma(\mathbb{S})$ that satisfy $\bar{f} = -f$ are those whose square pairs positively to every normal ingoing vectors field:*

$\bar{f} = f \qquad \Leftrightarrow \qquad f^2\Big($  $\Big)_{v} \geq 0 \qquad \forall\, v$ *normal outgoing*

that's a 1-form under the isomorphism $\mathbb{S}^{\otimes 2} \cong T^*$.

that's a function on S^1

$\bar{f} = -f \qquad \Leftrightarrow \qquad f^2\Big($  $\Big)_{v} \geq 0 \qquad \forall\, v$ *normal ingoing*

Proof: The condition of pairing positively with normal outgoing vectors defines a ray bundle (a ray is half of a line) inside $T^*\mathbb{D}|_{S^1}$. Its preimage under the squaring map $f \mapsto f^2 : \mathbb{S} \to T^*\mathbb{D}$ is a real line bundle $\mathbb{S}^+ \subset \mathbb{S}$. Similarly, the condition of pairing positively with normal ingoing vectors defines a ray bundle inside $T^*\mathbb{D}|_{S^1}$ (the negative of the previous ray bundle) whose preimage under the squaring map is a real line bundle $\mathbb{S}^- \subset \mathbb{S}$. Since $(\mathbb{S}^-)^2 = -(\mathbb{S}^+)^2$, we have $\mathbb{S}^- = i\mathbb{S}^+$, and in particular $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$.

Now, $z^n\sqrt{dz} + z^{-n-1}\sqrt{dz}$ and $iz^n\sqrt{dz} - iz^{-n-1}\sqrt{dz}$ form a basis of $\{f \in \Gamma(\mathbb{S}) \mid \bar{f} = f\}$. Any normal outgoing vector field is of the form $g(z)z\,\partial_z$ with g an $\mathbb{R}_+$-valued function, so we can check:

$$\big(z^n\sqrt{dz} + z^{-n-1}\sqrt{dz}\big)^2\big(g(z)z\,\partial_z\big) = g(z)(z^{2n+1} + 2 + z^{-(2n+1)}) = g(z)|1 + z^{2n+1}|^2 \geq 0$$
$$\big(iz^n\sqrt{dz} - iz^{-n-1}\sqrt{dz}\big)^2\big(g(z)z\,\partial_z\big) = g(z)(-z^{2n+1} + 2 - z^{-(2n+1)}) = g(z)|1 - z^{2n+1}|^2 \geq 0.$$

It follows that $\{\bar{f} = f\} \subseteq \Gamma(\mathbb{S}^+)$.

Similarly, $iz^n\sqrt{dz} + iz^{-n-1}\sqrt{dz}$ and $z^n\sqrt{dz} - z^{-n-1}\sqrt{dz}$ form a basis of $\{f \in \Gamma(\mathbb{S}) \mid \bar{f} = -f\}$ and we have

$$\big(iz^n\sqrt{dz} + iz^{-n-1}\sqrt{dz}\big)^2(-g(z)z\,\partial_z) = g(z)|1 + z^{2n+1}|^2 \geq 0$$
$$\big(z^n\sqrt{dz} - z^{-n-1}\sqrt{dz}\big)^2(-g(z)z\,\partial_z) = g(z)|1 - z^{2n+1}|^2 \geq 0.$$

It follows that $\{\bar{f} = -f\} \subseteq \Gamma(\mathbb{S}^-)$.

Finally, since $\Gamma(\mathbb{S})$ can be written as both $\{\bar{f} = f\} \oplus \{\bar{f} = -f\}$ and $\Gamma(\mathbb{S}^+) \oplus \Gamma(\mathbb{S}^-)$, we must have $\{\bar{f} = f\} = \Gamma(\mathbb{S}^+)$ and $\{\bar{f} = -f\} = \Gamma(\mathbb{S}^-)$. $\square$

Let $\Gamma_{>0}(\mathbb{S})$ denote the space of section of $\mathbb{S}$ over S^1 that extend to holomorphic sections over $\mathbb{D}$. It is the subspace spanned by $z^n\sqrt{dz}$ for $n \geq 0$. Here, we write $\Gamma_{>0}(\mathbb{S})$ instead of $\Gamma_{\geq 0}(\mathbb{S})$ because we think of $\sqrt{dz}$ as having degree $1/2$.

Lemma *If f and g are in $\Gamma_{>0}(\mathbb{S})$, then $\frac{1}{2\pi i}\int_{S^1} fg = 0$.*

Proof: The 1-form fg on S^1 extends to a holomorphic 1-form on $\mathbb{D}$, and its integral along $S^1 = \partial\mathbb{D}$ is then zero by Cauchy's theorem. $\square$

The CAR algebra therefore contains the exterior algebra $\bigwedge^\bullet \Gamma_{>0}(\mathbb{S})$ as a subalgebra. Let $\mathbb{C}$ be the trivial module of $\bigwedge^\bullet \Gamma_{>0}(\mathbb{S})$ (where all the generators act as zero), and let Ω denote its standard basis element (i.e., the element $1 \in \mathbb{C}$). We then consider the induced module

$$\mathrm{Ind}_{\bigwedge \Gamma_{>0}}^{CAR}\mathbb{C} \;:=\; \mathrm{Ind}_{\bigwedge^\bullet \Gamma_{>0}(\mathbb{S})}^{CAR(S^1)}\mathbb{C} \;=\; CAR(S^1) \otimes_{\bigwedge^\bullet \Gamma_{>0}(\mathbb{S})} \mathbb{C}.$$

Lemma *The module* $\mathrm{Ind}^{CAR}_{\bigwedge\Gamma_{>0}}\mathbb{C}$ *is spanned by* $c(f_1)c(f_2)\ldots c(f_n)\Omega$ *with* $f_i \in \Gamma_{<0}(\mathbb{S})$.

Proof: Identical to the proof of the lemma on page 28. □

Let $\Gamma_{<0}(\mathbb{S}) := \mathrm{Span}\{z^n\sqrt{dz}\,|\,n < 0\}$ be the orthogonal complement of $\Gamma_{>0}(\mathbb{S})$ with respect to the inner product $\langle f, g\rangle := \frac{1}{2\pi i}\int_{S_1} f\bar{g}$ on $\Gamma(\mathbb{S})$. The subspace $\Gamma_{<0}(\mathbb{S})$ is most conveniently describing by means of the spinor bundle over $\mathbb{CP}^1$ (which we'll again call $\mathbb{S}$). The latter is the line bundle over $\mathbb{CP}^1$ defined by declaring its sections over some open $U \subset \mathbb{CP}^1$ to be expressions of the form $f(z)\sqrt{dz}$, where f is a function over $U \setminus \{\infty\}$ subject to the condition that if $\infty \in U$ then $\lim_{z\to\infty} z\, f(z)$ should exist. This bundle goes by the name $\mathcal{O}(-1)$ in algebraic geometry. As before, the isomorphism $\mathbb{S}^{\otimes 2} \cong T^*\mathbb{CP}^1$ sends $(f(z)\sqrt{dz})(g(z)\sqrt{dz})$ to $f(z)g(z)dz$.

To verify that the above map is indeed an isomorphism, we need to check that a 1-form $f(z)dz$ extends over $\infty \in \mathbb{CP}^1$ iff the limit $\lim_{z\to\infty} z^2 f(z)$ exists. **Exercise:** Verify that a 1-form $f(z)dz$ defined on $\mathbb{C}$ extends over $\infty \in \mathbb{CP}^1$ iff the limit $\lim_{z\to\infty} z^2 f(z)$ exists. *Hint*: Define $r(z) := z^{-1}$ and check whether $r^*(f(z)dz)$ extends to 0.

With the above preliminaries in place, we can now describe $\Gamma_{<0}(\mathbb{S})$ geometrically. The elements of $\Gamma_{<0}(\mathbb{S})$ are the sections of $\mathbb{S}$ over S^1 that extend to holomorphic section over $\mathbb{D}' := \{z \in \mathbb{C}\,|\,z \geq 1\} \cup \{\infty\} \subset \mathbb{CP}^1$.

Lemma *If* f *and* g *are in* $\Gamma_{<0}(\mathbb{S})$*, then* $\frac{1}{2\pi i}\int_{S^1} fg = 0$.

Proof: If f and g extend to holomorphic sections over $\mathbb{D}'$, then so does the 1-form fg. Its contour integral is therefore zero by Cauchy's theorem. □

Proposition *The module* $\mathrm{Ind}^{CAR}_{\bigwedge\Gamma_{>0}}\mathbb{C}$ *admits a unique positive definite inner product such that* $\langle\Omega, \Omega\rangle = 1$ *and for which the action of* $CAR(S^1)$ *is compatible with the* $*$ *operation. Its completion is the* <u>*Fock space*</u>*:*

$$\boxed{\mathcal{F} := \text{Hilbert space completion of } \mathrm{Ind}^{CAR}_{\bigwedge\Gamma_{>0}}\mathbb{C}}$$

Proof: The proof of uniqueness is identical to the one presented on page 26 and we shall not repeat it here. Consider the map

$$\begin{array}{rcl} \bigwedge^\bullet \Gamma_{<0}(\mathbb{S}) & \to & \mathrm{Ind}^{CAR}_{\bigwedge\Gamma_{>0}}\mathbb{C} \\ f_1 \wedge \ldots \wedge f_n & \mapsto & c(f_1)\ldots c(f_n)\Omega. \end{array} \tag{20}$$

That map is well defined because $[c(f_i), c(f_j)]_+ = 0$ for any $f_i, f_j \in \Gamma_{<0}(\mathbb{S})$, and it is surjective by the previous lemma.

The vector space $\bigwedge^\bullet \Gamma_{<0}(\mathbb{S})$ has an inner product given by

$$\langle f_1 \wedge \ldots \wedge f_n, g_1 \wedge \ldots \wedge g_m\rangle = \delta_{n,m}\sum_{\sigma\in S_n}(-1)^\sigma\langle f_i, g_{\sigma(i)}\rangle$$

where the right hand side uses the inner product $\langle f, g\rangle := \frac{1}{2\pi i}\int_{S_1} f\bar{g}$ on $\Gamma_{<0}(\mathbb{S})$. It is easy to check that this formula is well defined, and gives a positive definite inner product:

the formula is made so that an orthonormal basis $\{e_i\}$ of a Hilbert space H induces a corresponding orthonormal basis $\{e_{i_1} \wedge \ldots \wedge e_{i_k}\}_{i_1<\ldots<i_k}$ of $\bigwedge^\bullet H$.

The space $\bigwedge^\bullet \Gamma_{<0}(\mathbb{S})$ has an action of $CAR(S^1)$ given by

$$\text{letting } c(f) \text{ act by} \begin{cases} f \wedge - & \text{if } f \in \Gamma_{<0}(\mathbb{S}) \\ (\bar{f} \wedge -)^* & \text{if } f \in \Gamma_{>0}(\mathbb{S}). \end{cases} \tag{21}$$

It will be convenient to abbreviate $(f \wedge -)^*$ by $f \lrcorner -$; here's the formula for that operation:

$$f \lrcorner (g_1 \wedge \ldots \wedge g_n) = \sum_{i=1}^{n} (-1)^{i+1} \langle g_i, f \rangle\, g_1 \wedge \ldots \widehat{g_i} \ldots \wedge g_n.$$

A few verifications are in order before we can claim that (21) defines an action.

First of all, the compatibility with the $*$-structure is satisfied by definition (indeed, formula (21) can be deduced by looking at just that compatibility). We now check the main relation:

$$\Big[c(z^n\sqrt{dz}), c(z^m\sqrt{dz})\Big]_+ = \begin{cases} 0 \quad \text{if } n, m \geq 0 \text{ or if } n, m < 0 \\ \Big[(z^{-n-1}\sqrt{dz}) \lrcorner -, (z^m\sqrt{dz}) \wedge -\Big]_+ = \delta_{-n-1,m}, \end{cases}$$

WLOG $n \geq 0$ and $m < 0$.

where we have used:

Mini-lemma: *(1) If* $\|s\| = 1$*, then* $[s \lrcorner -, s \wedge -]_+ = 1$.
(2) If $s_1 \perp s_2$*, then* $[s_1 \lrcorner -, s_2 \wedge -]_+ = 0$.

Proof: (1) Write a general element on which this acts as $A + B$ with $A = s \wedge (\ldots)$ and $B = (\ldots)$, where the $(\ldots)$ only involves stuff that is orthogonal to s. The term $(s \lrcorner -) \circ (s \wedge -)$ acts as 0 on A and as 1 on B. Similarly, the term $(s \wedge -) \circ (s \lrcorner -)$ acts as 1 on A and as 0 on B. The sum of these two operators is therefore equal to 1.
(2) **Exercise:** Write a general element as $s_1 \wedge s_2 \wedge (\ldots) + s_1 \wedge (\ldots) + s_2 \wedge (\ldots) + (\ldots)$ and check that the relation holds. $\square$

Since $1 \in \bigwedge^\bullet \Gamma_{<0}(\mathbb{S})$ is annihilated by all the $c(f)$ with $f \in \Gamma_{>0}(\mathbb{S})$, by the universal property of induced modules, there exists a map of $CAR(S^1)$-modules

$$\begin{array}{ccc} \mathrm{Ind}_{\wedge\Gamma_{>0}}^{CAR}\mathbb{C} & \to & \bigwedge^\bullet \Gamma_{<0}(\mathbb{S}) \\ \Omega & \mapsto & 1 \end{array}$$

It is easy to check that the composite

$$\bigwedge^\bullet \Gamma_{<0}(\mathbb{S}) \to \mathrm{Ind}_{\wedge\Gamma_{>0}}^{CAR}\mathbb{C} \to \bigwedge^\bullet \Gamma_{<0}(\mathbb{S})$$

is the identity. The map (20) is therefore injective. We already knew that it is surjective. It is therefore an isomorphism. To finish the proof, we use that isomorphism to transport the inner product on $\bigwedge^\bullet \Gamma_{<0}(\mathbb{S})$ to an inner product on $\mathrm{Ind}_{\wedge\Gamma_{>0}}^{CAR}\mathbb{C}$. $\square$

Lemma *The Fock space $\mathcal{F}$ is an irreducible $CAR(S^1)$ module.*

Proof: Let us recall the statement of Schur's lemma: if a module is irreducible, then its endomorphism algebra is one dimensional.

In the context of $*$-algebras acting on Hilbert spaces, the converse also holds: if a module is not irreducible, then its endomorphism algebra is non-trivial. Indeed, the orthogonal projection onto any submodule always commutes with the algebra. (This can be checked as follows: if p is an orthogonal projection onto a submodule, then we have

$$\langle ap\xi, \eta\rangle = \langle pap\xi, \eta\rangle = \langle \xi, pa^*p\eta\rangle = \langle \xi, a^*p\eta\rangle = \langle pa\xi, \eta\rangle$$

for every a in the algebra and every vectors ξ and η in the Hilbert space. Hence $ap = pa$.)

Let $a : \mathcal{F} \to \mathcal{F}$ be a module endomorphism. The vacuum vector Ω is the only vector up to scalar that is annihilated by all the operators $c(f)$ with $f \in \Gamma_{>0}(\mathbb{S})$. Therefore, $a\Omega = \lambda\Omega$ for some $\lambda \in \mathbb{C}$. The vector Ω being furthermore cyclic, it follows that $a = \lambda \cdot \mathrm{Id}$. $\square$

Now that we have constructed the state space $\mathcal{F}$, we can define the local algebras of the Majorana Free Fermion conformal net:

$$\boxed{\mathcal{A}_{Fer}(I) := \{c(f) \,|\, \mathrm{Supp}(f) \subset I\}''.}$$

We now see a striking difference with the case of the WZW conformal nets. In the case of the WZW conformal nets, the local algebras $\mathcal{A}_{G,k}$ (defined on page 27) could **<u>not</u>** be described as

$$\xcancel{\mathcal{A}_{G,k}(I) := \{f \in \widetilde{L\mathfrak{g}}_k \,|\, \mathrm{Supp}(f) \subset I\}''}$$

because the Lie algebra $\widetilde{L\mathfrak{g}}_k$ does not consist of bounded operators. That's why we had to use the Lie group $\widetilde{LG}_k$ instead (thus introducing a whole new set of technical difficulties). In the case of the free fermions conformal net, we have instead:

Lemma ($\star$) *The operators $c(f)$ are bounded.*

Proof: For every $\xi \in \mathcal{F}$, we have

$$\|c(f)\xi\|^2 \le \|c(f)\xi\|^2 + \|c(f)^*\xi\|^2 = \big\langle (c(f)^*c(f) + c(f)c(f)^*)\xi, \xi\big\rangle = \tfrac{1}{2\pi i}\int_{S^1} \bar{f}f \,\|\xi\|^2.$$

Therefore $\|c(f)\|^2 \le \|f\|^2 < \infty$. $\square$

We now turn our attention to the action of the Möbius group on the Fock space $\mathcal{F}$. The free fermion being a super-conformal net, it is only the following double cover of the Möbius group that acts on $\mathcal{F}$:

$$\mathrm{M\ddot{o}b}^{(2)}(S^1) := \left\{ \begin{array}{l|l} g : S^1 \to S^1 & g \text{ extends to an (anti)holomorphic map } G : \mathbb{D} \to \mathbb{D}, \\ g^{(2)} : \mathbb{S} \to \mathbb{S} & g^{(2)} \otimes g^{(2)} = \underbrace{T^*G^{-1}}_{\text{covariant in } G} \text{ under the identification } \mathbb{S}^{\otimes 2} \cong T^*\mathbb{D} \end{array} \right\}$$

Note that $g^{(2)}$ is complex linear when g is orientation preserving and complex antilinear when g is orientation reversing. By abuse of notation, we'll denote an element $(g, g^{(2)})$ of $\text{Möb}^{(2)}(S^1)$ simply by g.

The action of $\text{Möb}^{(2)}(S^1)$ on $CAR(S^1)$ is given by:

$$\boxed{g \cdot c(f) = c(g^{(2)} \circ f \circ g^{-1}).}$$

Note: if you think of $f \in \Gamma(\mathbb{S})$ in terms of its graph, then $g^{(2)} \circ f \circ g^{-1}$ is what you get when you apply $g^{(2)}$ to the graph of f.

The fact that the action respects the CAR relations is obvious for $g \in \text{Möb}^{(2)}_+(S^1)$, but does require verification for $g \in \text{Möb}^{(2)}_-(S^1)$:

$$\Big[g \cdot c(f_1), g \cdot c(f_2)\Big]_+ = \tfrac{1}{2\pi i}\int_{S^1} (T^*G^{-1})(f_1 f_2) = -\tfrac{1}{2\pi i}\overline{\int_{S^1} f_1 f_2} = \overline{\tfrac{1}{2\pi i}\int_{S^1} f_1 f_2} = g \cdot \Big(\tfrac{1}{2\pi i}\int_{S^1} f_1 f_2\Big).$$

We get a minus sign here because the change of variables is orientation reversing, and a bar because T^*G^{-1} is antilinear.

The compatibility of the action with the $*$-operation follows from the lemma on page 50: self-adjoint generators go to self-adjoint generators and skew-adjoint generators go to skew-adjoint generators (note that this fact it would be quite unclear had we only introduced the definition on page 50).

Finally, the above action of $\text{Möb}^{(2)}(S^1)$ respects the subspace $\Gamma_{>0}(\mathbb{S})$, and so we get an induced action on the Fock space in the obvious way:

$$g \cdot \big(c(f_1) \dots c(f_n)\Omega\big) = c\big(g^{(2)} \circ f_1 \circ g^{-1}\big) \dots c\big(g^{(2)} \circ f_n \circ g^{-1}\big)\Omega.$$

Note: The computations that we made above also show that the following double cover of the diffeomorphism group

complexification if g is orientation preserving, anticomplexification if g is orientation reversing.

$$\text{Diff}^{(2)}(S^1) := \left\{ \begin{array}{c|c} g : S^1 \to S^1 & g^{(2)} \otimes g^{(2)} \text{ is the (anti)complexification of} \\ g^{(2)} : \mathbb{S} \to \mathbb{S} & T^* g^{-1} \text{ under the identification } \mathbb{S}^{\otimes 2} \cong T^*S^1 \otimes_{\mathbb{R}} \mathbb{C} \end{array} \right\}$$

acts on $CAR(S^1)$. Here, the complexification of an $\mathbb{R}$-linear map $f : V \to V$ is the map $f \otimes (\text{Id}_{\mathbb{C}}) : V \otimes_{\mathbb{R}} \mathbb{C} \to V \otimes_{\mathbb{R}} \mathbb{C}$, and its anticomplexification is the map $f \otimes (z \mapsto \bar{z}) : V \otimes_{\mathbb{R}} \mathbb{C} \to V \otimes_{\mathbb{R}} \mathbb{C}$.

Now that that we finished describing the Majorana Free Fermion, let us give the general definition of a super-conformal net:

Definition: A **Super-conformal net** is what you get if you take the definition on page 15 and perform the following modifications:

- Everything is now $\mathbb{Z}/2$-graded: both the Hilbert space and the local algebras.
- The groups $\text{Möb}(S^1)$ and $\text{Diff}(S^1)$ are replaced by their double covers $\text{Möb}^{(2)}(S^1)$ and $\text{Diff}^{(2)}(S^1)$, respectively. Both of them act by degree preserving maps on H.

The axioms should then be modified in the following way. Let $s \in \ker(\text{Möb}^{(2)}(S^1) \to \text{Möb}(S^1))$ be the so-called *spin involution*, given by the identity on S^1 and by -1 on $\mathbb{S}$. Let $\gamma : H \to H$ be the so-called *grading involution*, given by $+1$ on even part of H and by -1 on its odd part.

- The vacuum vector Ω is in the even part of H.
- The spin involution is required to act like the grading involution on H.
- Locality is replaced by *super-locality*: if I and J are disjoint and if $a \in \mathcal{A}(I)$ and $b \in \mathcal{A}(J)$, then we require that $ab = (-1)^{|a||b|}ba$. Here, $|a|$ is 0 for a even, and 1 for a odd (and let us agree that whenever we use that notation, we are implicitly assuming that a is homogeneous).

The Majorana Free fermion is $\mathbb{Z}/2$-graded by declaring the operators $c(f)$ to be odd. The grading on Fock space is then given by declaring

$$c(f_1)\dots c(f_n)\Omega \in \mathcal{F}$$

to be even if n is even and odd if n is odd.

Recap: The Majorana Free Fermion conformal net is given by:

generators:	$c(f)$ for $f \in \Gamma(\mathbb{S})$
relations:	$[c(f), c(g)]_+ = \langle f, \bar{g}\rangle$
$*$-operation:	$c(f)^* = c(\bar{f})$
action:	$c(f)$ kills Ω if $f \in \Gamma_{>0}(\mathbb{S})$

Table 1: The Majorana Free Fermion

We mentioned earlier the Dirac Free fermion. Here's its description:

generators:	$a^\dagger(f)$ and $a(f)$ for $f \in \Gamma(\mathbb{S})$ $\quad a^\dagger(\lambda f) = \lambda a^\dagger(f),\ a(\lambda f) = \bar{\lambda} a(f)$
relations:	$[a^\dagger(f), a^\dagger(g)]_+ = [a(f), a(g)]_+ = 0$, $[a^\dagger(f), a(g)]_+ = \langle f, g\rangle$
$*$-operation:	$a(f)^* = a^\dagger(f)$
action:	$a^\dagger(f)$ kills Ω if $f \in \Gamma_{>0}(\mathbb{S})$, $a(f)$ kills Ω if $f \in \Gamma_{<0}(\mathbb{S})$

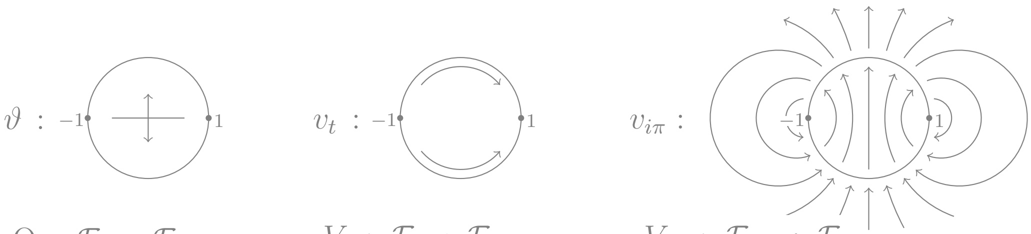
Table 2: The Dirac Free Fermion

The isomorphism (Dirac Free Fermion) $\cong$ (Majorana Free Fermion)$^{\otimes 2}$ is given by

$$\begin{aligned}
a^\dagger(f) &\longmapsto \tfrac{1}{\sqrt{2}}\big(c(f)\otimes 1 + 1 \otimes ic(f)\big) \\
a(f) &\longmapsto \tfrac{1}{\sqrt{2}}\big(c(\bar{f})\otimes 1 - 1 \otimes ic(\bar{f})\big) \\
\tfrac{1}{\sqrt{2}}\big(a^\dagger(f) + a(\bar{f})\big) &\longleftarrow c(f)\otimes 1 \\
\tfrac{1}{i\sqrt{2}}\big(a^\dagger(f) - a(\bar{f})\big) &\longleftarrow 1 \otimes c(f).
\end{aligned}$$

***Extra material*: Haag duality for the Free Fermions.**

It will be important later to know that the Free Fermion conformal net satisfies Haag duality. The latter is only a super-conformal net, and so there are a couple of substantial differences from the non-super case. The argument is nevertheless mostly similar. Let



$\Theta : \mathcal{F} \to \mathcal{F}$ $\qquad$ $V_t : \mathcal{F} \to \mathcal{F}$ $\qquad$ $V_{i\pi} : \mathcal{F} \dashrightarrow \mathcal{F}$

be as on page 39. Actually, in order for Θ and V_z to be really well defined, the maps ϑ and v_z should also be specified at the level of the spinor bundle: ϑ is defined to be $z \mapsto \bar{z}$ on both S^1 and $\mathbb{S}$, and the transformations $\{v_z\}_{z\in\mathbb{C}}$ are defined by the requirement that they depend continuously on z and that it's the identity when $z = 0$.

If a is an element of some $\mathbb{Z}/2$-graded $*$-algebra, let us define

$$a^\# := \begin{cases} a^* & \text{if } a \text{ is even} \\ -ia^* & \text{if } a \text{ is odd.} \end{cases}$$

As before, we let $I_- \subset S^1$ be the lower half of the circle, and we also let $\mathbb{D}' := \{z \in \mathbb{C} \,|\, z \geq 1\} \cup \{\infty\}$.

Proposition *For every $a \in \mathcal{A}_{Fer}(I_-)$, we have*

$$\Theta V_{i\pi} a\Omega = a^\#\Omega,$$

equivalently, $V_{i\pi}a\Omega = \Theta a^\#\Omega$.

Proof: Let $\mathcal{D}_0 \subset \mathcal{F}$ be the dense domain given by

$$\mathcal{D}_0 := \mathrm{Span}\left\{ c(f_1)\ldots c(f_n)\Omega \;\middle|\; \begin{array}{l} \mathrm{supp}(f_i) \subset I_- \text{ and } f_i|_{I_-} \text{ extends to} \\ \text{a holomorphic section } F_i \in \Gamma(\mathbb{D}', \mathbb{S}). \end{array} \right\}$$

We check $V_{i\pi}a\Omega = \Theta a^\#\Omega$ against elements of $\mathcal{D}_0$:

$$\big\langle V_{i\pi}a\Omega, c(f_1)\ldots c(f_n)\Omega\big\rangle \stackrel{?}{=} \big\langle \Theta a^\#\Omega, c(f_1)\ldots c(f_n)\Omega\big\rangle.$$

Let

$$g_i := (F_i \circ v_{-i\pi})|_{I_-} \qquad \text{and} \qquad h_i := (F_i \circ v_{-i\pi})|_{I_+}$$

(really, the more correct thing to write should be $v^{(2)}_{i\pi} \circ F_i \circ v_{-i\pi}$ instead of $F_i \circ v_{-i\pi}$, where $v^{(2)}_{i\pi}$ is the action of $v_{i\pi}$ on $\mathbb{S}$), so that

$$V_{i\pi}c(f_1)\ldots c(f_n)\Omega = c(g_1)\ldots c(g_n)\Omega \qquad \text{and}$$

$$\Theta c(f_1)\ldots c(f_n)\Omega \overset{\triangle}{=} c(i\bar{h}_1)\ldots c(i\bar{h}_n)\Omega = i^n c(h_1)^*\ldots c(h_n)^*\Omega.$$

This is super-confusing... I hope I got it right.

We then have (by the same calculation as on page 43, but with $\binom{n}{2}$ extra minus signs):

$$\begin{aligned}\big\langle V_{i\pi}a\Omega, c(f_1)\dots c(f_n)\Omega\big\rangle &= \big\langle a\Omega, V_{i\pi}c(f_1)\dots c(f_n)\Omega\big\rangle \\ &= \big\langle a\Omega, c(g_1)\dots c(g_n)\Omega\big\rangle \\ &= (-1)^{\binom{n}{2}}(-1)^n\big\langle a\Omega, c(h_n)\dots c(h_1)\Omega\big\rangle \\ &= (-1)^{\binom{n}{2}}(-1)^n\big\langle c(h_1)^*\dots c(h_n)^*\Omega, a^*\Omega\big\rangle =\end{aligned}$$

and

$$\begin{aligned}\big\langle \Theta a^{\#}\Omega, c(f_1)\dots c(f_n)\Omega\big\rangle &= \big\langle \Theta c(f_1)\dots c(f_n)\Omega, a^{\#}\Omega\big\rangle \\ &= i^n\big\langle c(h_1)^*\dots c(h_n)^*\Omega, a^{\#}\Omega\big\rangle =\end{aligned}$$

To check this, it's best to split things into four cases: $n \equiv 0, 1, 2, 3$ (mod 4).

Since $\mathcal{D}_0$ is invariant under $\{V_t\}_{t\in\mathbb{R}}$, it is a core of $V_{i\pi}$. It follows from the above computation that $a\Omega \in \mathcal{D}_{V_{i\pi}}$ (in particular, the first term $\langle V_{i\pi}a\Omega, c(f_1)\dots c(f_n)\Omega\rangle$ is well defined) and that $V_{i\pi}a\Omega = \Theta a^{\#}\Omega$. □

Now, since $\mathcal{A}_{Fer}(I_-)\Omega$ is invariant under $\{V_t\}_{t\in\mathbb{R}}$, it is a core of $V_{i\pi}$ and $\Theta V_{i\pi}$ is the polar decomposition of (the closure of) the operator

$$\begin{array}{ccc}\mathcal{A}_{Fer}(I_-)\Omega & \to & \mathcal{A}_{Fer}(I_-)\Omega. \\ a\Omega & \mapsto & a^{\#}\Omega\end{array}$$

Letting $\kappa := \left\{\begin{smallmatrix} 1 & \text{on the even part} \\ i & \text{on the odd part}\end{smallmatrix}\right. : \mathcal{F} \to \mathcal{F}$, so that $S = \kappa\Theta V_{1\pi}$, it follows that

$$J = \kappa\Theta.$$

(where S and J are the modular operators of Tomita-Takesaki theory, defined on page 35). Recall the definition of the commutant $A' := \{b \,|\, ab = ba, \forall a \in A\}$ of a von Neumann algebra A. If A is $\mathbb{Z}/2$-graded, let us define its super-commutant by:

$$A^{\blacktriangledown} := \{b \,|\, ab = (-1)^{|a||b|}ba,\ \forall a \in A\}.$$

Lemma *Let κ be as above. Then* $\kappa A^{\blacktriangledown}\kappa^{-1} = A'$.

Proof: $\kappa A^{\blacktriangledown}\kappa^{-1} = A^{\blacktriangledown}_{even} \oplus i\gamma A^{\blacktriangledown}_{odd} = A^{\blacktriangledown}_{even} \oplus \gamma A^{\blacktriangledown}_{odd} = A'_{even} \oplus A'_{odd} = A'$, where γ is the grading involution. Here, we have used that:
If b is an odd operator, then $\big(ab = (-1)^{|a||b|}ba,\ \forall a \in A\big) \iff \big(a(\gamma b) = (\gamma b)a,\ \forall a \in A\big)$. □

We are now in position to prove Haag duality for the case $\mathcal{A} = \mathcal{A}_{Fer}$. By the above proposition, we have

$$\begin{aligned}J\mathcal{A}(I_-)J = \kappa\Theta\mathcal{A}(I_-)\kappa\Theta &= \kappa\Theta\mathcal{A}(I_-)\Theta\kappa^{-1} \\ &= \kappa\mathcal{A}(I_+)\kappa^{-1} \subseteq \kappa\mathcal{A}(I_-)^{\blacktriangledown}\kappa^{-1} = \mathcal{A}(I_-)'\end{aligned} \tag{22}$$

and we have already seen (see proof on p.49) that $(JAJ' \subseteq A) \Rightarrow (JAJ' = A)$, therefore $J\mathcal{A}(I_-)J = \mathcal{A}(I_-)'$ and the inclusion in (22) is an equality: $\mathcal{A}_{Fer}(I_+) = \mathcal{A}_{Fer}(I_-)^{\blacktriangledown}$.

The Segal quantization criterion

Let's now focus on the abstract setup needed to define the Majorana CAR algebra and its Fock representation. The basic inputs are:

- A complex Hilbert space H (in our case, $H = \Gamma(\mathbb{S})$)
- A real structure $f \mapsto \bar{f}$ on H (satisfies $\langle \bar{f}, \bar{g} \rangle = \overline{\langle f, g \rangle}$)
- A complex subspace $H_{>0} \subset H$ such that $\overline{H_{>0}} = H_{>0}^{\perp}$ (in our case, $H_{>0} = \Gamma_{>0}(\mathbb{S})$).

The first two items are equivalent to having a real Hilbert space $H_{\mathbb{R}} := \{f \in H \mid f = \bar{f}\}$. They're what one needs to define the CAR algebra: we have generators $c(f)$ for every $f \in H$ subject to the relations $[c(f), c(g)]_+ = \langle f, \bar{g} \rangle$ and $c(f)^* = c(\bar{f})$.

The third item is called a *polarization*, and is needed to construct the Fock space:

$$\mathcal{F} \;:=\; \text{Hilbert space completion of } \textstyle\bigwedge^{\bullet} H_{<0}.$$

Here, $H_{<0} := H_{>0}^{\perp}$. The action of the CAR algebra on $\mathcal{F}$ is given by

$$\begin{cases} c(f) \;\mapsto\; f \wedge - & \text{for } \; f \in H_{<0} \\ c(f) \;\mapsto\; \bar{f} \lrcorner - & \text{for } \; f \in H_{>0}. \end{cases}$$

In our case of interest, we have an action of the double cover $\mathrm{Diff}^{(2)}(S^1)$ of the diffeomorphism group of S^1 on the Hilbert space $H = \Gamma(\mathbb{S})$, and thus on its CAR algebra. We want an action of that goup on $\mathcal{F}$ such that

$$(*) \qquad \boxed{g \cdot \big(c(f)\xi\big) = c(g \cdot f)(g \cdot \xi)} \qquad \begin{pmatrix} f \in H,\ \xi \in \mathcal{F} \\ g \in \mathrm{Diff}^{(2)}(S^1) \end{pmatrix}$$

At this point, we'll no longer worry about the orientation reversing elements of $\mathrm{Diff}^{(2)}(S^1)$ because we've already shown that $\text{Möb}^{(2)}(S^1)$ acts on $\mathcal{F}$. So, from now on, we'll restrict our attention to the subgroup

$$\mathrm{Diff}_+^{(2)}(S^1) = \left\{ \begin{array}{l} g : S^1 \to S^1 \text{ orientation preserving} \\ \text{plus a choice of square root of } g'(z) \end{array} \right\}$$

The general setup in which to investigate the solutions of $(*)$ is the following:

- A unitary $g \in U(H)$ compatible with the real structure, meaning that $g \cdot \bar{f} = \overline{g \cdot f}$. Equivalently, g is the complexification of an orthogonal operator on $H_{\mathbb{R}}$.

We want to understand when there exist maps $\mathcal{F} \to \mathcal{F} : \xi \mapsto g \cdot \xi$ that satisfy $(*)$. An equivalent way of reformulating those equations is to say that $\xi \mapsto g \cdot \xi$ should be a module map between

$$\mathcal{F} = \Big(\mathcal{F},\, c(f) \text{ acts as } c(f)\Big) \qquad \text{and} \qquad {}^{g}\mathcal{F} := \Big(\mathcal{F},\, c(f) \text{ acts as } c(g \cdot f)\Big).$$

Note that since g is compatible with the real structure on H, the map $c(f) \mapsto c(g \cdot f)$ is an algebra automorphism and so ${}^g\mathcal{F}$ is indeed a CAR module.

We've already seen that $\mathcal{F}$ is an irreducible module. By Schur's lemma, the set of maps $\mathcal{F} \to \mathcal{F}$ that satisfy equation $(*)$ is therefore either zero or one dimensional. It is one dimensional if the modules $\mathcal{F}$ and ${}^g\mathcal{F}$ are isomorphic, and zero otherwise. We want to show that for $g \in \mathrm{Diff}_+^{(2)}(S^1)$ that set is always one dimensional, i.e., that the representations $\mathcal{F}$ and ${}^g\mathcal{F}$ are always isomorphic. This will then give us the map

$$\boxed{\begin{array}{ccc} \mathrm{Diff}_+^{(2)}(S^1) & \longrightarrow & PU(\mathcal{F}) \\ \Downarrow & & \Downarrow \\ g & \mapsto & \left\{\begin{matrix}\text{unitaries } \mathcal{F} \to \mathcal{F} \text{ that} \\ \text{are solutions of } (*).\end{matrix}\right\} \end{array}} \tag{23}$$

needed for the definition of a super-conformal net.

As we have seen, the question boils down to whether $\mathcal{F}$ and ${}^g\mathcal{F}$ are isomorphic as representations of the CAR algebra. We have the following general result:

Theorem (Segal quantization criterion) *Let H, $\bar{\cdot}$, $H_{>0}$, g be as above, and let P be the orthogonal projection onto $H_{>0}$. Then*

$$\Big(\text{completion of } \textstyle\bigwedge^\bullet H_{<0}\Big) \quad \underset{\substack{\uparrow \\ \boxed{\text{as CAR modules}}}}{\cong} \quad {}^g\Big(\text{completion of } \textstyle\bigwedge^\bullet H_{<0}\Big)$$

iff $\boxed{gPg^{-1} - P \text{ is Hilbert-Schmidt}}$. *In that case, the isomorphism $\mathcal{F} \cong {}^g\mathcal{F}$ is either even (= degree preserving) or odd (= interchanges even and odd parts).*

Let us explain the terms that appear in the above theorem:

Definition: A operator between Hilbert spaces H_1 and H_2 is <u>*Hilbert-Schmidt*</u> if it is in the image of the map

$$\begin{array}{ccc} \bar{H}_1 \otimes H_2 & \to & B(H_1, H_2) \\ \bar{\eta} \otimes \xi & \mapsto & \langle -, \eta \rangle \xi \end{array}$$

from the Hilbert space tensor product $\bar{H}_1 \otimes H_2$ to the space of bounded operators $H_1 \to H_2$. The norm on Hilbert-Schimdt operators inherited from $\bar{H}_1 \otimes H_2$ is called the Hilbert-Schimdt norm.

Note that for the above definition to be fully justified, we need to check that the map $\bar{\eta} \otimes \xi \mapsto \langle -, \eta \rangle \xi$ (a priori only defined on the algebraic tensor product of $\bar{H}_1$ and H_2) extends to the Hilbert space tensor product $\bar{H}_1 \otimes H_2$. Indeed, that map is bounded:

$$\Big\|\sum_{i=1}^{n}\langle\zeta,\eta_i\rangle\xi_i\Big\| \underset{\text{WLOG the } \xi_i \text{ are orthogonal to each other}}{=} \sqrt{\sum_i |\langle\zeta,\eta_i\rangle|^2\|\xi_i\|^2} \le \sqrt{\sum_i \|\zeta\|^2\|\eta_i\|^2\|\xi_i\|^2} = \|\zeta\|\sqrt{\sum_i \|\eta_i\|^2\|\xi_i\|^2}$$

and therefore extends.

Note that a self-adjoint operator is Hilbert-Schmidt if and only if it has discrete spectrum and its eigenvalues (counted with multiplicities) form an ℓ^2 sequence.

Lemma *Hilbert-Schimdt operators forms an ideal inside of bounded operators. (I.e., if $a : H_2 \to H_3$ is bounded and $b : H_1 \to H_2$ is Hilbert-Schimdt, then $ab : H_1 \to H_3$ is Hilbert-Schimdt. Similarly, if a is Hilbert-Schimdt and b bounded, then ab is Hilbert-Schimdt.)*

Proof: We'll prove that if $a : H_2 \to H_3$ is Hilbert-Schimdt and $b : H_1 \to H_2$ is bounded, then $ab : H_1 \to H_3$ is Hilbert-Schimdt. (The other claim is similar.) Consider the commutative diagram

$$\begin{tikzcd}
\bar{H}_1 \otimes H_2 \arrow[r] \arrow[d, "1\otimes a"'] & B(H_1, H_2) \arrow[d, "a\circ -"] \\
\bar{H}_1 \otimes H_3 \arrow[r] & B(H_1, H_3)
\end{tikzcd}$$

If b is Hilbert-Schimdt, then it's in the image of $\bar{H}_1 \otimes H_2 \to B(H_1, H_2)$. Hence $a \circ b$ is in the image of $\bar{H}_1 \otimes H_3 \to B(H_1, H_3)$, and so it's Hilbert-Schimdt. □

We won't give the full proof of the Segal quantization criterion in these notes; we'll only prove the implication

$$\Big(gPg^{-1} - P \text{ is Hilbert Schimdt}\Big) \;\Rightarrow\; \Big(\mathcal{F} \cong {}^g\mathcal{F} \text{ as CAR modules}\Big) \tag{24}$$

because that's the one that's needed to construct the action of $\mathrm{Diff}_+^{(2)}(S^1)$ on the Fock space of the Majorana free fermion.

From now on, we w'll assume that $gPg^{-1} - P$ is Hilbert Schimdt.

Since $\bigwedge^\bullet H_{<0}$ is dense in $\mathcal{F}$, and since the inner products on $\mathcal{F}$ and on ${}^g\mathcal{F}$ are uniquely determined (up to scalar) by the action of the CAR algebra, having a module map

$$\mathcal{F} \to {}^g\mathcal{F}$$

is equivalent to having a module map

$$\textstyle\bigwedge^\bullet H_{<0} \to {}^g\mathcal{F}.$$

Moreover, since $\bigwedge^\bullet H_{<0}$ is an induced module, constructing such a map is tantamount to finding a vector $\Phi \in {}^g\mathcal{F}$ that is annihilated by all $c(f)$ for $f \in H_{>0}$.

If we write $\tilde{\cdot}$ for the action on ${}^g\mathcal{F}$, then the condition becomes:

$$\begin{array}{rcl} & c(f)\,\tilde{\cdot}\,\Phi = 0 & \forall\, f \in H_{>0} \\ \Leftrightarrow & c(g\cdot f)\,\Phi = 0 & \forall\, f \in H_{>0} \\ \Leftrightarrow & \underbrace{c(f)}_{\text{acts as } (f_- \wedge -) + (\bar{f}_+ \lrcorner -)}\,\Phi = 0 & \forall\, f \in gH_{>0} \end{array}$$

Therefore, all in all, what we need is a vector $\boxed{\Phi \in \mathcal{F}}$ satisfying the equation

$$\boxed{f_- \wedge \Phi \,+\, \bar{f}_+ \lrcorner\, \Phi = 0 \qquad \forall\, f \in gH_{>0}} \qquad (\star)$$

At this point, we could take that equation and work out Φ degree by degree. I'll skip those steps and give you the answer directly:

Anzatz: let us take $\Phi = \exp(\varphi)$ for some $\varphi \in \bigwedge^2 H_{<0}$.

Here, $\exp : \bigwedge^2 H_{<0} \to \bigwedge^\bullet H_{<0}$ is defined by the usual power series. Let us check that that power series is always convergent:

Lemma *For any Hilbert space H, the map $\exp : \varphi \mapsto \sum \frac{1}{n!}\varphi^n$ from $\bigwedge^2 H$ to the Hilbert space completion of $\bigwedge^\bullet H$ converges. Moreover, $\|\exp(\varphi)\|^2 \le \exp(\|\varphi\|^2)$.*

Proof: Write $\varphi = \sum_{i=1}^\infty \varphi_i$ where φ_i are pure wedges that are orthogonal to each other. Since $\varphi_i^2 = 0$, we have

$$\varphi^n = n! \sum_{i_1<\ldots<i_n} \varphi_{i_1} \wedge \ldots \wedge \varphi_{i_n}.$$

It follows that

$$\exp(\varphi) = \sum_{n=0}^\infty \sum_{i_1<\ldots<i_n} \varphi_{i_1} \wedge \ldots \wedge \varphi_{i_n},$$

and so

$$\begin{aligned} \|\exp(\varphi)\|^2 &= \sum_n \sum_{i_1<\ldots<i_n} \|\varphi_{i_1}\|^2 \ldots \|\varphi_{i_n}\|^2 = \sum_n \frac{1}{n!} \sum_{i_1\neq\ldots\neq i_n} \|\varphi_{i_1}\|^2 \ldots \|\varphi_{i_n}\|^2 \\ &\le \sum_n \frac{1}{n!} \sum_{\substack{i_1,\ldots,i_n \\ \uparrow \\ \text{not necessarily} \\ \text{distinct}}} \|\varphi_{i_1}\|^2 \ldots \|\varphi_{i_n}\|^2 = \sum_n \frac{1}{n!} \Big(\sum_i \|\varphi_i\|^2\Big)^n = \exp(\|\varphi\|^2). \end{aligned}$$

□

Recall that $\lrcorner$ satisfies the Leibniz rule with respect to $\wedge$

$$f \lrcorner (\alpha \wedge \beta) = (f \lrcorner \alpha) \wedge \beta + (-1)^{|\alpha|} \alpha \wedge (f \lrcorner \beta),$$

and let's work out the second term $\bar{f}_+ \lrcorner\, \Phi$ in $(\star)$ using our ansatz:

$$\begin{aligned} \bar{f}_+ \lrcorner \exp(\varphi) &= \bar{f}_+ \lrcorner \varphi + 2\cdot\tfrac{1}{2!}(\bar{f}_+ \lrcorner \varphi) \wedge \varphi + 3\cdot\tfrac{1}{3!}(\bar{f}_+ \lrcorner \varphi) \wedge \varphi \wedge \varphi + \ldots \\ &= (\bar{f}_+ \lrcorner \varphi) \wedge \big(1 + \varphi + \tfrac{1}{2!}\varphi\wedge\varphi + \tfrac{1}{3!}\varphi\wedge\varphi\wedge\varphi + \ldots\big) = (\bar{f}_+ \lrcorner \varphi) \wedge \exp(\varphi). \end{aligned}$$

Conclusion:

$$f_- + \bar{f}_+ \lrcorner \varphi = 0 \qquad \Rightarrow \qquad f_- \wedge \Phi \, + \, \bar{f}_+ \lrcorner \Phi = 0.$$

So we have reduced our problem to that of finding $\boxed{\varphi \in \bigwedge^2 H_{<0}}$ satisfying

$$\boxed{\bar{f}_+ \lrcorner \varphi \, = -f_- \qquad \forall \, f \in gH_{>0}} \tag{$\star\star$}$$

we may as well forget his minus sign

Lemma *An antilinear map $a : H \to H$ is of the form $- \lrcorner \varphi$ for some $\varphi \in \bigwedge^2 H$ iff it's Hilbert-Schmidt and satisfies $\langle a(\xi), \eta \rangle = -\langle a(\eta), \xi \rangle$ for all $\xi, \eta \in H$, equivalently, $\langle a(\xi), \xi \rangle = 0$ for all $\xi \in H$.*

Proof: An operator $a : H \to H$ is Hilbert-Schmidt if and only if it's of the form $- \lrcorner \varphi$ for some $\varphi \in H^{\otimes 2}$. (Here, $\xi \lrcorner (\varphi_1 \otimes \varphi_2) := \langle \varphi_1, \xi \rangle \, \varphi_2$). We then have

$$\begin{aligned} \varphi \in \textstyle\bigwedge^2 H \qquad &\Leftrightarrow \qquad \langle \varphi, \xi \otimes \eta \rangle = -\langle \varphi, \eta \otimes \xi \rangle \quad \forall \, \xi, \eta \in H \\ &\Leftrightarrow \qquad \langle \xi \lrcorner \varphi, \eta \rangle = -\langle \eta \lrcorner \varphi, \xi \rangle \quad \forall \, \xi, \eta \in H \\ &\Leftrightarrow \qquad \langle a(\xi), \eta \rangle = -\langle a(\eta), \xi \rangle \quad \forall \, \xi, \eta \in H \end{aligned}$$

□

At this point, let us make the extra assumption that the map $P : gH_{>0} \to H_{>0}$ is invertible. Using the above lemma, we get:

> $\exists \; \varphi \in \bigwedge^2 H_{<0}$ such that $(\star\star)$ holds iff the map
>
> $$\begin{array}{ccccccccc} a: & H_{<0} & \xrightarrow{\;\bar{\cdot}\;} & H_{>0} & \xrightarrow{(P|_{gH_{>0}})^{-1}} & gH_{>0} & \xrightarrow{1-P} & H_{<0} \\ & \cup & & \cup & & \cup & & \cup \\ & \bar{f}_+ & \longmapsto & f_+ & \longmapsto & f & \longmapsto & f_- \end{array}$$
>
> is Hilbert-Schmidt and satisfies $\langle a(\xi), \xi \rangle = 0$ for all $\xi \in H_{<0}$.

So we need to check that a is Hilbert-Schmidt and satisfies the antisymmetry condition:

Verification that a is Hilbert-Schmidt:

Let $Q := gPg^{-1}$. As an operator on H, the map $(1 - P) : gH_{>0} \to H_{<0}$ is given by

$$(1 - P)Q = (Q - P)Q,$$

which is Hilbert-Schmidt because $Q - P$ is. Therefore a is Hilbert-Schmidt. ◇

Verification of the antisymmetry condition:

We need to check that

$$\big\langle a(\xi), \xi \big\rangle = \big\langle (1 - P)(P|_{gH_{>0}})^{-1} \bar{\xi}, \xi \big\rangle$$

vanishes for all $\xi \in H_{>0}$. In terms of $f := (P|_{gH_{>0}})^{-1}\bar{\xi}$, that condition reads:

$$\big\langle (1-P)f, \overline{Pf}\,\big\rangle \stackrel{?}{=} 0 \qquad \forall f \in gH_{>0}.$$

Note that since $H_{>0}$ is orthogonal to $\overline{H_{>0}}$ and g is compatible with $\bar{\cdot}$, the space $gH_{>0}$ is orthogonal to $\overline{gH_{>0}}$. In particular, $f \perp \bar{f}$ for every $f \in gH_{>0}$. Let us expand $\langle f, \bar{f}\rangle$:

$$\langle f,\bar f\rangle = \langle (1-P)f, \overline{Pf}\rangle + \underbrace{\langle Pf, \overline{Pf}\rangle}_{=0} + \underbrace{\langle (1-P)f, \overline{(1-P)f}\rangle}_{=0} + \langle Pf, \overline{(1-P)f}\rangle.$$

$$\text{because } H_{>0}^{\perp} = \overline{H_{>0}}$$

Using that $\langle f_1, \bar f_2\rangle = \overline{\langle \bar f_1, f_2\rangle} = \langle f_2, \bar f_1\rangle$, we also have $\langle Pf, \overline{(1-P)f}\rangle = \langle (1-P)f, \overline{Pf}\rangle$. It follows that

$$0 = \big\langle f, \bar f\,\big\rangle = 2\big\langle (1-P)f, \overline{Pf}\,\big\rangle$$

and so $\langle (1-P)f, \overline{Pf}\rangle = 0$. $\diamond$

This finishes the proof of (24) in the case when $P : gH_{>0} \to H_{>0}$ is invertible. Really, at this point, the proof should have been finished and it's kind of unfair that there exist situations when the above operator is not invertible. We'll see later that that operator is always invertible modulo something finite dimensional.

technical end of proof →

If $P : gH_{>0} \to H_{>0}$ is not invertible, we'll have to modify our anzatz.

Claim: If $P : gH_{>0} \to H_{>0}$ is not invertible, then $(H, \bar{\cdot}, H_{>0}, gH_{>0})$ splits as a direct sum

$$\Big(H, \bar{\cdot}, H_{>0}, gH_{>0}\Big) = \Big(H^{(1)}, \bar{\cdot}, H^{(1)}_{>0}, (gH_{>0})^{(1)}\Big) \oplus \Big(H^{(2)}, \bar{\cdot}, H^{(2)}_{>0}, (gH_{>0})^{(2)}\Big)$$

such that $P^{(1)} : (gH_{>0})^{(1)} \to H^{(1)}_{>0}$ invertible and such that $H^{(2)}$ finite dimensional with $P^{(2)} : (gH_{>0})^{(2)} \to H^{(2)}_{>0}$ identically zero (the map g is not required to respect the decomposition).

Proof: Let $Q = gPg^{-1}$ be the orthogonal projection onto $gH_{>0}$, and let

$$H^{(2)}_{>0} := \ker\big(Q : H_{>0} \to gH_{>0}\big) = H_{>0} \cap (gH_{>0})^{\perp}$$
$$(gH_{>0})^{(2)} := \ker\big(P : gH_{>0} \to H_{>0}\big) = gH_{>0} \cap H_{>0}^{\perp}$$

and $H^{(2)} := H^{(2)}_{>0} \oplus (gH_{>0})^{(2)}$. Note also that, since $\bar{\cdot}$ interchanges $H^{(2)}_{>0}$ and $(gH_{>0})^{(2)}$, their direct sum $H^{(2)}$ is invariant under that conjugation operation.

The space $H^{(2)}_{>0}$ is the (-1)-eigenspace of $PQP - P = P(Q-P)P$, which s is Hilbert-Schmidt, and is therefore finite dimensional. Also, by construction, the spaces $H^{(2)}_{>0}$ and $(gH_{>0})^{(2)}$ are orthogonal to each other. The projection operator $P : (gH_{>0})^{(2)} \to H^{(2)}_{>0}$ is therefore zero.

Let $H^{(1)}_{>0}$ and $(gH_{>0})^{(1)}$ be the orthogonal complements of $H^{(2)}_{>0}$ and $(gH_{>0})^{(2)}$ inside $H_{>0}$ and $gH_{>0}$, respectively. To check that the operator

$$P : (gH_{>0})^{(1)} \to H^{(1)}_{>0}$$

is invertible, we look at the composite

$$PQP : \; H^{(1)}_{>0} \xrightarrow{Q} (gH_{>0})^{(1)} \xrightarrow{P} H^{(1)}_{>0}.$$

Its kernel is zero by construction. Moreover, it is congruent to $\mathrm{Id}_{H^{(1)}_{>0}}$ (which is P) modulo Hilbert-Schmidt operators. Its spectrum therefore avoids a neighborhood of zero, and so it is invertible. By the same argument, the composite

$$QPQ : (gH_{>0})^{(1)} \xrightarrow{P} H^{(1)}_{>0} \xrightarrow{Q} (gH_{>0})^{(1)}$$

is also invertible. It follow that both $P : (gH_{>0})^{(1)} \to H^{(1)}_{>0}$ and $Q : H^{(1)}_{>0} \to (gH_{>0})^{(1)}$ are invertible. $\diamond$

Recall that our ultimate goal is to find a vector $\Phi \in \mathcal{F}$ that is annihilated by all the $c(f)$ with $f \in gH_{>0}$. For that purpose, it will be enough to find vectors $\Phi^{(1)} \in \mathcal{F}^{(1)}$ and $\Phi^{(2)} \in \mathcal{F}^{(2)}$ such that

$$\begin{aligned} & c(f)\,\Phi^{(1)} = 0 \qquad \forall f \in (gH_{>0})^{(1)} \\ \text{and} \qquad & c(f)\,\Phi^{(2)} = 0 \qquad \forall f \in (gH_{>0})^{(2)}. \end{aligned}$$

Indeed, the vector

$$\Phi := \Phi^{(1)} \otimes \Phi^{(2)} \in \mathcal{F}^{(1)} \otimes \mathcal{F}^{(2)} = \mathcal{F}$$

will then be a solution of our main equation $(\star)$. So, all that remains to be done is to find vectors $\Phi^{(1)}$ and $\Phi^{(2)}$ that solve the above equations.

- *Construction of $\Phi^{(1)}$*: we already did that (that was the bulk of the proof).
- *Construction of $\Phi^{(2)}$*: We want a vector $\Phi^{(2)}$ that satisfies

  $$c(f)\,\Phi^{(2)} = 0$$

  for every $f \in (gH_{>0})^{(2)}$. But $(gH_{>0})^{(2)} = H^{(2)}_{<0}$. So the equation becomes

  $$f \wedge \Phi^{(2)} = 0 \qquad \forall f \in H^{(2)}_{<0}$$

  and any vector in the top degree piece $\bigwedge^{\mathrm{top}} H^{(2)}_{<0} \subset \bigwedge^{\bullet} H^{(2)}_{<0} = \mathcal{F}^{(2)}$ is a solution.

Note that if $H^{(2)}_{<0}$ is odd dimensional, the vector $\Phi^{(2)} \in \mathcal{F}^{(2)}$ will be odd, and the same will then hold for $\Phi^{(1)} \otimes \Phi^{(2)}$. In that case, the isomorphism $\mathcal{F} \to {}^{g}\mathcal{F}$ specified by $\Omega \mapsto \Phi^{(1)} \otimes \Phi^{(2)}$ will be odd.

This last argument finishes the proof of (24). $\square$

Let us now go back to our concrete situation, with $g \in \mathrm{Diff}^{(2)}(S^1)$. It will be convenient to use

$$J := 2iP - i$$

instead of P because it's easier to write down a closed formula for it. The operator J is given by $+i$ on PH and by $-i$ on $(PH)^\perp$ and therefore contains the exact same information as P. Moreover, we have the following easy facts:

$$\begin{array}{rrcl} & gPg^{-1} - P & \in & \textit{H.-S.} \\ \Leftrightarrow & gJg^{-1} - J & \in & \textit{H.-S.} \\ \Leftrightarrow & g^{-1}Jg - J & \in & \textit{H.-S.} \end{array}$$

Our new goal is to check that $\boxed{g^{-1}Jg - J \text{ is Hilbert-Schmidt}}$: this will ensure that (23) always has solutions.

Claim. *The operator J is given (at least on some dense domain) by:*

$$J : f(z)\sqrt{dz} \quad \mapsto \quad \frac{1}{\pi}\ \text{P.V.}\int_{S^1} \frac{f(w)dw}{w - z}\sqrt{dz}$$

Here, the symbol "P.V." indicates that one takes the <u>*principal value*</u> of the singular integral. The principal value is defined as follows. If α is a 1-form on some one dimensional manifold M, and α is not locally L^1 around some point $x_0 \in M$, then $\text{P.V.}\!\int\alpha$ is given by

$$\text{P.V.}\int_M \alpha \;:=\; \lim_{\epsilon\to 0}\int_{M - I(x_0,\epsilon)} \alpha\,,$$

where $I(x_0,\epsilon)$ is the interval of radius ϵ around x_0. In our situations of interest, the 1-form α blows up like $1/(x - x_0)$ and the principal value is independent of the choice of metric on M (used to define $I(x_0,\epsilon)$). As a consequence, the usual change of variables formula is available for our principal value integrals: if $g : M_1 \to M_2$ is an orientation preserving diffeomorphism and α is a 1-form on M_2 satisfying the above bound, then

$$\text{P.V.}\int_{M_2} \alpha \;=\; \text{P.V.}\int_{M_1} g^*\alpha. \tag{25}$$

In the context of complex contour integrals, we have the following alternative interpretation of principal value integrals. Let M be a contour in $\mathbb{C}$ and let $F(z)$ be a function that is analytic on a neighborhood of M, except for a first order pole at some point $z_0 \in M$, then:

$$\text{P.V.}\int F(z)dz \;=\; \tfrac{1}{2}\left(\oint + \oint\right)F(z)dz.$$

z_0

Indeed, if we let R be the residu of $F(z)$ at z_0, then the contributions of the two little half-circles are $\approx \frac{1}{2}2\pi i R$ and $\approx -\frac{1}{2}2\pi i R$ respectively, with an error that tends to zero as the radius $\epsilon \to 0$.

Proof of claim: (For the purpose of this proof, we'll just work with functions on S^1 as opposed to sections of $\mathbb{S}$. This will simplify our formulas.) If f extends to a holomorphic function on a neighborhood of $\mathbb{D}$, then

$$\tfrac{1}{\pi}\ \text{P.V.}\int_{S^1}\frac{f(w)dw}{w-z} \;=\; \tfrac{1}{\pi}\left[\tfrac12\left(\oint + \cancelto{0}{\oint}\right)\right]\frac{f(w)dw}{w-z}$$
$$=\; \tfrac{1}{2\pi}\cdot 2\pi i\,\mathrm{Res}_z\frac{f(w)}{w-z} \;=\; if(z).$$

If f extends to a holomorphic function on a neighborhood of $\mathbb{D}' := \{z \in \mathbb{C}\,|\, z \geq 1\}\cup\{\infty\}$ and vanishes at infinity, then the 1-form $\frac{f(w)dw}{w-z}$ is regular at infinity (dw has a double pole at infinity while $\frac{f(w)}{w-z}$ has at least a double zero). Then, by the same argument as above using now $\mathbb{D}'$ instead of $\mathbb{D}$, we get

$$\tfrac{1}{\pi}\ \text{P.V.}\int_{S^1}\frac{f(w)dw}{w-z} \;=\; \tfrac{1}{\pi}\left[\tfrac12\left(\cancelto{0}{\oint} + \oint\right)\right]\frac{f(w)dw}{w-z}$$
$$=\; \tfrac{1}{2\pi}\cdot\Big(-2\pi i\,\mathrm{Res}_z\frac{f(w)}{w-z}\Big) \;=\; -if(z).$$

The contour $\oint$ runs *clockwise* around $\mathbb{D}'$

So we see that, at least on some dense subsets of PH and of $(PH)^\perp$, the operators J and $f \mapsto \frac{1}{\pi}\text{P.V.}\int_{S^1}\frac{f(w)dw}{w-z}$ agree. $\square$

Recall that an element of $\mathrm{Diff}_+^{(2)}(S^1)$ consists of an orientation preserving diffeomorphism $g : S^1 \to S^1$, along with a chosen square root $\sqrt{g'(z)}$ of $g'(z)$. The action of such an element on a section $f(z)\sqrt{dz}$ on $\mathbb{S}$ is then given by

$$g\cdot\big(f(z)\sqrt{dz}\big) \;=\; \frac{1}{\sqrt{g'(g^{-1}(z))}}\,f\big(g^{-1}(z)\big)\sqrt{dz} \;=\; \sqrt{(g^{-1})'(z)}\,f\big(g^{-1}(z)\big)\sqrt{dz}.$$

Unsurprisingly, the formula for the action of g^{-1} is more pleasant:

$$g^{-1}\cdot\big(f(z)\sqrt{dz}\big) = \sqrt{g'(z)}\,f\big(g(z)\big)\sqrt{dz}.$$

Let us now work out the formula for $g^{-1}Jg$:

$$w=g^{-1}(u) \qquad u=g(w) \qquad du=g'(w)dw \qquad [(g^{-1})'(u)]^{1/2}=[g'(w)]^{-1/2}$$

$$\begin{aligned}
f(z)\sqrt{dz} \;&\overset{g}{\mapsto}\; \sqrt{(g^{-1})'(z)}\,f\big(g^{-1}(z)\big)\sqrt{dz}\\
&\overset{J}{\mapsto}\; \tfrac{1}{\pi}\ \text{P.V.}\int_{S^1}\frac{\sqrt{(g^{-1})'(u)}\,f(g^{-1}(u))}{u-z}du\ \sqrt{dz}\\
&=\; \tfrac{1}{\pi}\ \text{P.V.}\int_{S^1}\frac{\sqrt{g'(w)}\,f(w)}{g(w)-z}dw\ \sqrt{dz} \qquad \longleftarrow \text{note the use of (25)}\\
&\overset{g^{-1}}{\mapsto}\; \tfrac{1}{\pi}\ \text{P.V.}\int_{S^1}\frac{\sqrt{g'(w)}\sqrt{g'(z)}}{g(w)-g(z)}f(w)\,dw\ \sqrt{dz}
\end{aligned}$$

Putting this all together, we get:

$$g^{-1}Jg - J: \ f(z)\sqrt{dz} \ \mapsto \ \tfrac{1}{\pi}\ \text{P.V.}\int_{S^1}\left[\frac{\sqrt{g'(w)}\sqrt{g'(z)}}{g(w)-g(z)} - \frac{1}{w-z}\right] f(w)\,dw\,\sqrt{dz}$$

<u>*Note*</u>*:* Later, we'll also want to use the formula for $g^{-1}Jg - J$ in the case of non-smooth diffeomorphisms. If g is of class at least $\mathcal{C}^1$, then it's legal to do the change of variables (25), and the above formula is correct.

Now we can use the following ***wonderful fact***:

> An operator $L^2(X) \to L^2(Y)$ that is given by an integral kernel $K(x,y)$ is Hilbert-Schmidt if and only $K \in L^2(X\times Y)$. Indeed, by definition, Hilbert-Schmidt maps from $L^2(X)$ to $L^2(Y)$ come from $L^2(X)\otimes L^2(Y)$, and the latter is exactly $L^2(X\times Y)$!

In our case of interest, the integral kernel $K(x,y)$ is given by $\frac{\sqrt{g'(x)}\sqrt{g'(y)}}{g(x)-g(y)} - \frac{1}{x-y}$. Our question therefore boils down to determining those maps $g: S^1 \to S^1$ for which the corresponding kernel is in $L^2(S^1\times S^1)$.

Theorem *If g is smooth, then*

$$K_g(x,y) := \frac{\sqrt{g'(x)}\sqrt{g'(y)}}{g(x)-g(y)} - \frac{1}{x-y}$$

is in L^2.

Actually, we'll prove the following stronger statement:

If g is $\mathcal{C}^2$, then

$$K_g(x,y) = \frac{\sqrt{g'(x)}\sqrt{g'(y)}}{g(x)-g(y)} - \frac{1}{x-y}$$

is bounded.

Proof: We'll use the following bounds on g and on its derivative:

$$\begin{aligned} g(y) &= g(x) + g'(x)(y-x) + \mathcal{O}(x-y)^2 \\ g'(y) &= g'(x) + \mathcal{O}(x-y). \end{aligned}$$

Since g' is never zero, and since the square root function is smooth away from zero, the second equation implies

$$\sqrt{g'(y)} = \sqrt{g'(x)} + \mathcal{O}(x-y).$$

It follows that

$$\begin{aligned}
\frac{\sqrt{g'(x)}\sqrt{g'(y)}}{g(x)-g(y)} &= \frac{g'(x)+\mathcal{O}(x-y)}{g'(x)(x-y)+\mathcal{O}(x-y)^2} \\
&= \frac{g'(x)+\mathcal{O}(x-y)}{g'(x)+\mathcal{O}(x-y)} \cdot \frac{1}{x-y} \\
&= \big(1+\mathcal{O}(x-y)\big) \cdot \frac{1}{x-y} \;=\; \frac{1}{x-y} + \mathcal{O}(1),
\end{aligned}$$

and so

$$\frac{\sqrt{g'(x)}\sqrt{g'(y)}}{g(x)-g(y)} - \frac{1}{x-y} \;=\; \mathcal{O}(1).$$

□

To be honest, the statement we're really interested in is the following:

If g is of Hölder class $\mathcal{C}^{1+\alpha}$ for some $\alpha > 1/2$, then the integral kernel

$$K_g(x,y) = \frac{\sqrt{g'(x)}\sqrt{g'(y)}}{g(x)-g(y)} - \frac{1}{x-y}$$

is in L^2. Here, by definition, $g \in \mathcal{C}^{1+\alpha}$ if it's in $\mathcal{C}^1$ and if $g'(y) = g'(x) + \mathcal{O}(x-y)^\alpha$.

Proof: Once again, we have $\sqrt{g'(y)} = \sqrt{g'(x)} + \mathcal{O}(x-y)^\alpha$. By integrating the equation $g'(y) = g'(x) + \mathcal{O}(x-y)^\alpha$, we also get

$$g(y) \;=\; g(x) + \int_x^y g'(t)\,dt \;=\; g(x) + g'(x)(y-x) + \mathcal{O}(x-y)^{1+\alpha}.$$

It follows that

$$\begin{aligned}
\frac{\sqrt{g'(x)}\sqrt{g'(y)}}{g(x)-g(y)} &= \frac{g'(x)+\mathcal{O}(x-y)^\alpha}{g'(x)(x-y)+\mathcal{O}(x-y)^{1+\alpha}} \\
&= \frac{g'(x)+\mathcal{O}(x-y)^\alpha}{g'(x)+\mathcal{O}(x-y)^\alpha} \cdot \frac{1}{x-y} \\
&= \big(1+\mathcal{O}(x-y)^\alpha\big) \cdot \frac{1}{x-y} \;=\; \frac{1}{x-y} + \mathcal{O}(x-y)^{-1+\alpha},
\end{aligned}$$

and so

$$\frac{\sqrt{g'(x)}\sqrt{g'(y)}}{g(x)-g(y)} - \frac{1}{x-y} \;=\; \mathcal{O}(x-y)^{-1+\alpha},$$

which is in L^2 because $\alpha > 1/2$. □

From now on, let us work with the N-th tensor power of the Dirac free fermion for some fixed number N. This conformal net is generated by elements $a^\dagger(f)$ and $a(f)$ for $f \in \Gamma(\mathbb{C}^N \otimes \mathbb{S})$, subject to the relations

$$[a^\dagger(f), a^\dagger(g)]_+ = [a(f), a(g)]_+ = 0, \quad [a^\dagger(f), a(g)]_+ = \langle f, g\rangle = \tfrac{1}{2\pi i}\int_{S^1} \langle f, \bar{g}\rangle$$

and $*$-operation $a(f)^* = a^\dagger(f)$.

An element $\gamma \in LU(N)$ of the loop group of $U(N)$ induces an automorphism of the above CAR algebra by acting here $\Gamma(\mathbb{C}^N \otimes \mathbb{S})$ pointwise.

If you translate the Segal Quantization Criterion from the Majorana setup to the Dirac setup, then you'll see that:

$$\left[\begin{array}{c} \gamma \text{ acts on } \mathcal{F}\text{, i.e.,} \\ \text{there exists a solution } u_\gamma : \mathcal{F} \to \mathcal{F} \text{ of} \\ u_\gamma\, a^\dagger(v)\, u_\gamma^* = a^\dagger(\gamma \cdot v), \\ \forall\, v \in \Gamma(\mathbb{C}^N \otimes \mathbb{S}) \end{array}\right] \Leftrightarrow \left[\begin{array}{c} [\gamma, P] \text{ is a Hilbert-Schmidt operator} \\ \text{on } \Gamma(\mathbb{C}^N \otimes \mathbb{S}) \\ (P \text{ is the projection onto } \Gamma_{>0}(\mathbb{C}^N \otimes \mathbb{S})) \end{array}\right]$$

(Note that the condition $u_\gamma a(v) u_\gamma^* = a(\gamma \cdot v)$ is a formal consequence of $u_\gamma a^\dagger(v) u_\gamma^* = a^\dagger(\gamma \cdot v)$)

We want to check that automorphisms of the CAR algebra that come from elements $\gamma \in LU(N)$ satisfy that condition. If we expand $\gamma = \sum_{n\in\mathbb{Z}} \gamma_n z^n$ as a power series, then the matrix representing its action on $\Gamma(\mathbb{C}^N \otimes \mathbb{S})$ is given by:

$$\gamma = \begin{array}{c} \\ \Gamma_{>0}(\mathbb{C}^N \otimes \mathbb{S}) \left\{ \vphantom{\begin{array}{c}a\\a\\a\\a\end{array}} \right. \\ \Gamma_{<0}(\mathbb{C}^N \otimes \mathbb{S}) \left\{ \vphantom{\begin{array}{c}a\\a\\a\\a\end{array}} \right. \end{array} \begin{array}{c} \overbrace{\hphantom{\gamma_0 \; \gamma_1 \; \gamma_2 \;}}^{\Gamma_{>0}(\mathbb{C}^N \otimes \mathbb{S})} \; \overbrace{\hphantom{\gamma_0 \; \gamma_1 \; \gamma_2 \;}}^{\Gamma_{<0}(\mathbb{C}^N \otimes \mathbb{S})} \\ \left(\begin{array}{ccc|ccc} \ddots & & & & & \iddots \\ \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 \\ \gamma_{-1} & \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \gamma_{-2} & \gamma_{-1} & \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 \\ \hline \gamma_{-3} & \gamma_{-2} & \gamma_{-1} & \gamma_0 & \gamma_1 & \gamma_2 \\ \gamma_{-4} & \gamma_{-3} & \gamma_{-2} & \gamma_{-1} & \gamma_0 & \gamma_1 \\ \gamma_{-5} & \gamma_{-4} & \gamma_{-3} & \gamma_{-2} & \gamma_{-1} & \gamma_0 \\ \iddots & & & & & \ddots \end{array}\right) \end{array}$$

where each γ_n is itself an $N \times N$ matrix.

Now, since $\left[\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\right] = \left(\begin{smallmatrix} 0 & -B \\ C & 0 \end{smallmatrix}\right)$, we have

$$[\gamma, P] = \left(\begin{array}{cccc|cccc} \ddots & & & & & & & \iddots \\ & 0 & 0 & 0 & -\gamma_3 & -\gamma_4 & -\gamma_5 & \\ & 0 & 0 & 0 & -\gamma_2 & -\gamma_3 & -\gamma_4 & \\ & 0 & 0 & 0 & -\gamma_1 & -\gamma_2 & -\gamma_3 & \\ \hline & \gamma_{-3} & \gamma_{-2} & \gamma_{-1} & 0 & 0 & 0 & \\ & \gamma_{-4} & \gamma_{-3} & \gamma_{-2} & 0 & 0 & 0 & \\ & \gamma_{-5} & \gamma_{-4} & \gamma_{-3} & 0 & 0 & 0 & \\ \iddots & & & & & & & \ddots \end{array}\right)$$

The Hilbert-Schmidt norm of that commutator is therefore given by

$$\left\|[\gamma, P]\right\|^2_{H.\text{-}S.} = \sum_{n \in \mathbb{Z}} |n| \cdot \|\gamma_n\|^2$$

where the norms on the left hand side are the finite dimensional Hilbert-Schmidt norms of $N \times N$ matrices (and therefore equivalent to any other norm on $N \times N$ matrices).

If γ is $\mathcal{C}^\infty$, then its Fourier coefficients γ_n are rapidly decreasing and the above sum is clearly convergent. The smooth loop group of $U(N)$ therefore acts projectively on the Fock space $\mathcal{F}$ of the N-th tensor power of the Dirac fermion... but actually, we learn much more:

We learn that the Sobolev - $1/2$ loop group of $U(N)$ acts on $\mathcal{F}$!

Here, by definition, a function f is Sobolev-s if its Fourier coefficients f_n are such that $\sum |n|^{2s}|f_n|^2 < \infty$. The space of all such functions is denoted H^s. When s is a positive integer, H^s can also be defined as the space of all functions whose s-th derivative (in the sense of distributions) is in L^2. As we'll se later, the space $H^{1/2}$ also contains discontinuous functions. (Actually, $s = 1/2$ is the biggest s such that H^s contains discontinuous functions.)

Our next goal is to use the above result to prove that the positive energy representations of $\widetilde{L\mathfrak{g}}_k$ exponentiate to projective representations of LG. (Actually, the method that we'll present here cannot be used to prove this for all G and all k. Given a group G, only a certain subset of k's can be treated. For $G = SU(n)$ however, all the levels can be treated via this method). Here's a roadmap of what we'll do next:

1. Construct a representation of the algebraic part of $\widetilde{L\mathfrak{g}}_k$ on the algebraic part of $\mathcal{F}$.

2. Show that it extends to a rep of all of $\widetilde{L\mathfrak{g}}_k$ on the set of smooth vectors of $\mathcal{F}$.

3. Prove that rep of $\widetilde{L\mathfrak{g}}_k$ exponentiates elementwise (unfortunately, this will give us no information about the relations that those might satisfy).

4. Argue that the one-parameter subgroups so constructed satisfy the defining property

$$u_{\exp(f)}\, a^\dagger(v)\, u^*_{\exp(f)} = a^\dagger(\exp(f)\cdot v)$$

of $u_{\exp(f)}$ and therefore agree with the projective representation of LG that we constructed above.

Note: At first, we'll do it all for $\mathfrak{g} = \mathfrak{gl}(N)$ and $k = 1$. Later, we'll then explain how to extend it to other groups and levels. This will depend on the choice (and thus existence) of a representation (not necessarily irreducible) $\rho : \mathfrak{g} \to \mathfrak{gl}(N)$ such that $tr(\rho(X)\rho(Y)) = k\langle X, Y\rangle_{basic}$ for every $X, Y \in \mathfrak{g}$.

Free field realization of $\widetilde{L\mathfrak{gl}(N)}_1$

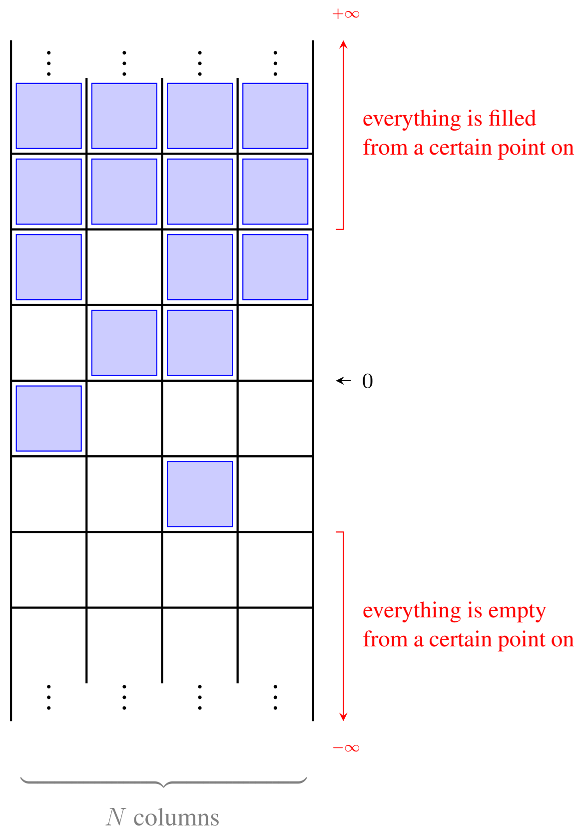
In this section, we'll construct an action of $\widetilde{L\mathfrak{gl}(N)}_1$ (the level 1 central extension of $L\mathfrak{gl}(N)$) on the state space of the N-th tensor power of the Dirac free fermion.

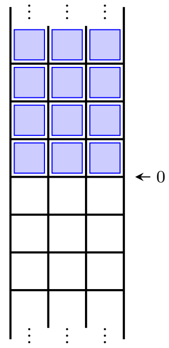
We begin with a graphical representation for the elements of that Fock space:

Definition: A *Maya diagram* is something that looks like this:

The boxes are indexed by pairs (i, n)

with $i \in \{1, \ldots, N\}$ and $n \in \mathbb{Z}' := \mathbb{Z} + \frac{1}{2}$.



The vacuum vector corresponds to this Maya diagram: $\Omega =$ 

We claim that there is an orthonormal basis of $\mathcal{F}$ indexed by Maya diagrams such that the operators

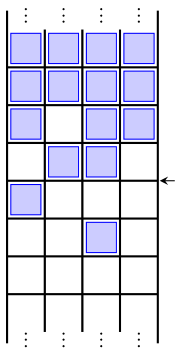
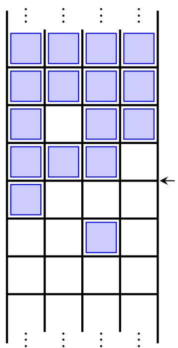
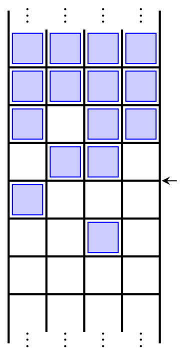
$$e_i^\dagger(m+\tfrac{1}{2}) := a^\dagger(e_i \otimes z^m\sqrt{dz}) \qquad \text{and} \qquad e_i(m+\tfrac{1}{2}) := a(e_i \otimes z^m\sqrt{dz})$$

act by

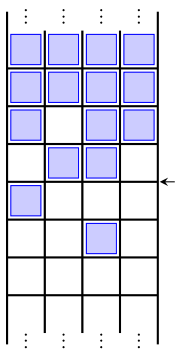
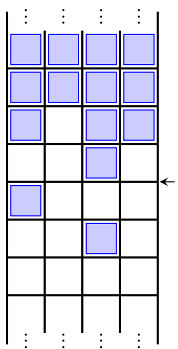
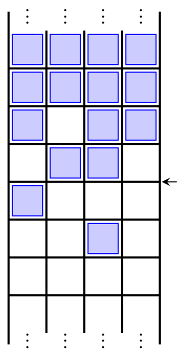
$$\begin{aligned} e_i^\dagger(n) &= (\ )\ \text{create a box at position } (i,n) \\ e_i(n) &= (\ )\ \text{kill the box at position } (i,n). \end{aligned}$$

Actually, it's a bit more subtle. We first need to pick an order on the set $\{1,\ldots,N\} \times \mathbb{Z}'$ (that's the set that indexes all the possible locations of boxes), and then insert the sign $(-1)^{\#\{\text{boxes that come before}\}}$ here .

If it is not possible to perform the operation of "creation" or "annihilation", then the outcome is zero. For example, we have:

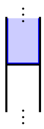
$e_1^\dagger(\tfrac{1}{2})$  $=$  but $e_2^\dagger(\tfrac{1}{2})$  $= 0$

Similarly, we have

$e_2(\tfrac{1}{2})$  $=$  but $e_2(\tfrac{3}{2})$  $= 0$

To see that this new description agrees with the old description of $\mathcal{F}$, it suffices to check that the operators $e_i^\dagger(n)$ and $e_i(n)$ satisfy the desired commutation relations

$$[e_i^\dagger(n), e_j^\dagger(m)]_+ = [e_i(n), e_j(m)]_+ = 0, \qquad [e_i^\dagger(n), e_j(m)]_+ = \delta_{ij}\delta_{nm},$$

that the vacuum vector $\Omega =$  is cyclic, and that it satisfies

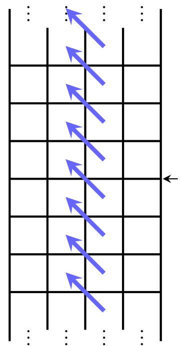
$$e_i^\dagger(n)\Omega = 0 \quad \text{for } n > 0 \qquad \text{and} \qquad e_i(n)\Omega = 0 \quad \text{for } n < 0.$$

Recall that $\mathbb{Z}' := \mathbb{Z} + \frac{1}{2}$. Given the above preliminaries, we can now define:

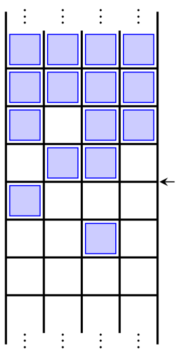
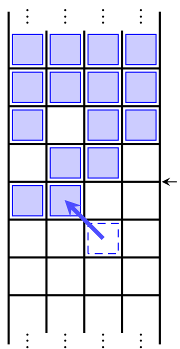
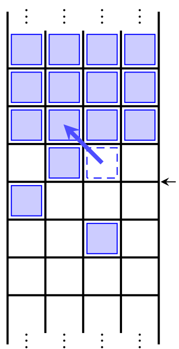
$$\boxed{\begin{aligned} E_{ij}(n) &:= \sum_{m\in\mathbb{Z}'} e_i^\dagger(n+m)e_j(m) \qquad \text{if } i \neq j \text{ or } n \neq 0 \\ E_{ii}(0) &:= \sum_{m\in\mathbb{Z}'_{<0}} e_i^\dagger(m)e_i(m) - \sum_{m\in\mathbb{Z}'_{>0}} e_i(m)e_i^\dagger(m) \end{aligned}}$$

Note that if $i \neq j$ or $n \neq 0$, the terms $e_i^\dagger(n+m)$ and $e_j(m)$ anticommute, and so it doesn't matter if one writes them as $e_i^\dagger(n+m)e_j(m)$ or as $-e_j(m)e_i^\dagger(n+m)$. In the expression for $E_{ii}(0)$, the order certainly matters. Note that $\sum_{m\in\mathbb{Z}'} e_i^\dagger(m)e_i(m)$ is *divergent* on every basis vector, and so we can't use it to define anything. The above definition of $E_{ii}(0)$ is a way of fixing that problem. This is called a *normally ordered product*.

If $i \neq j$ or $n \neq 0$, then $E_{ij}(n)$ is the operator that tries to move the boxes of the Maya diagram from column j to column i, while raising them by n, e.g.:

$$E_{23}(1) \quad = \quad \sum_{m\in\mathbb{Z}'}$$ 

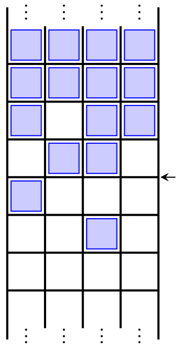
Note that on any given Maya diagram, the infinite sum that defines $E_{ij}(n)$ is finite, and thus well defined. For example,

$E_{23}(1)$  $= \pm$  $\pm$ 

has only two non-zero terms (all the other terms vanish).

The operator $E_{ii}(0)$ also has an interpretation in terms of Maya diagrams. It acts diagonally on $\mathcal{F}$ in its basis of Maya diagrams, and the eigenvalue of a basis vector is the number of “excess boxes” in the i-th column of the Maya diagram. For example, if

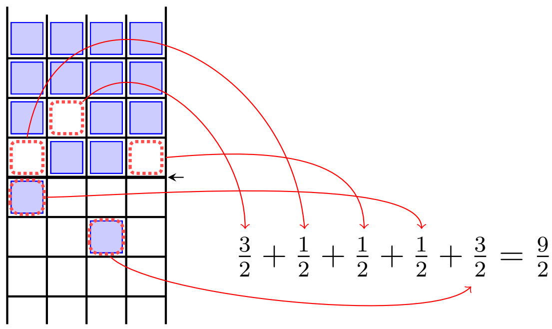
our Maya diagram is



then

$$\begin{cases} \text{the eigenvalue of } E_{11}(0) \text{ is } 0 \\ \text{the eigenvalue of } E_{22}(0) \text{ is } -1 \\ \text{the eigenvalue of } E_{33}(0) \text{ is } 1 \\ \text{the eigenvalue of } E_{44}(0) \text{ is } -1. \end{cases}$$

The energy operator L_0 also acts diagonally on the basis of Maya diagrams. Given a Maya diagram, we compare it to the one for Ω. A missing box in location (i, n) contributes n to the total energy and an extra box in position $(i, -n)$ also contributes n to the total energy. For example, the eigenvalue of L_0 on the above Maya diagram is $\frac{9}{2}$, computed as follows:



Proposition *The following relations are satisfied:*

$$\begin{aligned} [E_{ij}(n), e_k^\dagger(m)] &= \delta_{jk} e_i^\dagger(n+m) \\ [E_{ij}(n), e_k(m)] &= -\delta_{ik} e_j^\dagger(m-n) \end{aligned}$$

$$\begin{aligned} [E_{ij}(n), E_{kl}(m)] &= 0 \quad \text{if } j \neq k \text{ and } i \neq l \\ [E_{ij}(n), E_{jk}(m)] &= E_{ik}(n+m) \quad \text{if } i \neq k \\ [E_{ij}(n), E_{ji}(m)] &= E_{ii}(n+m) - E_{jj}(n+m) + n \cdot \delta_{n+m,0} \end{aligned}$$

Proof:

1. $$\sum_p \left[e_i^\dagger(n+p)e_j(p), e_k^\dagger(m)\right] = \sum_p e_i^\dagger(n+p) \underbrace{\left[e_j(p), e_k^\dagger(m)\right]_+}_{\delta_{jk}\delta_{pm}} = \delta_{jk} e_i^\dagger(n+m).$$

2. $$\sum_p \left[e_i^\dagger(n+p)e_j(p), e_k(m)\right] = -\sum_p \underbrace{\left[e_i^\dagger(n+p), e_k(m)\right]_+}_{\delta_{ik}\delta_{n+p,m}} e_j(p) = -\delta_{ik} e_j^\dagger(m-n).$$

3. $$\sum_{p,q}\left[e_i^\dagger(n+p)e_j(p), e_k^\dagger(m+q)e_l(q)\right] = 0 \qquad \text{(obvious)}$$

4. $$\sum_{p,q}\left[e_i^\dagger(n+p)e_j(p), e_j^\dagger(m+q)e_k(q)\right] = \sum_{p,q} e_i^\dagger(n+p)\underbrace{\left[e_j(p), e_j^\dagger(m+q)\right]_+}_{\delta_{p,m+q}} e_k(q)$$
$$= \sum_q e_i^\dagger(n+m+q)e_k(q) = E_{ik}(n+m).$$

5. $$\sum_{p,q}\left[e_i^\dagger(n+p)e_j(p), e_j^\dagger(m+q)e_i(q)\right]$$
$$= \sum_{p,q}\left(e_i^\dagger(n+p)\underbrace{\left[e_j(p), e_j^\dagger(m+q)\right]_+}_{\delta_{p,m+q}} e_i(q) - e_j^\dagger(m+q)\underbrace{\left[e_i(q), e_i^\dagger(n+p)\right]_+}_{\delta_{q,n+p}} e_j(p)\right)$$

$=$

$n+m\neq 0$:
$$\sum_q e_i^\dagger(n+m+q)e_k(q) - \sum_q e_j^\dagger(m+q)e_j(q-n)$$
$$= E_{ii}(n+m) - E_{jj}(n+m).$$

$n+m=0$:
$$\sum_q\left(e_i^\dagger(n+m+q)e_k(q) \;-\; e_j^\dagger(m+q)e_j(q-n)\right)$$
$$= \sum_q\left(e_i^\dagger(q)e_k(q) \;-\; e_j^\dagger(q+m)e_j(q+m)\right)$$

Note that if $n+m=0$, one cannot write $\sum_q e_i^\dagger(n+m+q)e_k(q) - \sum_q e_j^\dagger(m+q)e_j(q-n)$ because the individual sums are divergent!

$$= \sum_{q<0}\left(e_i^\dagger(q)e_k(q) \;-\; e_j^\dagger(q+m)e_j(q+m)\right) - \sum_{q>0}\left(e_i(q)e_k^\dagger(q) \;-\; e_j(q+m)e_j^\dagger(q+m)\right)$$
$$= \sum_{q<0} e_i^\dagger(q)e_k(q) - \sum_{q>0} e_i(q)e_k^\dagger(q) - \sum_{q<0} e_j^\dagger(q+m)e_j(q+m) + \sum_{q>0} e_j(q+m)e_j^\dagger(q+m)$$
$$= \underbrace{\left(\sum_{q<0} e_i^\dagger(q)e_k(q) - \sum_{q>0} e_i(q)e_k^\dagger(q)\right)}_{=E_{ii}(0)} - \underbrace{\left(\sum_{q<m} e_j^\dagger(q)e_j(q) - \sum_{q>m} e_j(q)e_j^\dagger(q)\right)}_{=E_{jj}(0)+m}$$
$$= \; E_{ii}(0) - E_{jj}(0) - m \;=\; E_{ii}(0) - E_{jj}(0) + n.$$

□

If one interprets the $E_{ij}(n)$ as an attempt to define an action of the algebraic part of $L\mathfrak{gl}(N)$ on the algebraic part of $\mathcal{F}$, then the relations in the above proposition can be rewritten as:

$$[f, a^\dagger(v)] = a^\dagger(fv) \qquad [f, a(v)] = -a(f^*v) \tag{26}$$
$$[f, g]_{\mathcal{F}} = [f, g]_{L\mathfrak{gl}(N)} + \tfrac{1}{2\pi i}\int_{S^1}\langle f, dg\rangle. \tag{27}$$

Here, $[\;,\;]_{\mathcal{F}}$ denotes the commutator as operators on Fock space, while $[\;,\;]_{L\mathfrak{gl}(N)}$ denotes the commutator in the Lie algebra $L\mathfrak{gl}(N)$ (and the inner product in the last term is

$\langle X, Y\rangle = -tr_{\mathbb{C}^N}(XY))$.

So we get an action of $\widetilde{L\mathfrak{gl}(N)}_1$ (the level one central extension of $L\mathfrak{gl}(N)$) on the Fock space of the N-th tensor power of the Dirac free fermion, or at least an algebraic action.

Our next goal is to extend that action of the algebraic part of $\widetilde{L\mathfrak{gl}(N)}_1$ to an action of all of $\widetilde{L\mathfrak{gl}(N)}_1$ on some suitable dense domain of $\mathcal{F}$. For that purpose, we define the <u>*Sobolev norms*</u> on $\mathcal{F}$:

$$\|\xi\|_s := \|(L_0 + 1)^s \xi\|.$$

For vectors that are not in the domain of $(L_0 + 1)^s$, we set $\|\xi\|_s := \infty$. Let $\mathcal{F}^s$ be the Hilbert space defined by the norm $\|\ \|_s$. It is given by

$$\mathcal{F}^s = \mathcal{D}_{(1+L_0)^s} = \mathcal{D}_{L_0^s}$$

for $s \geq 0$, and it is a completion of $\mathcal{F}$ for $s < 0$. The set of <u>*smooth vectors*</u> of $\mathcal{F}$ is then defined to be

$$\mathcal{F}^\infty := \left\{\xi \in \mathcal{F} \,\middle|\, \|\xi\|_s < \infty\ \forall s\right\} = \bigcap_s \mathcal{F}^s.$$

We equip it with the topology induced by all the Sobolev norms. The following theorem is our next goal:

Theorem *If $f = \sum f_n z^n$ is in the algebraic part of $L\mathfrak{gl}(N)$ and if ξ is in the algebraic part of $\mathcal{F}$, then*

$$\big\|f\xi\big\|_s \leq c\Big(\sum_{n\in\mathbb{Z}} (|n| + 1)^{\max(s,0)+1}\|f_n\|\Big)\|\xi\|_{s+\frac{1}{2}}.$$

Here, c is some constant that depends only on N.

As a consequence, every (smooth) element of $L\mathfrak{gl}(N)$ defines a bounded map from $\mathcal{F}^{s+\frac{1}{2}}$ to $\mathcal{F}^s$, and thus a continuous map from $\mathcal{F}^\infty$ to itself.

Note:
The sub $\widetilde{L\mathfrak{gl}(N)}_1$-module of $\mathcal{F}^\infty$ generated by Ω is isomorphic, after completion, to the vacuum module of the WZW model (for $L\mathfrak{gl}(N)$, at level 1). From the above Fock space realization of that module, we learn that its unique invariant inner product exists, and that it is positive definite.

Recall that we have constructed a representation

$$LU(N) \longrightarrow PU^\pm(\mathcal{F})$$

of the loop group of $U(N)$ on the Fermionic Fock space of the N-th tensor power of the Dirac free fermion characterized by the requirement that

$$u_\gamma a(v) u_\gamma^* = a(\gamma v) \quad \forall\, v \in \Gamma(\mathbb{C}^N \otimes \mathbb{S}).$$

Here, the superscript $^{\pm}$ on PU means that the unitaries can be either even (= grading preserving) or odd (= grading reversing). I don't want to deal with with odd unitaries at this point, so I'll pick a simply connected group G a representation $\rho : G \to U(N)$, and I'll restrict everything along the induced homomorphism

$$LG \to LU(N).$$

A good example to keep in mind is the case $G = SU(n)$, $N = kn$, where ρ is the direct sum of k copies of the standard n dimensional representation of $SU(n)$. If G is simply connected, then LG is connected and maps to even unitaries only.

The problem with that approach is that there is no easy way of computing the cocycle that controls the failure of that representation to land in $U(\mathcal{F})$.

On the Lie algebra side of things, we have constructed an action of the algebraic part of $\widetilde{L\mathfrak{gl}(N)}$ on the algebraic part of $\mathcal{F}$. As before, let us compose it with the map

$$\widetilde{L\mathfrak{g}} \to \widetilde{L\mathfrak{gl}(N)}$$

induced by some representation $\rho : \mathfrak{g} \to \mathfrak{gl}(N)$.

We've already computed in (27) the cocycle for the Lie algebra action, and we found that

$$[f,g]_{\mathcal{F}} = [f,g]_{L\mathfrak{g}} + \underbrace{\tfrac{1}{2\pi i}\int_{S^1} \langle f, dg\rangle}_{= -\tfrac{1}{2\pi i}\int_{S^1} tr_{\mathbb{C}^N}(f\,dg)}$$

In our specific example where $\mathfrak{g} = \mathfrak{sl}(n)$, $N = kn$, ρ is the direct sum of k copies of the standard n dimensional representation of $\mathfrak{sl}(n)$, the cocycle can be rewritten as

$$-\tfrac{1}{2\pi i}\int_{S^1} tr_{\mathbb{C}^{kn}}(f\,dg) \;=\; -\tfrac{k}{2\pi i}\int_{S^1} tr_{\mathbb{C}^{n}}(f\,dg),$$

which is exactly the one that defines the level k central extension of $L\mathfrak{sl}(n)$. More generally, we get an action of $\widetilde{L\mathfrak{g}}_k$ on $\mathcal{F}$, where

$$k := \frac{tr\big(\rho(X)\rho(Y)\big)}{\langle X, Y\rangle_{basic}}$$

is the so-called *Dynkin index* of the representation ρ. For the Lie algebra $\mathfrak{g} = \mathfrak{sl}(n)$, every positive integer k is realizable as the Dynkin index of some representation, but for other simple Lie algebras $\mathfrak{g}$ that is not always the case.

In order to extend the above action to an action of all of $\widetilde{L\mathfrak{g}}_k$ (not just its algebraic part), we need some analytic bounds.

Proposition 2 *If $\xi \in \mathcal{F}$ is vector with energy h (that is, an eigenvector of L_0 with eigenvalue h), then*

$$\|E_{ij}(n)\xi\| \leq (2\sqrt{h} + |n|)\|\xi\|.$$

Note that when restricted to vectors of a given energy h, the operator $E_{ij}(n)$ becomes a finite matrix. The proof of the above proposition is based on the following lemma:

Lemma *Let M be a matrix with entries in $\{0, 1, -1\}$ (arbitrary entries of norm ≤ 1 would also work). Assume that M has at most k non-zero entries in each row, and at most k non-zero entries each column. Then $\|M\| \leq k$.*

Proof: Our strategy will be to show that M is a sum of k signed partial permutation matrices (that is, a matrix with at most one ± 1 par row and per column). Each such partial permutation matrix has norm one, and so their sum has norm at most k.

Now we show that M is a sum of k signed partial permutation matrices. (Note that this is really a problem about edge colorings of bipartite graphs: the vertices of the graph are the rows and columns of M, and the edges are its non-zero entries. We'll formulate the proof in the language of matrices.)

We assume WLOG that the entries of M are in $\{0, 1\}$. **[***Notation:* Given matrices A, B with entries in $\mathbb{N}$, let us write $A \subseteq B$ if $A_{ij} \leq B_{ij}$ for every i, j. That is, $A \subseteq B$ if A is entry-wise smaller or equal than B.**]** Here's an algorithm for constructing the desired partial permutation matrices:

- Pad M with zeros to make it into a square matrix $\hat{M}$.
- If there is an i and a j such that the i-th row of $\hat{M}$ has sum $< k$ and its j-th column has sum $< k$, replace $\hat{M}_{ij}$ by $\hat{M}_{ij} + 1$. Do this until all the columns and rows add up to k. Call the resulting matrix M_1. Its entries are now in $\mathbb{N}$.
- By *sub-lemma 2* below, there exists a permutation matrix $P_1 \subseteq M_1$. Let $M_2 := M_1 - P_1$. Now, again there exists a permutation matrix $P_2 \subseteq M_2$. Let $M_3 := M_2 - P_2$. Etc. This way, we can write M_1 as a sum of k permutation matrices: $M_1 = \sum_{i=1}^{k} P_i$.
- By replacing some of the entries of the matrices P_i by zeros —call the resulting partial permutation matrices Q_i— we can arrange that $\hat{M} = \sum_{i=1}^{k} Q_i$.
- Finally, the last step is to truncate the matrices Q_i back to the size of M. □

***Sub-lemma 1*:** Let A be an $n \times n$ matrix with entires in $\mathbb{N}$. Assume that for every way of bringing A it into block upper triangular form by permuting its rows and columns

$$A \approx \begin{array}{cc} & \overset{t}{\leftrightarrow} \\ s\updownarrow & \left(\begin{array}{c|c} * & * \\ 0 & * \end{array}\right) \end{array} \tag{28}$$

we have $s \geq t$. Then there exists a permutation matrix P with $P \subseteq A$.

Proof: (Note: this result is equivalent to 'Hall's Matching Theorem' about the existence of perfect matchings in bipartite graphs). We reason by induction on n.

Case 1: There exists a way of bringing A into the above block upper triangular form with $s = t$. In that case, we may assume WLOG that our matrix is already in that form: $A = \left(\begin{smallmatrix} B & C \\ 0 & D \end{smallmatrix}\right)$. It is not difficult to see that B and D satisfy the hypothesis of the lemma. By induction, we can find permutation matrices $P \subseteq B$ and $Q \subseteq D$. It follows that $\left(\begin{smallmatrix} P & 0 \\ 0 & Q \end{smallmatrix}\right) \subseteq A$.

Case 2: Every way of bringing A into the above block upper triangular form has $s > t$. In that case, pick any non-zero entry of A, which we assume WLOG to be A_{1n}. Our matrix then has the following form: $A = \left(\begin{smallmatrix} B & A_{1n} \\ D & C \end{smallmatrix}\right)$. The submatrix D satisfies the hypothesis of the lemma (otherwise, this would contradict the fact that every way of making A block upper triangular has $s > t$), and so by induction, there exists a permutation matrix $P \subseteq D$. We then have $\left(\begin{smallmatrix} 0 & 1 \\ P & 0 \end{smallmatrix}\right) \subseteq A$. $\square$

Sub-lemma** 2: Let A be an $n \times n$ matrix with entires in $\mathbb{N}$. Assume that all of the rows and all of the columns of A add up to some fixed number $k \geq 1$. Then there exists a permutation matrix $P \subseteq A$.

Proof: It is enough to show that A satisfies the assumptions of the first sub-lemma. Consider a permutation of the rows and columns that brings A in block upper triangular form:

$$A \approx \begin{array}{c} \\ s\updownarrow \end{array} \!\!\begin{array}{c} \overset{t}{\leftrightarrow} \\ \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right) \end{array}.$$

The entries of our matirx being non-negative, we then have:

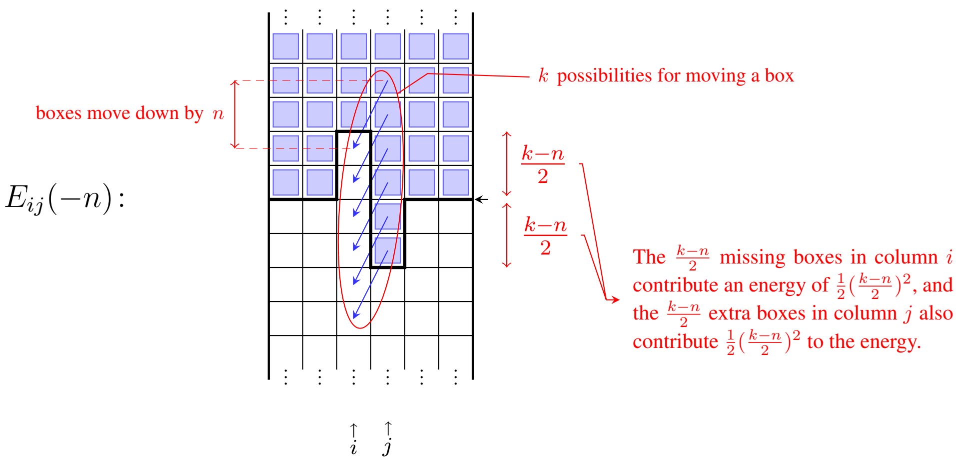
$$\begin{aligned} t &= \tfrac{1}{k}\cdot(\text{sum of all the entires in the first } t \text{ columns}) \\ &\leq \tfrac{1}{k}\cdot(\text{sum of all the entires in the first } s \text{ rows}) = s. \end{aligned}$$ $\square$

Proof of proposition 2: Let $\mathcal{F}_h \subset \mathcal{F}$ denote the subspace of vectors with energy h (that is, the eigenspace of L_0 with eigenvalue h). We are interested in the norm of the operator $E_{ij}(n) : \mathcal{F}_h \to \mathcal{F}_{h-n}$. Actually, let us consider the operator $E_{ij}(-n) : \mathcal{F}_h \to \mathcal{F}_{h+n}$ instead. It doesn't matter which one of the two we look at: they are related by $E_{ij}(-n) = E_{ji}(n)^*$ and so they have the same norm. We want to show that the norm is less than or equal to $2\sqrt{h} + n$:

$$\big\| E_{ij}(-n) : \mathcal{F}_h \to \mathcal{F}_{h+n} \big\| \overset{?}{\leq} 2\sqrt{h} + n.$$

We split our proof into two cases:

Case 1: If $i \neq j$, then the matrix representing $E_{ij}(-n)$ has entires in $\{0, 1, -1\}$. We'll show that this matrix has at most $2\sqrt{h} + n$ non-zero entries per row and per column. In order to arrange k non-zero entries in a given column, the most energetically economic configuration is this one:



The total energy of that configuration is then given by:

$$\tfrac{1}{2}(\tfrac{k-n}{2})^2 + \tfrac{1}{2}(\tfrac{k-n}{2})^2 = (\tfrac{k-n}{2})^2$$

(If $k-n$ is odd, then the most economic configuration that allows for k ways of moving a box has $\frac{k-n}{2} \pm \frac{1}{2}$ missing boxes in column i and $\frac{k-n}{2} \mp \frac{1}{2}$ extra boxes in column ij. Its energy is $\frac{1}{2}(\frac{k-n}{2} + \frac{1}{2})^2 + \frac{1}{2}(\frac{k-n}{2} - \frac{1}{2})^2 > (\frac{k-n}{2})^2$).

Given that the above configuration is *the most energetically economic*, and that h is our given energy we have:

$$(\tfrac{k-n}{2})^2 \leq h,$$

where k is now the maximal number of non-zero entires in a column of the matrix for $E_{ij}(-n)$. Solving for k, this gives:

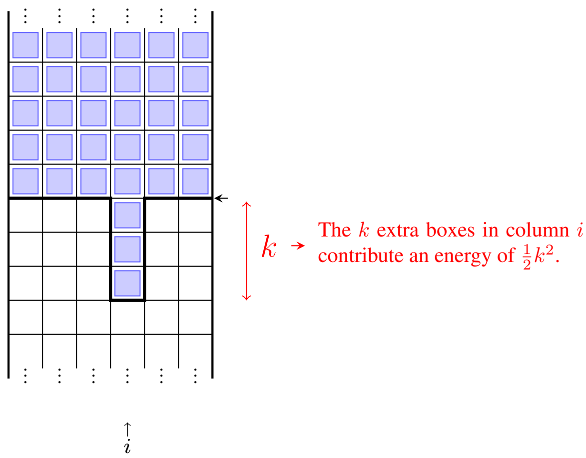
$$k \leq 2\sqrt{h} + n.$$

Now, we can do a similar kind of reasoning to find the maximal number of non-zero entires in a row of the matrix: that number turns out to be smaller (and so we don't care). *Conclusion:* The matrix for $E_{ij}(-n) : \mathcal{F}_h \to \mathcal{F}_{h+n}$ has at most $2\sqrt{h} + n$ non-zero entries per row and per column. Its norm is therefore bounded by $2\sqrt{h} + n$.

Case 2: $n \neq 0$ but $i = j$. In that case, the matrix for $E_{ii}(n) : \mathcal{F}_h \to \mathcal{F}_{h+n}$ is again filled with 0 and ± 1's. But in order to create k non-zero entries in a column of the matrix, the energy h needs to be higher than what it had to be in *Case 1*. The norm of the matrix is therefore smaller than the bound of $2\sqrt{h} + n$ that we got before.

Case 3: $E_{ij}(n)$ is of the form $E_{ii}(0)$. In that case, the matrix we're looking at is diagonal (and its entries are not restricted to the set $\{0, 1, -1\}$). The configuration that is most energetically economic and that realizes a k as an entry on the diagonal is the following one

$E_{ii}(0)$ has eigenvalue k:



(The other relevant configuration is when k boxes are missing from the i-th column. It has energy $\frac{1}{2}k^2$ and $E_{ii}(0)$ eigenvalue $-k$). Now, if h is our given energy and if k is the largest entry in our matrix, we get $\frac{1}{2}k^2 \leq h$, or equivalently $k \leq \sqrt{2h}$. Therefore, the norm of our matrix for $E_{ii}(0)$ is at most $\sqrt{2h} < 2\sqrt{h}$. $\square$

We can now prove the following theorem, already mentioned earlier (we phrase it here in a slightly weaker form). Recall the Sobolev norms: $\|\xi\|_s := \|(L_0 + 1)^s \xi\|$.

Theorem *If $f = \sum f_n z^n$ is in the algebraic part of $L\mathfrak{gl}(N)$ and ξ is in the algebraic part of $\mathcal{F}$, then*

$$\boxed{\|f\xi\|_s \leq c_f \|\xi\|_{s+\frac{1}{2}}} \tag{29}$$

for some constant c_f that depends on f and on s, but not on ξ.

Proof: **[*Notation:* let us write (s) for $\max(s,0)$.]** If ξ is homogeneous $L_0\xi = h\xi$, then

$$\begin{aligned}
\big\|E_{ij}(n)\xi\big\|_s &= \big(h-n+1\big)^s \big\|E_{ij}(n)\xi\big\| \\
&\leq \big(h+|n|+1\big)^s \big(2\sqrt{h}+|n|\big)\|\xi\| \\
&= \big(h+|n|+1\big)^s \big(h+1\big)^{-(s+\frac{1}{2})}\big(2\sqrt{h}+|n|\big)\,\|\xi\|_{s+\frac{1}{2}} \\
&= \left(\frac{h+1+|n|}{h+1}\right)^s \left(\frac{2\sqrt{h}+|n|}{\sqrt{h+1}}\right)\|\xi\|_{s+\frac{1}{2}} \\
&\leq \big(|n|+1\big)^{(s)}\big(|n|+2\big)\|\xi\|_{s+\frac{1}{2}} \leq 2\big(|n|+1\big)^{(s)+1}\|\xi\|_{s+\frac{1}{2}},
\end{aligned}$$

from which it follows that for $\xi = \sum \xi_h$ not necessarily homogeneous, we also have:

$$\begin{aligned}
\big\|E_{ij}(n)\xi\big\|_s = \sqrt{\sum_{h\geq 0}\|E_{ij}(n)\xi_h\|_s^2} &= \sqrt{\sum_{h\geq 0}\big(2(|n|+1)^{(s)+1}\|\xi_h\|_{s+\frac{1}{2}}\big)^2} \\
&= 2(|n|+1)^{(s)+1}\sqrt{\sum_{h\geq 0}\|\xi_h\|_{s+\frac{1}{2}}^2} = 2(|n|+1)^{(s)+1}\|\xi\|_{s+\frac{1}{2}}.
\end{aligned}$$

Now, given some arbitrary element $X = (x_{ij}) \in \mathfrak{gl}(N)$, we let $X(n) := \sum x_{ij}E_{ij}(n)$ stand for $Xz^n \in L\mathfrak{gl}(N)$. We then have

$$\begin{aligned}
\big\|X(n)\xi\big\|_s &\leq \sum |x_{ij}|\big\|E_{ij}(n)\xi\big\|_s \\
&\leq 2\sum |x_{ij}|(|n|+1)^{(s)+1}\|\xi\|_{s+\frac{1}{2}} \leq c\cdot(|n|+1)^{(s)+1}\,\|X\|\,\|\xi\|_{s+\frac{1}{2}},
\end{aligned}$$

where c is any constant that makes the inequality $2\sum|x_{ij}| \leq c\|X\|$ hold ($\mathfrak{gl}(N)$ being finite dimensional, any two norms on it are equivalent).

Finally, if $f = \sum f_n z^n$ is an arbitrary (algebraic) element of $L\mathfrak{gl}(N)$, then we get

$$\|f\xi\|_s \leq \sum_{n\in\mathbb{Z}}\|f_n z^n\cdot\xi\|_s \leq c\left(\sum_{n\in\mathbb{Z}}(|n|+1)^{(s)+1}\|f_n\|\right)\|\xi\|_{s+\frac{1}{2}},$$

which is exactly equation (29) with $c_f := c\sum_n(|n|+1)^{(s)+1}\|f_n\|$. $\square$

Corollaries:

- $\widetilde{L\mathfrak{gl}(N)}_1$ acts on the set $\mathcal{F}^\infty$ of smooth vectors of $\mathcal{F}$ (not just its algebraic part).

Since $\widetilde{L\mathfrak{g}}_k \subset \widetilde{L\mathfrak{gl}(N)}_1$, we then also get an action of $\widetilde{L\mathfrak{g}}_k$. Here, $k = \frac{tr(\rho(X)\rho(Y))}{\langle X,Y\rangle_{basic}}$ is the Dynkin index (explained on page 78) of our representation $\rho : \mathfrak{g} \to \mathfrak{gl}(N)$.

• Note that $\widetilde{L\mathfrak{g}_k}$ acts by *closeable* operators: the adjoint of $f \in \widetilde{L\mathfrak{g}_k}$ is densely defined (its domain contains $\mathcal{F}^\infty$). [*Notation:* let $o(f)$:= the *closed operator* (with core $\mathcal{F}^\infty$) associated to f.] We then have the following non-trival statement:

$$\boxed{o(f^*) = o(f)^*}$$

Here, f^* is the formal adjoint of f in $\widetilde{L\mathfrak{g}_k}$, while $o(f)^*$ denotes the adjoint of $o(f)$ in the sense of unbounded operators. Given our bounds (29), and the observation that $[L_0, f] = -z\frac{d}{dz}f$, the above formula is a consequence of **Nelson's commutator theorem** (actually our bounds are better than what is needed to be able to apply Nelson's theorem). The following text is taken from E. Nelson's paper 'Time ordered operator products of sharp-time quadratic forms'. It contains the statement and the proof of his theorem (and it is very well written!):

For all real k, let $\mathscr{H}^k$ be the completion of $\mathscr{D}(H^{k/2})$ (the symbol $\mathscr{D}$ denotes the domain of an operator) with respect to the inner product

$$\langle v, u\rangle_k = \langle (1+H)^{k/2}v, (1+H)^{k/2}u\rangle.$$

We let $\mathscr{L}(\mathscr{H}^k, \mathscr{H}^l)$, for $-\infty \leqslant k, l \leqslant \infty$, be the space of continuous linear transformations from $\mathscr{H}^k$ to $\mathscr{H}^l$, and we abbreviate $\mathscr{L}(\mathscr{H}^k, \mathscr{H}^k)$ by $\mathscr{L}(\mathscr{H}^k)$. Somewhat pedantically, we let $\mathscr{L}_0(\mathscr{H}^k, \mathscr{H}^l)$, for $-\infty < k, l < \infty$ be the set of linear operators with domain $\mathscr{H}^\infty$ (which is dense in $\mathscr{H}^k$) which extend by continuity to be in $\mathscr{L}(\mathscr{H}^k, \mathscr{H}^l)$. The norm of an operator A in $\mathscr{L}(\mathscr{H}^k, \mathscr{H}^l)$ or $\mathscr{L}_0(\mathscr{H}^k, \mathscr{H}^l)$ is denoted by $\| A \|_{k,l}$. The spaces $\mathscr{H}^k$ are called the *scale* associated with H. Throughout this paper we will be concerned with the scale associated with a fixed positive self-adjoint operator H.

If A is in $\mathscr{L}_0(\mathscr{H}^k, \mathscr{H}^l)$, we define its formal adjoint A^+ to be the operator in $\mathscr{L}_0(\mathscr{H}^{-l}, \mathscr{H}^{-k})$ such that $\langle A^+v, u\rangle = \langle v, Au\rangle$ for all v and u in $\mathscr{H}^\infty$. We say that A is Hermitean in case $A = A^+$.

PROPOSITION 2. *Suppose that A and $[H, A]$ are in $\mathscr{L}_0(\mathscr{H}^1, \mathscr{H}^{-1})$. Then $A^* = \overline{A^+}$. In particular, if A is Hermitean it is essentially self-adjoint.*

Proof. We use the method of mollifiers. If $\epsilon > 0$ then $e^{-\epsilon H} : \mathscr{H} \to \mathscr{H}^\infty$. On $\mathscr{H}^\infty$ we have the familiar identity

$$[A, e^{-\epsilon H}] = -\int_0^\epsilon e^{-(\epsilon - s)H}[A, H]\, e^{-sH}\, ds.$$

(Both sides are in $\mathscr{L}(\mathscr{H}^\infty)$, and their equality on the dense set of vectors which are in some spectral subspace of H corresponding to a bounded interval can be verified by power series expansion.)

We wish to estimate the norm of this commutator as an operator on $\mathscr{H}$. We estimate the norm of the integrand as being

$$\leqslant \| e^{-(\epsilon-s)H} \|_{-1,0} \|[A, H]\|_{1,-1} \| e^{-sH} \|_{0,1} .$$

But

$$\| e^{-sH} \|_{0,1} = \|(1 + H)^{1/2} e^{-sH} \| \leqslant \sup_{0 \leqslant \lambda < \infty} (1 + \lambda)^{1/2} e^{-s\lambda} = (2s)^{-1/2} e^{-1/2+s},$$

and

$$\| e^{-(\epsilon-s)H} \|_{-1,0} = \| e^{-(\epsilon-s)H} \|_{0,1} \leqslant (2(\epsilon - s))^{-1/2} e^{-1/2+\epsilon-s}.$$

Thus

$$\begin{aligned}\|[A, e^{-\epsilon H}]\| &\leqslant \frac{1}{2} e^{\epsilon-1} \int_0^\epsilon s^{-1/2}(\epsilon - s)^{-1/2}\, ds\, \|[A, H]\|_{1,-1} \\ &= \frac{1}{2} e^{\epsilon-1} \int_0^1 u^{-1/2}(1 - u)^{-1/2}\, du\, \|[A, H]\|_{1,-1} = \frac{\pi}{2} \|[A, H]\|_{1,-1}\, e^{\epsilon-1},\end{aligned}$$

which remains bounded as $\epsilon \to 0$. Let R_ϵ in $\mathscr{L}(\mathscr{H})$ be the closure of $[A, e^{-\epsilon H}]$. For u in $\mathscr{H}^\infty$,

$$R_\epsilon u = Ae^{-\epsilon H}u - e^{-\epsilon H}Au \to Au - Au = 0.$$

Since $\mathscr{H}^\infty$ is dense in $\mathscr{H}$ and $\| R_\epsilon \|$ remains bounded as $\epsilon \to 0$, R_ϵ converges strongly to 0 as $\epsilon \to 0$.

If w is in $\mathscr{D}(A^*)$, then

$$e^{-\epsilon H}A^*w = A^+e^{-\epsilon H}w + R_\epsilon^*w,$$

since the two sides have equal inner products with all elements in $\mathscr{H}^\infty$. But R_ϵ^*w converges weakly (even strongly) to 0, so that the pair (w, A^*w) is in the weak closure of the graph of A^+, and so is in the graph of $\overline{A^+}$. Q.E.D.

As a consequence of the above analysis, if an element $f \in \widetilde{L\mathfrak{g}}_k$ is skew-adjoint, then the operator $o(f)$ on $\mathcal{F}$ is skew-adjoint (not just formally skew-adjoint), and so its exponential $\mathrm{Exp}(o(f))$ is a well defined unitary operator on Fock space.

Recapitulation: We have constructed an action

$$\gamma \mapsto u_\gamma$$

of LG on $\mathcal{F}$ by projective unitary operators. By completely different means, we have also constructed an action

$$f \mapsto o(f)$$

of $\widetilde{L\mathfrak{g}}_k$ on $\mathcal{F}$ by closed unbounded operators, with common core $\mathcal{F}^\infty$. Moreover, we have argued that $\mathrm{Exp}(o(f)) \in U(\mathcal{F})$ makes sense for skew-adjoint elements of $\widetilde{L\mathfrak{g}}_k$. Our next task is to relate those two constructions.

Let $\exp : L\mathfrak{g} \to LG$ denote the exponential map.

Theorem *Let f be a skew-adjoint element of $\widetilde{L\mathfrak{g}}_k$. Then the projective unitary $u_{\exp([f])}$ associated to the image $[f]$ of f in $L\mathfrak{g}$ is equal to the image $[\mathrm{Exp}(o(f))]$ of $\mathrm{Exp}(o(f))$ in $PU(\mathcal{F})$. In formulas: $u_{\exp([f])} = [\mathrm{Exp}(o(f))]$. If you allow me to be sloppy in my notation and omit all the confusing brackets, this is:*

$$\boxed{u_{\exp(f)} = \mathrm{Exp}(o(f))}$$

Proof: Let me drop the cumbersome o's from the notation, and just write $\mathrm{Exp}(f)$ instead of $\mathrm{Exp}(o(f))$ ⚠ and make sure not to confuse $\mathrm{Exp}(f) \in U(\mathcal{F})$ with $\exp(f) \in LG$ ⚠. We need to check that the element $\mathrm{Exp}(f)$ satisfies the defining property of $u_{\exp(f)}$. Namely, we need to check that

$$\mathrm{Exp}(f)\, a^\dagger(v)\, \mathrm{Exp}(f)^* \overset{?}{=} a^\dagger(\exp(f)\, v)$$

for every $v \in \Gamma(\mathbb{C}^N \otimes \mathbb{S})$. Let us add a parameter to the above equation, and consider instead the equation

$$\mathrm{Exp}(tf)\, a^\dagger(v)\, \mathrm{Exp}(tf)^* \overset{?}{=} a^\dagger(\exp(tf)\, v). \tag{30}$$

Let us assume for the moment that both the left hand side (abbreviated *LHS*), and the right hand side (abbreviated *RHS*) of the above equation are differentiable. We can then check, at least formally, that:

$$\tfrac{d}{dt}\big(\mathit{LHS}\big) = [f, \mathit{LHS}] \qquad \text{and} \qquad \tfrac{d}{dt}\big(\mathit{RHS}\big) = a^\dagger\big(f\cdot \exp(tf)v\big) = [f, \mathit{RHS}],$$

where we have used (26) (on p.76) for the last equality. Both the *LHS* and the *RHS* are solutions of the same ODE, they agree at $t = 0$, and so they are equal. The problem with the above approach, is that it is not clear in what sense the above operator valued functions are differentiable. So that doesn't work.

Instead of (30), it turn out that it's easier to deal with the following equivalent form:

$$a^\dagger(v) \stackrel{?}{=} \operatorname{Exp}(tf)^* a^\dagger(\exp(tf)\,v)\operatorname{Exp}(tf).$$

We test that equation against vectors ξ and η in $\mathcal{F}^\infty$:

$$\begin{aligned}\langle a^\dagger(v)\xi,\eta\rangle &\stackrel{?}{=} \big\langle \operatorname{Exp}(tf)^* a^\dagger\big(\exp(tf)v\big)\operatorname{Exp}(tf)\xi,\eta\big\rangle \\ &= \big\langle a^\dagger\big(\exp(tf)v\big)\operatorname{Exp}(tf)\xi,\operatorname{Exp}(tf)\eta\big\rangle.\end{aligned} \tag{31}$$

Recall that if A is a skew adjoint operator, then the function $t \mapsto \operatorname{Exp}(A)\xi$ is differentiable for every vector ξ in the domain of A (indeed, $\mathcal{D}_A$ is exactly the set of vectors ξ for which that map is differentiable). The functions $t \mapsto \operatorname{Exp}(tf)\xi$, $t \mapsto \operatorname{Exp}(tf)\eta$ and $t \mapsto \exp(tf)v$ are all differentiable. By Lemma ($\star$) on page 54, the expression $\langle a^\dagger(\bullet)\,\bullet,\bullet\rangle$ depends trilinearly and continuously on its three arguments. So we see that the function

$$t \mapsto \big\langle a^\dagger\big(\exp(tf)v\big)\operatorname{Exp}(tf)\xi,\operatorname{Exp}(tf)\eta\big\rangle \tag{32}$$

is differentiable. Using equation (26) on page 76, the derivative of the above function is given by:

$$\begin{aligned}\big\langle a^\dagger\big(\exp(tf)v\big)f\operatorname{Exp}(tf)\xi,\operatorname{Exp}(tf)\eta\big\rangle + \big\langle a^\dagger\big(\exp(tf)v\big)\operatorname{Exp}(tf)\xi, f\operatorname{Exp}(tf)\eta\big\rangle \\ +\big\langle \underbrace{a^\dagger\big(f\exp(tf)v\big)}_{=[f,a^\dagger(\exp(tf)v)]}\operatorname{Exp}(tf)\xi,\operatorname{Exp}(tf)\eta\big\rangle = \\ \big\langle \underbrace{\big(a^\dagger(\exp(tf)v)f + f^*a^\dagger(\exp(tf)v) + [f,a^\dagger(\exp(tf)v)]\big)}_{=0}\operatorname{Exp}(tf)\xi,\operatorname{Exp}(tf)\eta\big\rangle = 0.\end{aligned}$$

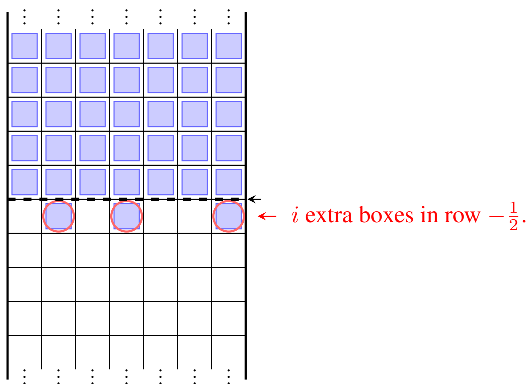
So (32) is an everywhere differentiable function with zero derivative, hence constant. Evaluating at $t = 0$, its value is $\langle a^\dagger(v)\xi,\eta\rangle$, and equation (31) follows. $\square$

Let us now think about which $\widetilde{L\mathfrak{g}}_k$-reps occur as subreps of $\mathcal{F}$. The first thing to note is that the charge grading of $\mathcal{F}$ is preserved by the action of $\widetilde{L\mathfrak{g}}_k$. Here, the *charge grading* is the grading by eigenspaces of the 'total charge' operator $\sum_i E_{ii}(0)$. It counts the total number "excess boxes" compared to the Maya diagram for the vacuum vector. Let

$$\mathcal{F}_{(i)} := \big(\text{charge } i \text{ summand of } \mathcal{F}\big).$$

Then for $i \in \{0,\ldots,N\}$, the minimal energy subspace of $\mathcal{F}_{(i)}$ is $\Lambda^i\mathbb{C}^N$. Its basis is given by all the $\binom{i}{N}$ ways of taking the Maya diagam for Ω and adding i extra boxes to it in the next available row:

The basis of the minimal energy subspace of $\mathcal{F}_{(i)}$:



Recall that $\mathbb{C}^N$ is equipped with our fixed representation ρ of $\mathfrak{g}$. Let V_λ be an irreducible representation of $\mathfrak{g}$ that occurs in $\Lambda^i\mathbb{C}^N$. We then get a map

$$\begin{tikzcd} V_\lambda \arrow[r] & \mathcal{F}_{(i)} \\ \cap & \\ W_{\lambda,k} = \operatorname{Ind}_{L\mathfrak{g}_{\geq 0}\oplus\mathbb{C}}^{\widetilde{L\mathfrak{g}}_k} V_\lambda. \arrow[uur, dashed] & \end{tikzcd} \tag{33}$$

by the universal property of induced modules. Pulling back the inner product of $\mathcal{F}_{(i)}$ along the dashed arrow, we get an $\widetilde{L\mathfrak{g}}_k$ invariant inner product on $W_{\lambda,k}$. It is positive semi-definite, and its null-vectors are the kernel of the map to $\mathcal{F}_{(i)}$. Therefore, we get a copy of

$$H_{\lambda,k} \;=\; \text{Hilbert space completion of } W_{\lambda,k}/(\text{null-vectors})$$

inside $\mathcal{F}_{(i)}$.

Let us work out the example where $\mathfrak{g} = \mathfrak{sl}(2)$, and $\mathbb{C}^N = \underbrace{\mathbb{C}^2 \oplus \ldots \oplus \mathbb{C}^2}_{k \text{ times}}$ is the direct sum of k copies of the 2-dimensional representation of $\mathfrak{sl}(2)$.
In that case, as explained on page 78, the Lie algebra that acts on the Fock space $\mathcal{F}$ is the level k central extension of $L\mathfrak{sl}(2)$. If $0 \leq i \leq k$, then $\Lambda^i\mathbb{C}^N = \Lambda^i(\mathbb{C}^2 \oplus \ldots \oplus \mathbb{C}^2)$ contains a vector of weight i (namely, the wedge product of the highest weight vectors of the first i copies of $\mathbb{C}^2$). As a consequence, $\Lambda^i\mathbb{C}^N$ contains a copy of the $i+1$ dimensional representation V_i of $\mathfrak{sl}(2)$. Using that representation in the diagram (33), we learn that there are non-trivial $\widetilde{L\mathfrak{sl}(2)}_k$-equivariant maps $W_{i,k} \to \mathcal{F}$ for every $i \in \{0, \ldots, k\}$. From the above discussion, we learn that: in the case $\mathfrak{g} = \mathfrak{sl}(2)$, the conditions $k \in \mathbb{Z}_{\geq 0}$ and $\lambda \in \{0, 1, \ldots, k\}$ which were shown on page 33 to be necessary for $W_{i,k}$ to admit an invariant positive semi-definite inner product are also *sufficient*. On our way, we have also proved the first two lemmas on page 32 for the case $\mathfrak{g} = \mathfrak{sl}(2)$.

Let us mention without proof that the very same constrictions also work for $\mathfrak{sl}(n)$. The Fock space associated to $\mathbb{C}^{kn}$ contains every V_λ as the lowest energy subspace of some sub- $\widetilde{L\mathfrak{sl}(n)}_k$-rep. So we get maps $W_{\lambda,k} \to \mathcal{F}$ that can then be used to show that all the $W_{\lambda,k}$ admit invariant positive semi-definite inner products.

Now I'd like to adress the third lemma on page 32. That's the claim that the von Neumann algbera $\mathcal{A}_{G,k}(I)$ (generated by L_IG on $H_{0,k}$) is canonically isomorphic to the von Neumann algbera generated by that same group on some other level k representation. Once again, we'll only treat the case $\mathfrak{g} = \mathfrak{sl}(2)$, but our method would also work for $\mathfrak{g} = \mathfrak{sl}(n)$.[8]

[8]For a general simple Lie algebra $\mathfrak{g}$, it is my understanding that this statement is still an open problem. The case $\mathfrak{g} = \mathfrak{sl}(n)$ has been treated by Wassermann, and a couple of other partial results exist in the literature.

Consider the following two von Neumann algebras:

Double commutant in $H_{0,k}$

$$\mathcal{A}(I) := \{u_\gamma \,|\, \gamma \in L_I G\}''$$

Double commutant in $\mathcal{F}$.

$$\mathcal{B}(I) := \{u_\gamma \,|\, \gamma \in L_I G\}''$$

By definition, $\mathcal{A}(I)$ is the algebra $\mathcal{A}_{G_k}(I)$ of local observables for the chiral WZW model. If we could should that the canonical map from $\mathcal{B}(I)$ to $\mathcal{A}(I)$ is an isomorphism, this would show that the Fock space is a representation of $\mathcal{A}_{G_k}$. In particular, all the $H_{\lambda,k}$'s that occur in $\mathcal{F}$ would then also automatically be representation of $\mathcal{A}_{G_k}$.

Theorem *The canonical map $\mathcal{B}(I) \to \mathcal{A}(I)$ is an isomorphism.*

Proof: • *Surjectivity*: This follows from a general fact about von Neumann algebras. If $A \subset B(H \oplus K)$ is a von Neumann algbera that respects the direct sum decomposition (that is, if A is a subalgebra of $B(H) \oplus B(K)$), then its image in $B(H)$ is also a von Neumann algebra. We'll prove this as a lemma after this theorem.

• *Injectivity*: (That's the hard part.) Recall the statement of Haag duality for the free fermions: $\mathcal{A}_{Fer}(I')^{\boldsymbol{\prime}} = \mathcal{A}_{Fer}(I)$, where the fat prime $^{\boldsymbol{\prime}}$ denotes the graded commutant. Recall also that $u_\gamma a(v) u_\gamma^* = a(\gamma v)$. By Haag duality for the free fermions, we therefore have:

$$\begin{aligned} \mathrm{supp}(\gamma) \subset I \quad &\Rightarrow \quad [u_\gamma, a(v)] = 0 \quad \text{for all } v \in \Gamma(\mathbb{C}^N \otimes \mathbb{S}) \text{ with support in } I' \\ &\Rightarrow \quad u_\gamma \in \mathcal{A}_{Fer}(I')^{\boldsymbol{\prime}} = \mathcal{A}_{Fer}(I), \end{aligned}$$

from which it follows that $\mathcal{B}(I) \subset \mathcal{A}_{Fer}(I)$.

Now let $x \in \mathcal{B}(I)$ be in the kernel of the map to $\mathcal{A}(I)$. Such an element acts as zero on $H_{0,k} \subset \mathcal{F}$. In particular, $x\Omega = 0$. But we have already seen that Ω is separating for $\mathcal{A}_{Fer}(I)$ (that's the Reeh-Schlieder theorem). Combining the facts that $x \in \mathcal{A}_{Fer}(I)$ and $x\Omega = 0$, it follows that $x = 0$. □

Corollary: The third lemma on page 32 holds for $\mathfrak{g} = \mathfrak{sl}(2)$.

Lemma *Let $A \subset B(H) \oplus B(K) \subset B(H \oplus K)$ be a von Neumann algebra. Then its image under the projection $B(H) \oplus B(K) \twoheadrightarrow B(H)$ is also a von Neumann algebra.*

Proof: Let $p \in A'$ be the orthogonal projection onto H. The commutant of any $*$-algebra is a von Neumann agebra. Therefore, it's enough to show that $pA = (pA'p)'$, where the first commutant is taken on $H \oplus K$, and second commutant is taken on H. The inclusion $pA \subseteq (pA'p)'$ is obvious, so we concentrate on the other inclusion.

Let x be an operator on H that commutes with $pA'p$, and let us immediately identify it with the corresponding operator $\left(\begin{smallmatrix} x & 0 \\ 0 & 0 \end{smallmatrix}\right)$ on $H \oplus K$. We want to show that x is of the form

$p\tilde{x}$ for some element $\tilde{x} \in A$. We define $\tilde{x}$ as follows:

$$\underbrace{A'H} \oplus \underbrace{(A'H)^{\perp}}$$

On this piece $\tilde{x} = 0$,

and on this piece, $\tilde{x}$ is given by the formula

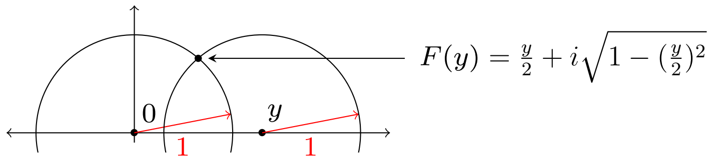
$$\sum b_i \xi_i \mapsto \sum b_i x \xi_i$$

for $b_i \in A'$ and $\xi_i \in H$.

It is not clear that the above expression defines a bounded operator on $H \oplus K$ (it might be ill-defined; it might be unbounded). We need to check that $\|\sum b_i x \xi_i\| \le C \cdot \|\sum b_i \xi_i\|$ for some constant C that only depends on x (this will solve both issues).

Sub-lemma: Any element x of a von Neumann algebra A can be written as a linear combination $x = \sum_{i=1}^{4} \lambda_i u_i$ of four unitaries $u_i \in A$.

Proof: We assume WLOG that $\|x\| \le 2$. First write x as a linear combination $x = \frac{1}{2}(x + x^*) - \frac{i}{2}((ix) + (ix)^*)$ of two self-adjoint elements of norm ≤ 2. Every self-adjoint element $y \in A$ of norm ≤ 2 can in turn be written it as the sum of two unitaries: $y = F(y) + F(y)^*$, where $F : [-2, 2] \to \mathbb{C}$ is the function defined by the following figure



The spectrum of $F(y)$ lies on the unit circle, and so it is indeed unitary. $\square$

Write x as a sum of four unitaries $x = \sum_{\alpha=1}^{4} \lambda_\alpha u_\alpha$. We then have:

$$\begin{aligned}
\Big\|\sum_i b_i x \xi_i\Big\| &\le \sum_{\alpha=1}^{4} |\lambda_\alpha| \Big\|\sum_i b_i u_\alpha \xi_i\Big\| \\
&= \sum_{\alpha=1}^{4} |\lambda_\alpha| \Big\langle \sum_i b_i u_\alpha \xi_i, \sum_j b_j u_\alpha \xi_j \Big\rangle^{1/2} \\
&= \sum_{\alpha=1}^{4} |\lambda_\alpha| \Big(\sum_{i,j} \Big\langle b_i p u_\alpha \xi_i, b_j p u_\alpha \xi_j \Big\rangle \Big)^{1/2} \\
&= \sum_{\alpha=1}^{4} |\lambda_\alpha| \Big(\sum_{i,j} \Big\langle \cancel{u_\alpha^*} p b_j^* b_i p \cancel{u_\alpha} \xi_i, \xi_j \Big\rangle \Big)^{1/2} \\
&= \sum_{\alpha=1}^{4} |\lambda_\alpha| \Big(\sum_{i,j} \langle b_i \xi_i, b_j \xi_j \rangle \Big)^{1/2} = \sum_{\alpha=1}^{4} |\lambda_\alpha| \Big\|\sum_i b_i \xi_i\Big\|.
\end{aligned}$$

As a consequence of the above calculation, we learn that $\tilde{x}$ is well defined, and that it is bounded. The operator $\tilde{x}$ commutes with A' and is therefore an element of A. Moreover, by construction, we have $x = \tilde{x}p \,(= p\tilde{x})$. This finishes the proof that $(pA'p)' \subset pA$. $\square$

We did a lot of hard work... Now is the moment to collect the fruit of our efforts. Let us recall what we have constructed:

—— ◀◀ ① ▶▶ ——

First of all, we have constructed a Fock space $\mathcal{F}$ (that depended on the choice of some complex vector space $\mathbb{C}^N$), and a projective action

$$\mathrm{Diff}_+^{(2)}(S^1) \to PU(\mathcal{F}) : g \mapsto u_g$$

characterized by the requirement that $u_g\, a^\dagger(v)\, u_g^* = a^\dagger(g \cdot v)$ for every $g \in \mathrm{Diff}_+^{(2)}(S^1)$ and $v \in \Gamma(\mathbb{C}^N \otimes \mathbb{S})$. Pulling back that representation along the projection map $U(\mathcal{F}) \twoheadrightarrow PU(\mathcal{F})$, we get a homomorphism

$$\widetilde{\mathrm{Diff}}_+^{(2)}(S^1) \to U(\mathcal{F})$$

from some central extension of the double cover of the diffeomorphism group to the group of unitary operators on $\mathcal{F}$.

Given an interval $I \subset S^1$, let $\mathrm{Diff}_0(I)$ be the subgroup of $\mathrm{Diff}_+(S^1)$ consisting of elements that fix the complement I' of I pointwise. It's also isomorphic to the subgroup of $\mathrm{Diff}_+^{(2)}(S^1)$ that fixes I' (and $\mathbb{S}|_{I'}$) pointwise. If $g \in \mathrm{Diff}_0(I)$, then u_g commutes with $a^\dagger(v)$ for every v with support in I'. By Haag duality, u_g must then be an element of $\mathcal{A}_{Fer}(I)$, and so we get an embedding

$$\widetilde{\mathrm{Diff}}_0(I) \hookrightarrow \mathcal{A}_{Fer}(I).$$

Here, $\widetilde{\mathrm{Diff}}_0(I)$ denotes the central extension of $\mathrm{Diff}_0(I)$ obtained by restricting the central extension $\widetilde{\mathrm{Diff}}_+^{(2)}(S^1)$ of $\mathrm{Diff}_+^{(2)}(S^1)$. Moreover, the above story works with diffeomorphisms of regularity as low as Hölder class $\mathcal{C}^{1,\alpha}$ with $\alpha > \frac{1}{2}$, that is, continuously differentiable with derivative satisfying $g'(x+h) = g'(x) + \mathcal{O}(h^\alpha)$.

—— ◀◀ ② ▶▶ ——

Second, after having fixed a representation ρ of G on $\mathbb{C}^N$, we have constructed a projective action

$$LG \to PU(\mathcal{F}) : \gamma \mapsto u_\gamma$$

characterized by the requirement that $u_\gamma\, a^\dagger(v)\, u_\gamma^* = a^\dagger(\gamma \cdot v)$ for every $\gamma \in LG$ and $v \in \Gamma(\mathbb{C}^N \otimes \mathbb{S})$. Once again, pulling back that representation along the projection map $U(\mathcal{F}) \twoheadrightarrow PU(\mathcal{F})$, we get a homomorphism

$$\widetilde{LG} \to U(\mathcal{F})$$

from some central extension of LG to the group of unitary operators on $\mathcal{F}$.

Recall that $L_I G \subset LG$ denotes the subgroup of loops with support in some interval $I \subset S^1$. If γ is in $L_I G$, then u_γ commutes with $a^\dagger(v)$ for every v with support in I'. By Haag duality, u_γ must then be an element of $\mathcal{A}_{Fer}(I)$, and so we get an embedding

$$\widetilde{L_I G} \hookrightarrow \mathcal{A}_{Fer}(I).$$

Moreover, we were able to verify that the above map extends to homomorphism of von Neumann algebras $\mathcal{A}_{G,k}(I) \to \mathcal{A}_{Fer}(I)$ from the local algebras of the WZW model to the local algebras of the Dirac free fermion (that was the theorem on p.88), where k is the Dynkin index of the representation ρ.

Finally, the above story works with loops of regularity as low as Sobolev-$1/2$. As we'll see later, those need not be continuous.

③

We now would like to see how the local diffeomorpisms relate to the local loops, directly.