SUBFACTORS, THE BASIC CONSTRUCTION AND KNOT INVARIANTS

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All of the things discussed in this note can be found in Jones' article [1] and the book Jones and Sunde [2].

1. Subfactors of type II_1 factors and their index

In this section we define subfactors and use the coupling constant to define a numerical invariant of subfactors. The construction will be similar to that of subgroups and their index.

1.1. **Subfactors.** We will be working with type II_1 factors. These are von Neumann-algebras with a positive normalised trace $\mathrm{t}r : M \to \mathbb{C}$:

Positive: $\mathrm{t}r(x^*x) \geq 0$ for all $x \in M$.
Normalised: $\mathrm{t}r(1) = 1$.
Tracial: $\mathrm{t}r(xy) = \mathrm{t}r(yx)$ for all $x, y \in M$.

It is a fact that this trace is then unique, and additionally it is faithful and normal.

Normal: For every increasing net of positive elements $\{a_i\}$ we have $\mathrm{t}r(\bigvee_i a_i) = \limsup \mathrm{t}r(a_i)$. It is equivalent to $\mathrm{t}r$ being ultraweakly continuous.

Uniqueness: The map $\mathrm{t}r$ is unique. The map $\mathrm{t}r$ is determined by its values on projections and uniqueness for matrix algebras shows that it is uniquely determined on projections corresponding to $\mathbb{Q} \cap [0, 1] \subset [0, 1]$. Now apply normality for the projections corresponding to irrational numbers.

Faithful: If $\mathrm{t}r(x^*x) = 0$ then $x = 0$. The proof is given by showing that this is an ideal and there is a central projection onto it. This projection must be zero because $\mathrm{t}r$ is non-zero (hence the ideal is proper) and M is a factor (hence there are no non-zero proper ideals).

This trace allows us to construct the standard form of M: a Hilbert space $L^2(M)$ such that the action for M has a cyclic and seperating vector. It is given by the completion of M with respect to the inner product $\langle x, y\rangle = \mathrm{t}r(y^*x)$. There is a linear map $M \to L^2(M)$ and we denote the image of $x \in M$ by $\hat{x}$. The image $\hat{1}$ of $1 \in M$ is the cyclic seperating vector and will denoted Ω.

We will need some of these properties later, but the actual definition of a subfactor doesn't require them.

Definition 1.1. A subfactor N of M is a unital inclusion $N \subset M$ of type II_1 factors. Here unital means that N contains the unit of M.

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A simple example of a subfactor is the inclusion $M_2(\mathbb{R}) \to M_4(\mathbb{R})$ (every matrix algebra is a factor as in the Artin-Wedderburn theorem) given by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix}$$

In other words, a subfactor is nothing but a subalgebra which is a factor itself. Note that because subfactors have no non-trivial ideals, any map of von Neumann-algebras between subfactors is an inclusion. In this sense, factors are somewhat like non-commutative fields and subfactors like subfields.

1.2. **The index of a subfactor.** Recall that we have seen a bit of the theory of the coupling constant $\dim_M(\mathcal{H})$ of a module $\mathcal{H}$ over a type II_1 factor M. This has properties which make it analogous to the dimension of $\mathcal{H}$ over M, although this dimension can be any positive real.

The *coupling constant*, a number in $[0, \infty]$ associated to a M-module $\mathcal{H}$, was defined as follows: if $\mathcal{H}$ is a M-module then up to equivalence there is a unique projection $p \in M \otimes B(\ell^2(\mathcal{N}))$ such that $\mathcal{H}$ is isomorphic to $\mathcal{H} \otimes \ell^2(\mathcal{N})p$ as a M-module. We define the coupling constant to be the trace of this projection: $\dim_M \mathcal{H} = \mathrm{tr}(p)$.

By noting that $M \otimes B(\ell^2(\mathcal{N}))$ is of type II_∞ which has projections of all traces in $[0, \infty]$ and taking $\mathcal{H} = L^2(M) \otimes \ell^2(\mathcal{N})q$ for a projection q of trace $d \in [0, \infty]$, we see that the coupling constant attains all values in $[0, \infty]$ for all M.

We list the properties of the coupling constant here:

Proposition 1.2. *The coupling constant* $\dim_M(-)$ *satisfies the following properties:*

- **Normalisation:** $\dim_M(L^2(M)) = 1$.
- **Faithful:** *If* $\dim_M(\mathcal{H}) = \dim_M(\mathcal{K})$ *then* $\mathcal{H}$ *and* $\mathcal{K}$ *are unitarily equivalent.*
- **Addivity:** *If* $\{\mathcal{H}_i\}$ *is a countable collection of* M*-modules, then* $\dim_M(\bigoplus_i \mathcal{H}_i) = \sum_i \dim_M(\mathcal{H}_i)$.
- **Multiplicativity:** *If* $\mathcal{K}$ *is any Hilbert space and* M *acts on* $\mathcal{H} \otimes \mathcal{K}$ *by tensoring the original action with the identity, then* $\dim_M(\mathcal{H} \otimes \mathcal{K}) = \dim_M(\mathcal{H}) \dim_{\mathbb{C}}(\mathcal{K})$.
- **Compatibility with projections:** *For* p *a projection in* M*, we have that* $\dim_M(\mathcal{H}p) = \mathrm{tr}(p)$ *and* $\dim_{pMp}(p\mathcal{H}) = \mathrm{tr}(p)^{-1} \dim_M(\mathcal{H})$.

If $\dim_M(\mathcal{H}) < \infty$*, then* M' *(relative to* $\mathcal{H}$*) is again type* II_1*. The following properties hold if* M' *is a type* II_1 *factor*[1]*:*

- **Commutant:** *We have* $\dim_{M'}(\mathcal{H}) < \infty$ *and* $\dim_{M'}(\mathcal{H}) = (\dim_M(\mathcal{H}))^{-1}$.
- **Projections of commutant:** *For* p *a projection of* M'*, we have that* $\dim_{Mp}(p\mathcal{H}) = \mathrm{tr}_{M'}(p) \dim_M(\mathcal{H})$.

Using the coupling constant and the standard form $L^2(M)$ we can define the index of N. To do this, note that the inclusion $N \hookrightarrow M$ endows $L^2(M)$ with the structure of an N-module.

Definition 1.3. The index $[M : N]$ of N in M is defined to be $\dim_N(L^2(M))$.

The properties of the coupling constant imply several properties of the index.

Proposition 1.4. *The index has the following properties. Let* $N \subset P \subset M$ *be a sequence of subfactors.*

- **Normalisation:** $[M : M] = 1$.
- **Positivity:** $[M : N] \geq 1$ *and* $[M : N] \geq [M : P]$.
- **Tower rule:** $[M : N] = [M : P][P : N]$.
- **Faithful:** $[M : P] = [M : Q]$ *implies* $P \cong Q$.
- **Multiplicativity:** $[M_1 \otimes M_2 : N_1 \otimes N_2] = [M_1 : N_1][M_2 : N_2]$.
- **Commutant:** *If* N' *is a* II_1 *factor (relative to* $L^2(M)$*) then* $[M : N] = [N' : M']$.

[1] In fact, the commutant M' can only be a type II_1 or II_∞ factor.

Proof. A very useful tool is the fact that we can also calculate the index using other modules that $L^2(M)$. Let $\mathcal{H}$ be any M-module of finite non-zero dimension. Then we claim that

$$[M:N] = \frac{\dim_N(\mathcal{H})}{\dim_M(\mathcal{H})}$$

For a proof, see [2, proposition 2.3.5], but the idea is to use the faithfulness of the coupling constant to pick a nice isomorphic M-module $\mathcal{H}'$ for $\mathcal{H}$ and do the calculations with that.

Normalisation: We have $[M:M] = 1$ because the coupling constant satisfies $\dim_M(L^2(M)) = 1$.

Positivity: We do the first statement, then the second part is a consequence of this and the tower rule. If the index is infinite, we are done and hence we can assume it is finite. Because the index is finite, it follows that $\dim_N(L^2(M))$ is finite and hence N' is type II_1. Note that the projection q onto N is an element of N' factor. This means that

$$1 = \dim_N(L^2(N)) = \dim_{Nq}(qL^2(M)) = \mathrm{tr}_{N'}(q)\dim_N(L^2(M)) \leq \dim_N(L^2(M))$$

where we have used the compatibility of the coupling constant with projections in the commutant (when the commutant is type II_1) and the fact that all non-zero projections of a type II_1 factor have trace in $(0,1]$.

Tower rule: For the tower rule, we will use that we can use any module of finite dimension to compute the index. We have that:

$$[M:P] = \dim_P(L^2(M)) = \dim_N(L^2(M))\frac{\dim_P(L^2(M))}{\dim_N(L^2(M))} = [M:N][N:P]$$

Faithful: We do not need this, so we will not prove it.

Multiplicativity: We also do not need this.

Commutant: This is a direct consequence of the commutant formula of the coupling constant:

$$[M:N] = \frac{\dim_N(L^2(N'))}{\dim_M(L^2(N'))} = \frac{\dim_{M'}(L^2(N'))}{\dim_{N'}(L^2(N'))} = \dim_{M'}(L^2(N')) = [N':M']$$

Because we can use any finite dimensional module, this formula in fact holds for taking the commutant with respect to any $\mathcal{H}$ such that P' is type II_1.

□

2. The basic construction and the tower

In this section we will use the standard form to construct a tower of type II_1 factors associated to a subfactor $N \subset M$. To investigate its properties we will use the index. We start with a single step of the tower, known as the basic construction.

2.1. The basic construction.

The idea of the basic construction is as follows: starting with a subfactor $N \subset M$, we add the projection e_N onto N to M to get a subfactor $M \subset \langle M, e_N\rangle$, where the latter is the von Neumann-algebra generated by M and e_N in $B(L^2(M))$.

We begin by considering the linear projection $e_N : L^2(M) \to L^2(N)$ which exists because $L^2(N)$ is a closed subspace. It is a bounded linear map which commutes with everything that commutes with N and hence maps $M\Omega$ to $N\Omega$. Because Ω is seperating, we thus have an induced operation $E_N : M \to N$ defined by $e_N x\Omega = E_N(x)\Omega$ for all $x \in M$. The map E_N is called the *conditional expectation* of M onto N. Alternatively, one can characterize E_N by the property $tr(E_N(m)n) = tr(mn)$ for $m \in M$, $n \in N$ such that all traces are defined. However, with this definition it is less clear that such a map actually exists.

For convenience, we will write $e = e_N$ and $E = E_N$, only to use the subscripts in situation where there are several subfactors floating around.

Lemma 2.1. *The map E satisfies the following properties:*

(1) $Je = eJ$ *and* $E(m)^* = E(m^*)$ *for all* $m \in M$.
(2) $E(n_1 m n_2) = n_1 E(m) n_2$ *for all* $m \in M$, $n_1, n_2 \in N$.

(3) *For all $m \in M$ we have an equality $eme = E(m)e$ as operators on $L^2(M)$.*
(4) $\mathrm{tr} \circ E = \mathrm{tr}$.

Proof. (1) It suffices to prove the first statement on the dense subspace $M\Omega$. We have that $em\Omega$ is equal to the limit $m\Omega - n_\alpha\Omega$ where $n_\alpha \in N$ is a sequence such that the limit of the distances is the infimum of all distances from $m\Omega$ to points in $L^2(N)$. Taking the $*$ preserves gives a sequence with this property for $m^*\Omega$ and hence $J(em\Omega) = eJ(m\Omega)$. The second statement is a direct consequence of the first:

$$E(m)^*\Omega = J(E(m)\Omega) = J(em\Omega) = eJ(m\Omega) = em^*\Omega = E(m^*)\Omega$$

(2) We first note that e commutes with elements of N. This is because $e|_{N\Omega}$ is by definition the identity and in particular we have that $e(\Omega) = \Omega$ because the unit of N is the unit of M. This implies that

$$E(n_1x)\Omega = en_1x\Omega = n_1ex\Omega = n_1E(x)\Omega$$

which implies $E(n_1x) = n_1E(x)$.

For the general case, it now suffices to show $E(mn_2) = E(m)n_2$. To prove this, we use the J. Because J is bijective, it suffices to prove that two vectors are equal after applying J. Let's apply J to $E(mn_2)\Omega$:

$$\begin{aligned} J(E(mn_2)\Omega) &= J(emn_2\Omega) = eJ(mn_2\omega) = en_2^*m^*\Omega = n_2^*em^*\Omega \\ &= n_2^*E(m)^*\Omega = J(E(m)n_2\Omega) \end{aligned}$$

(3) It suffices to prove $eme = E(m)e$ on the dense subspace $M\Omega$. In this case e maps $M\Omega$ surjectively onto $N\Omega$, so it suffices to show that $emn\Omega = E(m)n\Omega$. But this is easy: the former is by definition $E(mn)$ and this is equal to the latter by the second property we have proven.
(4) We have that $\mathrm{tr}(E(m)) = \langle E(m)\Omega, \Omega\rangle$. The latter is equal to $\langle em\Omega, \Omega\rangle$. Since e is a bounded and a projection, it has an adjoint and is equal to its adjoint. So we can write $\langle m\Omega, e\Omega\rangle$. But N is unital, so $e\Omega = \Omega$. Thus we get $\langle m\Omega, \Omega\rangle = \mathrm{tr}(m)$.

□

We now do the basic construction and then discuss some of its properties.

Definition 2.2. $\langle M, e_N\rangle$ is the von Neumann algebra generated by M and e_N, i.e. it is given by $(M \cup \{e_N\})''$. This is called the basic construction.

Proposition 2.3. *The basic construction has the following properties:*
(1) *$\langle M, e_N\rangle = JN'J$ where N' is taken with respect to $L^2(M)$.*
(2) *$\langle M, e_N\rangle$ is a factor if N is.*
(3) *Furthermore $\langle M, e_N\rangle$ is a factor of type II_1 if $[M : N] < \infty$.*

For the following properties we will assume that $[M : N] < \infty$.

(4) *M is a subfactor of $\langle M, e_N\rangle$ with index $[\langle M, e_N\rangle : M] = [M : N]$.*
(5) *The canonical trace $\tilde{\mathrm{tr}}$ of $\langle M, e_N\rangle$ extends the canonical traces of N and M. It satisfies the Markov property*

$$\tilde{\mathrm{tr}}(xe_N) = [M : N]^{-1}\mathrm{tr}(x)$$

for all $x \in M$ and in particular we therefore have $\tilde{\mathrm{tr}}(e_N) = [M : N]^{-1}$.

Proof. (1) We first prove that $N' = (M' \cup \{e_N\})''$. For this it suffices to prove that $N = (M' \cup \{e_N\})'$. But if $x \in (M' \cup \{e_N\})'$ in particular $x \in M''$ and hence $x \in M$. If $x \in M$ commutes with e_N, then $x = E_N(x)$ and we conclude that $x \in N$.

Now it suffices to note that

$$JN'J = J(M' \cup \{e_N\})''J = (JM'J \cup \{Je_NJ\})'' = (M \cup \{e_N\})''$$

(2) We have that $N \subset \langle M, e_N \rangle$ and we claim that $x \in \langle M, e_N \rangle$ commutes with e_N if and only if $x \in N \subset M$. Then the claim is clear, because it implies that

$$\langle M, e_N \rangle' \cap \langle M, e_N \rangle = N \cap \langle M, e_N \rangle' \subset N \cap N' = \mathbb{C}$$

To prove this, we note that $x \mapsto xe_N$ is an isomorphism of N onto $e_N\langle M, e_N\rangle e_N$. It is surjective because $\langle M, e_N \rangle$ is the weak closure of $a_0 + \sum_i a_i e_N b_i$ for $a_i, b_i \in M$ and clearly $e_N a_i e_N b_i e_N \in N e_N$ by the previous proposition. If $xe_N = 0$ then $xe_N\Omega = x\Omega = 0$ implies $x = 0$ because Ω is seperating, so it is also injective.

(3) If $\dim_N(L^2(M)) = [M : N] < \infty$, then we know that N' (with respect to $L^2(M)$) is of type II_1. Because $\langle M, e_N \rangle = JN'J$, this implies that it is of type II_1 as well.

(4) Clearly the inclusion of M in $\langle M, e_N \rangle$ is unital and hence M is a subfactor. For the index we have

$$\begin{aligned}
[\langle M, e_N \rangle : M] &= [M' : (\langle M, e_N \rangle)'] \\
&= \frac{\dim_{(\langle M, e_N \rangle)'}(L^2(M))}{\dim_{M'}(L^2(M))} \\
&= \dim_{JNJ}(L^2(M)) \\
&= \dim_N(L^2(M)) \\
&= [M : N]
\end{aligned}$$

where we have used that we pick any module of finite dimension to compute the index, in this case $L^2(M)$, that $\dim_{M'}(L^2(M)) = (\dim_M(L^2(M)))^{-1} = 1$ and that the commutant of $\langle M, e_n \rangle$ with respect to $L^2(M)$ is JNJ, whose action on $L^2(M)$ is isomorphic to the opposite action of N, which gives the same dimension.

(5) Because it is of type II_1, there exists a canonical trace $\tilde{tr}$ on $\langle M, e_N \rangle$. Because $N \subset M \subset \langle M, e_N \rangle$ and normalized traces are unique on type II_1 factors, the trace $\tilde{tr}$ extends both the trace of N and M.

We can consider the trace $\tilde{tr}(ne_N)$ on N and because N is a factor we must have that this is $C\mathrm{tr}(n)$. By definition of the dimension $\bar{tr}(e_N) = [M : N]^{-1}$ for the trace $\bar{tr}$ on N' hence $\tilde{tr}(e_N) = [M : N]^{-1}$ as well and we obtain $C = [M : N]^{-1}$. Finally for the Markov property in general:

$$\begin{aligned}
\tilde{tr}(me_N) &= \tilde{tr}(me_N e_N) = \tilde{tr}(e_N m e_N) = \tilde{tr}(E_N(m)e_N) \\
&= [M : N]^{-1}\mathrm{tr}(E_N(m)) = [M : N]^{-1}\mathrm{tr}(m)
\end{aligned}$$

□

2.2. **The tower.** The trick is now to look at the input and output of the basic construction. The input was:

- A subfactor $N \subset M$ of type II_1 factors with finite index $\tau^{-1} = [M : N]$.

The output was:

- A subfactor $M \subset \langle M, e_N \rangle$ of type II_1 factors with finite index $\tau^{-1} = [M : N]$.

  This has the additional property that trace of $\langle M, e_N \rangle$ satisfies the Markov property $\mathrm{tr}_{\langle M, e_N \rangle}(me_N) = \tau \mathrm{tr}_M(m)$ for all $m \in M$.

This means that we can iterate the construction.

Definition 2.4. Iterate the basic construction of $M_{-1} = N$, $M_0 = M$ to get $M_1 = \langle M, e_N \rangle$ etc. Set $\tau^{-1} = [M : N]$. We obtain a tower of subfactors

$$M_{-1} \subset M_0 \subset M_1 \subset M_2 \subset \dots$$

such that each von Neumann-algebra is a type II_1 factor which has index τ^{-1} in the next one. Each algebra M_n contains n distuingished elements $e_1, \dots, e_n$ and these satisfy $\mathrm{tr}_n(xe_n) = \tau \mathrm{tr}_{n-1}(x)$ for all $x \in M_{n-1}$.

We will say a bit more about the elements e_i for later purpose.

Proposition 2.5. *The e_i satisfy the following relations:*

- $e_i^2 = e_i$.
- $e_n e_m = e_m e_n$ *if* $|m-n| > 1$.
- $e_{n+1} e_n e_{n+1} = \tau e_{n+1}$.
- $e_n e_{n+1} e_n = \tau e_n$.

Proof.
- This is clear as e_i is a projection.
- This is a consequence of the fact that in the basic construction $e_N \in N'$ which here translates to $e_n \in M'_{n-2}$.
- This is a consequence of the fact that in the basic construction we have $e_N m e_N = E_N(m) e_N$. This translates to $e_{n+1} x e_{n+1} = E_{n+1}(x) e_{n+1}$ for $x \in M_n$. We have that $E_{n+1}(e_n) = \tau e_{n+1}$, because both are projections and have the same trace.
- We have that $u = \tau^{-1/2} e_n e_{n+1}$ is a partial isometry such that $u^* u = e_n$ using the previous property and clearly $uu^* \leq e_n$. Taking traces in the inequality, we get an equality and hence in fact $uu^* = e_n$. But $uu^* = \tau^{-1} e_n e_{n+1} e_n$, proving the second property

□

3. Possible values for the index

This index has some remarkable properties. For example, if one look at all numbers that can appear as an index in any M, it turns out that this set is given by

$$\mathcal{A} = \{4\cos^2(\pi/n) | n \in \{3, 4, \ldots\}\} \cup [0, \infty]$$

This set is in fact obtained for M a hyperfinite type II_1 factor. A II_1 factor is hyperfinite if there is a sequence of finite-dimensional subalgebras whose union is weakly dense, e.g. $\mathbb{R}\Sigma_n$ in the von Neumann-algebra associated to the group algebra $\mathbb{R}\Sigma_\infty$. A side remark: all hyperfinite type II_1 factors are isomorphic.

It is easy to describe the construction of subfactors for the part $\{4\cos^2(\pi/n) | n \in \{3, 4, \ldots\}\}$ of $\mathcal{A}$. These can be constructed from finite-dimensional C^*-algebras. In special cases, the index can be computed by the inclusion matrix describing the Bratteli diagram. If the matrix is Λ, then the index is $||\Lambda||^2$. Finite-dimensional matrices with non-negative integral entries are in bijection with finite bipartite graphs and those matrices with norm ≤ 2 can be classified with a A-D-E classification. In particular, picking A_{m-1} gives $||\Lambda|| = 2\cos(\pi/m)$. Note that applying the tower construction to these gives a hyperfinite $M_\infty = \bigcup M_n$ and taking "$M_{\infty-1}$" as a subfactor of M_∞ one obtains a subfactor of a hyperfinite type II_1 factor.

Kronecker in fact proves that we always have $||\Lambda||^2 \in \mathcal{A}$. I do not know whether also the part $[4, \infty] \in \mathcal{A}$ can be obtained from a finite-dimensional pair.

4. Temperley-Lieb algebras and knot invariants

Finally, we will describe the relation between the tower of a subfactor and Temperley-Lieb algebras. We will then relate this to knot invariants, in particular the Jones polynomial. This is a very interesting topic and unfortunately I won't have much time to talk about it. Maybe we could do a talk about planar algebras and the classification of subfactors at some point.

Fixing a ring R and an element $\beta \in R$, one can define the Temperley-Lieb algebras $T_n(\beta)$ for all $n \geq 2$. There are natural inclusions $T_n(\beta) \hookrightarrow T_{n+1}(\beta)$ and $T_\infty(\beta) = \lim_n T_n(\beta)$.

Definition 4.1. Let $T_n(\beta)$ be the algebra over R generated by elements $U_1, \ldots, U_{n-1}$, subject to the following relations:

- $U_i^2 = \beta U_i$.
- $U_i U_{i-1} U_i = U_i$ and $U_i U_{i+1} U_i = U_i$
- $U_i U_j = U_j U_i$ if $|i-j| \geq 2$.

There is an alternative definition in the literature, which is equivalent if τ is invertible and admits a square root:

Definition 4.2. Let $\tilde{T}_n(\tau)$ be the algebra over R generated by elements $V_1, \dots, V_{n-1}$, subject to the following relations:

- $V_i^2 = V_i$.
- $V_i V_{i-1} V_i = \tau V_i$ and $V_i V_{i+1} V_i = \tau V_i$
- $V_i V_j = V_j V_i$ if $|i - j| \geq 2$.

The algebra $\tilde{T}_n(\tau)$ is isomorphic to $T_n(\beta)$ with $\beta = \tau^{-1/2}$ by sending V_i to $\tau^{1/2} U_i$. It is more suited to dealing with subfactors, because the relations in this alternative definition are closer to the relations one gets in the tower.

The relation with knots is made through braids. There is a well-known presentation of the braid group given as follows:

Definition 4.3. The braid group on n strands B_n is generated by elements $\sigma_1, \dots, \sigma_{n-1}$ subject to the relations:

- $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$,
- $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i - j| \geq 2$.

We claim that under certain mild conditions a Temperley-Lieb algebra $\tilde{T}_n(\tau)$ gives one a representation of the braid group B_n.

Proposition 4.4. *If q is invertible, has a square root and satisfies $q + q^{-1} + 2 = \tau^{-1}$, then the following map gives a representation B_n in $\tilde{T}_n(\tau)$:*

$$\pi : \sigma_i \mapsto q^{1/2}((q + 1)V_i - 1)$$

Proof. Let's try sending σ_i to $a(bV_i - 1)$. We will find that there are certain conditions on $a, b \in R$ such that this gives a representation of the braid group. The only thing we need to do is to make the two relations of the braid group hold. The second always holds because in a Temperley-Lieb algebra $V_i V_j = V_j V_i$ if $|i - j| \geq 2$. So let's look at the first relation:

$$\pi(\sigma_i)\pi(\sigma_{i+1})\pi(\sigma_i) = a^3(b^3 V_i V_{i+1} V_i - b^2(V_i^2 + V_i V_{i+1} + V_{i+1} V_i) + b(2V_i + V_{i+1}) - 1)$$

Using the relations of a Temperley-Lieb algebra, this simplifies to

$$a^3(-b^2(V_i V_{i+1} + V_{i+1} V_i) + (2b - b^2 + b^3\tau)V_i + bV_{i+1} - 1)$$

This should be equal to $\pi(\sigma_{i+1})\pi(\sigma_i)\pi(\sigma_{i+1})$, which by symmetry is equal to:

$$a^3(-b^2(V_{i+1} V_i + V_i V_{i+1}) + (2b - b^2 + b^3\tau)V_{i+1} + bV_i - 1)$$

So the only condition we derive from this is that we must have that b satisfies $2b - b^2 + b^3\tau = b$ or equivalently that $1 - b + b^2\tau = 0$. If we write $b = q + 1$, this equation is equivalent to $-q + (q + 1)^2\tau = 0$, which can be rewritten as

$$q + q^{-1} + 2 = \tau^{-1}$$

Our choice of a is unconstrained, except that the a should be invertible. This holds for $a = q^{1/2}$, which will turn out to be a useful choice of normalization later on. □

Representations of braids can given invariants of knots. The idea is that a braid can be closed to give a knot. If $b \in B_n$ is a braid we denote this knot by $b^{(n)}$. Then Markov and Alexander have proven the following theorem:

Theorem 4.5. *Every knot is obtained as the closure of a braid and two braids give the same knot if and only if they are related by a finite sequence of the following two moves:*

- *Replace cbc^{-1} by b in B_n or vice versa.*
- *Replace $b^{(n)}$ by $b^{(n+1)}(\sigma_n^{(n+1)})^{\pm 1}$ or vice versa.*

Thus, we get a knot invariant if we find a set of functions $f_n : B_n \to X$ which are constant on conjugacy classes and satisfy $f_n(b^{(n)}) = f_{n+1}(b^{(n+1)}(\sigma_n^{(n+1)})^{\pm 1})$.

The first condition is automatically satisfied if we take the trace of a representation of B_n. We say that a series of Temperley-Lieb algebras $\tilde{T}_n(\tau)$ has a set of compatible traces $\mathrm{tr}_n : \tilde{T}_n(\tau) \to R$

if we have traces satisfy the equations $\mathrm{tr}_n(xV_n) = \tau \mathrm{tr}_{n-1}(x)$ for $x \in \tilde{T}_{n-1}(\tau)$. This induces a trace on $\tilde{T}_\infty(\tau)$.

Proposition 4.6. *If we have a series of Temperley-Lieb algebras with compatibles traces, then the functions $f_n : B_n \to R$ given by*

$$f_n(b) = (-(q^{1/2} + q^{-1/2}))^{n-1} \mathrm{tr}_n(\pi_n(b))$$

are constant on conjugacy classes and satisfy $f_n(b^{(n)}) = f_{n+1}(b^{(n+1)}(\sigma_n^{(n+1)})^{\pm 1})$, hence give invariants of knots.

Proof. That the f_n are constant on conjugacy classes is a simply consequence of the properties of a trace. The second property is simply a matter of checking, using the compatibility of the traces $\mathrm{tr} + n$ and tr_{n+1}. □

The reason for picking $a = q^{1/2}$ is to make the formula for the f_n as symmetric as possible.

Corollary 4.7. *If $N \subset M$ is a subfactor of type II_1 factors of finite index, then the tower $\ldots \hookrightarrow M_i \hookrightarrow M_{i+1} \hookrightarrow \ldots$ gives one an algebra $\bigcup M_i$ which contains the subalgebra generated by $e_1, e_2, \ldots$. This subalgebra is a Temperley-Lieb algebra for $\tau = [M : N]^{-1}$ with trace coming from a series of compatible traces. Hence we obtain an invariant of knots.*

In fact, one can try to see q as a formal parameter. The result is the so-called Jones polynomial, a powerful invariant of knots which can detect chirality. In fact, one can derive a simple algorithm to calculate these invariants using Skein relations and nowadays the Jones polynomial is usually defined using this algorithm.

References

[1] Jones, V., *Index for subfactors.*
[2] Jones, V. & Sunder, V.S., *Introduction to subfactors, LMS Lecture Note Series 234.*