THE PREDUAL OF A VON NEUMANN ALGEBRA.

A.P.M. KUPERS

Contents

1. The predual 1
2. The case $B(\mathcal{H})$: trace-class and Hilbert-Schmidt operators 2
3. Standard forms for II_1 factors 5

We will always assume that our Hilbert spaces are separable.

1. The predual

Let's fix some notation: if $\mathcal{B}$ is a Banach space, then $\mathcal{B}^*$ is the vector space of norm-continuous linear functionals. This has two possible topologies: the operator norm topology $(\mathcal{B})^*_n$ and the weak-* topology $(\mathcal{B})^*_w$. In general we will look at the latter.

Let $\mathcal{A}$ be a von Neumann algebra. The goal of this talk will be naturally associate to a pair $(\mathcal{A}, \mathcal{A} \hookrightarrow B(\mathcal{H}))$ a Banach space $\mathcal{A}_*$ such that $\mathcal{A} = (\mathcal{A}_*)^*_w$, where the former is considered as having the ultraweak topology as a subspace of $B(\mathcal{H})$.

We claim that $\mathcal{A}_*$ can be given by the space of ultraweakly continuous functionals. We will show that this is a Banach space and that $(\mathcal{A}_*)^*_w \cong \mathcal{A}$. The Banach space $\mathcal{A}_*$ will be called the predual. To get a better feeling for $\mathcal{A}_*$, let $\mathcal{A}^*$ be the vector space of norm continuous linear functionals $\phi : \mathcal{A} \to \mathbb{C}$ (with induced norm). Because norm convergence implies ultraweak convergence, it follows that $\mathcal{A}_*$ is a subspace of $\mathcal{A}^*$.

We will reduce the proof to the case that $\mathcal{A} = B(\mathcal{H})$ using the following results. Recall that Hahn-Banach says that if one has a linear functional ϕ on a subspace of a vector space dominated by a sublinear functional $\mathcal{N}$ (think of it as a norm), then one can extend ϕ to the entire space in a way that it is still dominated by $\mathcal{N}$.

Proposition 1.1. *If $x \in B(\mathcal{H})$ satisfies the $\phi(x) = 0$ for each ultraweakly continuous ϕ with $\phi(\mathcal{A}) = 0$, then $x \in \mathcal{A}$.*

Proof. This works for V any ultraweakly closed subspace of $B(\mathcal{H})$. If $x \notin V$, then apply Hahn-Banach or one of its corollaries (e.g. Conway IV.3.15) to the linear functional ϕ defined by $\phi(x) = 1$ and $\phi(v) = 0$ for all $v \in V$. This is continuous with respect to the restriction of the ultraweak topology on $V + \mathbb{C}x$, hence extends to a ultraweakly continuous functional ϕ on $B(\mathcal{H})$ with the property that $\phi(x) = 1$. □

We prove the reduction to the case $B(\mathcal{H})$ in the next section. However, we show that the embedding in $B(\mathcal{H})$ is not essential for the theorem, by given an intrinsic definition of the predual.

Theorem 1.2. *Suppose that $B(\mathcal{H})_{uw} \cong (\mathcal{B})^*_w$. Then $\mathcal{A}_{uw} \cong (\mathcal{B}/\mathcal{A}^\perp)^*_w$ and furthermore $\mathcal{B}/\mathcal{A}^\perp$ can be identified as a vector space with the subspace $\mathcal{A}_*$ of ultraweakly continuous linear functionals on $\mathcal{A}$.*

Date: March 20, 2011.

Proof. We first identify $\mathcal{A}_{uw}$ with $(\mathcal{B}/\mathcal{A}^\perp)^*_w$. By definition there is an continuous embedding $\mathcal{A}_{uw} \to B(\mathcal{H})_{uw} = \mathcal{B}^*_w$. Because $\mathcal{A}$ is ultraweak closed, it is weak-* closed in $B(\mathcal{H}) = (\mathcal{B})^*_w$. This implies that there is a homeomorphism between the subspace of the norm-continuous functionals vanishing on $\mathcal{A}$ with the subspace topology of the weak-* one, and $(\mathcal{B}/\mathcal{A}^\perp)^*_w$. This can be seen as follows: the embedding $\mathcal{A} \to (\mathcal{B}/\mathcal{A}^\perp)^*_w$ is clearly continuous, open and injective, and surjective using the previous proposition.

Next we identify the vectorspace $(\mathcal{B}/\mathcal{A}^\perp)^*$ with the space of ultraweakly continuous functionals on $\mathcal{A}$. This means that we must identify the $\mathcal{B}/\mathcal{A}^\perp$ with the weak-* continuous linear functionals $\mathcal{A}_*$ in the double dual $(\mathcal{B}/\mathcal{A}^\perp)^{**} = (\mathcal{A}_{uw})^*$. See section V.1 of Conway: clearly the elements of $\mathcal{B}/\mathcal{A}^\perp$ are weak-* continuous considered as elements of $(\mathcal{A}_{uw})^*$. For the converse, one notes that a weak-* continous functional is bounded in absolute value by a finite sum of seminorms $i_v = b \mapsto |b(v)|$ for $v \in \mathcal{B}/\mathcal{A}^\perp$ and inductively use this to write the functional as a linear combination of the i_v's. □

2. The case $B(\mathcal{H})$: trace-class and Hilbert-Schmidt operators

In this section we will see that the predual of $B(\mathcal{H})$ is given by the Banach space $\mathcal{T}(\mathcal{H})$ of traceclass operators with the trace-norm $|-|_1$.

Definition 2.1. Let $a \in B(\mathcal{H})$ and let (ξ_i) be an orthonormal basis. Then a is said to be of trace class if the following sum is finite

$$|a|_1 = \sum_i \langle |a|\xi_i, \xi_i\rangle$$

The set of trace-class operators is denoted by $\mathcal{T}(\mathcal{H})$.

To show that this is well-defined, one proves that the sum is independent of the choice of orthonormal basis.

Lemma 2.2. *$|a|_1$ is independent of the choice of orthonormal basis.*

Proof. Note that $|a| = \sqrt{|a|}\sqrt{|a|}$ and write

$$\begin{aligned}\sum_i \langle |a|\xi_i, \xi_i\rangle &= \sum_i \langle \sqrt{|a|}\xi_i, \sqrt{|a|}\xi_i\rangle \\ &= \sum_i \sum_j |\langle \sqrt{|a|}\xi_i, \eta_i\rangle|^2 \\ &= \sum_j \sum_i |\langle \sqrt{|a|}\eta_i, \xi_i\rangle|^2 \\ &= \sum_j \langle |a|\eta_j, \eta_j\rangle\end{aligned}$$

where we were allowed to interchange the sum because all terms are positive. □

We now discuss several useful properties of the trace-class operators.

Proposition 2.3. *$\mathcal{T}(\mathcal{H})$ is an ideal which is closed under $*$. The map $|-|_1$ is a complete norm on $\mathcal{T}(\mathcal{H})$ satisfying $||a|| \leq |a|_1$, making it into a Banach space. The finite rank operators are dense and hence $\mathcal{T}(\mathcal{H})$ is in fact a sub-ideal of the compact operators $\mathcal{K}(\mathcal{H})$.*

Proof. To show that it is an ideal in $B(\mathcal{H})$, use that $|xa|_1 \leq ||x|||a|_1$ for $x \in B(\mathcal{H})$ and similarly $|ax|_1 \leq ||x|||a|_1$. That it is closed under $*$ is clear, as $|a| = |a^*|$.

To show that $||a|| \leq |a|_1$ one remarks that $||a|| \leq ||\sqrt{|a|}||^2 \leq |a|_1$. The first inequality follows from the functional calculus. For the second inequality, note that $||\sqrt{a}||^2 = \sup_{||\xi||=1}\langle |a|\xi, \xi\rangle$ and hence we pick any ξ of norm 1 for which this supremum is almost obtained as the first vector of an orthonormal basis to bound $|a|_1 \geq ||\sqrt{|a|}||^2 - \epsilon$ for all $\epsilon > 0$. Taking ϵ to zero then proves the statement.

To show that $|-|_1$ is a complete norm, one notes that if a_n is $|-|_1$-Cauchy, it is $||-||$-Cauchy to get a candidate a. We need to show that a is trace-class and $|a-a_n|_1 \to 0$. However, the second statement implies the first since $|a|_1 \leq |a_n|_1 + |a-a_n|_1$. So let's prove the second: if $||a-a_n|| \to 0$, then in particular $|a_m - a_n|\xi_i \to |a - a_n|\xi_i$ for all vectors ξ_i of length 1. This implies that if $|a_m - a_n|_1 = \sum \langle |a_m - a_n|\xi_i, \xi_i\rangle \leq \epsilon$ for m, n large enough, then $|a - a_n|_1 \leq \epsilon$ as well.

Finally, note that any $a \in \mathcal{T}(\mathcal{H})$ is the $|-|_1$-limit of ap_n, where p_n is the projection on the first n basis vectors ξ_i for $1 \leq i \leq n$. All ap_n are of finite rank, hence these are dense. Because $|-|_1$ dominates $||-||$, $\mathcal{T}(\mathcal{H})$ lies in the norm-closure of the finite-rank operators, which are the compact operators. □

Next we define the trace.

Lemma 2.4. *If a is a trace-class operator, then the following sum converges absolutely*

$$\mathrm{Tr}(a) = \sum_i \langle a\xi_i, \xi_i\rangle$$

It satisfies $\mathrm{Tr}(a) \leq |a|_1$ *and* $\mathrm{Tr}(xa) \leq ||x||\mathrm{Tr}(a)$ *for each* $x \in B(\mathcal{H})$.

Proof. It suffices to prove the second statement. We will only prove the first part, as the second is then easy. To prove this, write $a = u\sqrt{|a|}\sqrt{|a|}$ using the polar decomposition and consider the following:

$$\begin{aligned}
0 &\leq ||(\sqrt{|a|} - \lambda\sqrt{|a|}u^*)\xi_i||^2 \\
&= ||\sqrt{|a|}\xi_i||^2 - 2\mathrm{Re}\left(\lambda\langle\sqrt{|a|}\xi_i, \sqrt{|a|}u^*\xi_i\rangle\right) + |\lambda|^2||\sqrt{|a|}u^*\xi_i||^2 \\
&= \langle |a|\xi_i, \xi_i\rangle - 2\mathrm{Re}\left(\lambda\langle a\xi_i, \xi_i\rangle\right) + |\lambda|^2\langle |a|u^*\xi_i, u^*\xi_i\rangle
\end{aligned}$$

Now write $\lambda = e^{i\phi}$, where the phase is chosen such that $\mathrm{Re}\left(e^{i\phi}\langle a\xi_i, \xi_i\rangle\right) = |\langle a\xi_i, \xi_i\rangle|$. Then we get:

$$2|\langle a\xi_i, \xi_i\rangle| \leq \langle |a|\xi_i, \xi_i\rangle + \langle |a|u^*\xi_i, u^*\xi_i\rangle$$

Now note that the sequences (ξ_i) and $(u\xi_i)$ both orthonormal bases. Thus summing over i we obtain

$$\mathrm{Tr}(a) \leq |a|_1$$

□

We will now prove that there is an isomorphism of Banach spaces between $\mathcal{T}(\mathcal{H})^*$ with the operator norm topology and $B(\mathcal{H})$ with the norm topology. An important role will be played by the map $\phi : B(\mathcal{H}) \to \mathcal{T}(\mathcal{H})^*$ given by

$$\phi_x(a) = \mathrm{Tr}(xa)$$

Theorem 2.5. *We have that* $(\mathcal{T}(\mathcal{H}))^*_n \cong B(H)_n$.

Proof. By the previous proposition, we know that each ϕ_x is norm continuous. Thus, only the following statement is left: if $\lambda : \mathcal{T}(\mathcal{H}) \to \mathbb{C}$ is a linear functional which is bounded with respect to $|-|_1$ then there is a $x \in B(\mathcal{H})$ such that $\lambda(a) = \phi_x(a)$ and $||\lambda|| = ||x||$.

Note that we have a sesquilinear form on $B(\mathcal{H})$:

$$\langle \xi, \eta\rangle_\lambda = \lambda(v \mapsto \langle v, \xi\rangle\eta)$$

By Riesz representability, $\langle \xi, \eta\rangle_\lambda = \langle x\xi, \eta\rangle$ for some $x \in B(\mathcal{H})$ with

$$||x|| = \sup_{||\xi||=||\eta||=1} |\langle \xi, \eta\rangle_\lambda| = \sup_{||\xi||=||\eta||=1} |\lambda(v \mapsto \langle v, \xi\rangle\eta)| = ||\lambda||$$

where only the last equality is not obvious. Clearly the left hand side is less than or equal to the right hand side, because we should evaluate on more trace-class operators to get the usual definition. However, because $|\lambda(-)|$ is continuous it suffices to check it on the dense subspace of finite rank operators. Linearity then implies we can restrict to trace-class operators of the form $v \mapsto \langle v, \xi\rangle\eta$ if we want to show that the left hand side is greater than or equal to the right hand side.

To show that $\lambda(h) = \phi_x(h)$, it suffices to note that both are continuous with respect to $|-|_1$ and coincide on finite rank operators, thus must be equal. □

Of course, we are more interested in the weak-* and ultraweak topologies. We claim that the same statement holds with these.

To prove this we will need to introduce the Hilbert-Schmidt operators. The idea is that every trace-class operator splits as a product of Hilbert-Schmidt operators and Hilbert-Schmidt operators sends an orthonormal basis to a ℓ^2-convergent sequence of vectors, hence allowing us to make the connection with the ultraweak topology. One should think of a Hilbert-Schmidt operator as a square root of a trace-class operator.

Definition 2.6. A bounded operator $b \in B(\mathcal{H})$ is said to be Hilbert-Schmidt if b^*b is trace-class. We let $\mathcal{HS}(\mathcal{H})$ denote the set of Hilbert-Schmidt operators.

We will need the following lemma's, which are elementary, but very useful for calculations.

Lemma 2.7. *Every trace-class operator is Hilbert-Schmidt and in fact every trace-class operator is a product of two Hilbert-Schmidt operators. Every ℓ^2-convergent sequence (η_i) of vectors is the image of a basis (ξ_i) under a Hilbert-Schmidt operator.*

Proof. If a is a trace-class operator, then a^*a is trace-class as well, because $\mathcal{T}(\mathcal{H})$ is an ideal.

Let $a = u|a|$ be the polar decomposition of a trace-class operator. Then a can be written as the product of trace-class operators $u\sqrt{|a|}$ and $\sqrt{|a|}$.

For the second assertion, note that $\sum_i \langle -, \xi_i \rangle \eta_i$ is a Hilbert-Schmidt operator. □

Lemma 2.8. *For $b, c \in \mathcal{HS}(\mathcal{H})$, $bc \in \mathcal{T}(\mathcal{H})$ and $\mathrm{Tr}(bc) = \mathrm{Tr}(cb)$.*

Proof. For the first statement, we note that in our proof that the trace is well-defined, we have shown that $\mathrm{Tr}(bc) \leq \sum_i ||b\xi_i||^2 + ||c\xi_i||^2$. It is easy to check that $b \in \mathcal{HS}(\mathcal{H})$ is equivalent to $\sum_i ||b\xi_i||^2 < \infty$.

The second statement one proves that $| - |_2$ given by $|b|_2 = \sqrt{\sum_i ||b\xi_i||^2}$ is a complete norm for the Hilbert-Schmidt operators such that the finite-rank operators are dense and Tr is continuous in both components. The equation then follows by continuity from the corresponding equation for finite-rank operators, where it is the usual invariance under cyclic permutation of the trace of a matrix. □

We can now prove the result about the ultraweak topology on $B(\mathcal{H})$ and the weak-* topology on $\mathcal{T}(\mathcal{H})$.

Theorem 2.9. *We have that $(\mathcal{T}(\mathcal{H}))^*_w \cong B(\mathcal{H})_{uw}$.*

Proof. The only thing that is left to do is prove that ultraweak topology on $B(\mathcal{H})$ comes from the seminorms $x \mapsto |\mathrm{Tr}(xa)|$ for $a \in \mathcal{T}(\mathcal{H})$. The ultraweak topology is obtained by looking at seminorms $| - |_{\xi,\eta}$ for sequences $(\xi_i), (\eta_i) \in \ell^2(\mathcal{H})$ given by

$$| - |_{\xi,\eta} = \sum_{i=1}^{\infty} |\langle x\xi_i, \eta_i \rangle|$$

To relate this to the seminorm $|x|_h = |\mathrm{Tr}(xh)| = |\sum_{i=1}^{\infty} \langle xh\zeta_i, \zeta_i \rangle|$ with $(\zeta_i)_i$ an orthonormal basis. We note that $h = h_1 h_2^*$ for h_1, h_2 Hilbert-Schmidt operators. Then we have that $\mathrm{Tr}(xh_1h_2^*) = \mathrm{Tr}(h_2^* x h_1)$. Send h_2 to the other side of the inner product and note that both elements $(\xi_i), (\eta_i) \in \ell^2(\mathcal{H})$ can be obtained by applying a Hilbert-Schmidt operator to the fixed set of orthonormal vectors $(\zeta_i)_i$. By including appropriate phases, we can also get the absolute values. □

Finally, we show that $\mathcal{T}(\mathcal{H})$ can be identified with the ultraweakly continuous functionals on $B(\mathcal{H})$. To do, we take the transpose of ϕ. This is a map $\psi : \mathcal{T}(\mathcal{H}) \to B(\mathcal{H})^*$ given by $\psi_a(x) = \mathrm{Tr}(xa)$, where the latter has the operator norm topology. We will show that the image of this map is the space of functionals which are ultraweakly continuous in addition to being norm continuous.

Theorem 2.10. *Each ψ_a is ultraweakly continuous and the map $a \mapsto \psi_a$ is continuous. To be precise, $||\psi_a|| = |a|_1$.*

Conversely, if ω is an ultraweakly continuous linear functional on $B(\mathcal{H})$, then there is a trace-class operator a such that $\omega = \psi_a$ and $||\omega|| = |a|_1$.

Proof. For the statement about ultraweak continuity, use statement that a functional ω is ultraweakly continuous if and only if it is of the form

$$\omega(x) = \sum_i \langle x\xi_i, \eta \rangle$$

for $(\xi_i), (\eta_i)$ ℓ^2-convergent sequences of vectors. By the same reasoning as in the previous theorem, this exactly means that ω of the form $\mathrm{Tr}(a)$.

To prove that $||\psi_a|| = |a|_1$, we use that because $|a|$ is compact and positive, there exists a orthonormal basis ξ_i of $\mathcal{H}$ consisting of eigenvectors for $|a|$ with eigenvalues $\lambda_i \geq 0$. A simple estimate shows using this basis shows that $\psi_a(x) \leq ||x|||a|_1$ and entering $x = u^*$ for the polar decomposition $a = u|a|$ shows equality.

For the converse statement one uses similar reasoning as in the proposition. If ω is ultraweakly continuous then it can be written as $\omega(x) = \mathrm{Tr}(a_2^* x a_1)$ for $a_1, a_2 \in \mathcal{HS}(\mathcal{H})$. Use the cyclic property of the trace to show that $\omega(x) = \psi_{a_1 a_2}(x)$. □

The conclusion is that $\mathcal{T}(\mathcal{H})$ is the norm topology from $| - |_1$ is isometrically isomorphic to the subspace of $B(\mathcal{H})^*$ of the ultraweakly continuous functionals with the subspace topology from the operator norm topology. Furthermore, the weak-* topology on $B(\mathcal{H})$ is exactly the ultraweak topology.

3. Standard forms for II$_1$ factors

$\mathcal{A}$ is a type II$_1$ factor if it is an infinite dimensional factor and has a ultraweakly continuous trace tr, i.e. $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ and $\mathrm{tr}(a^*a) \leq 0$. There is a unique normalized trace: $\mathrm{tr}(1) = 1$.

We want to apply the GNS-construction to this. To do this, one lets $L^2(\mathcal{A}, \mathrm{tr})$ be the Hilbert space completion of the vector space given by $\mathcal{A}/\{a \in \mathcal{A} | \mathrm{tr}(a^*a) = 0\}$ with inner product induced by $a, b \mapsto \mathrm{tr}(b^*a)$.

The action of $\mathcal{A}$ on $L^2(\mathcal{A}, \mathrm{tr})$ induced by multiplication in the algebra turns out to be continuous and is called the standard form of $\mathcal{A}$.

To link with the previous section, we note that there exist spaces $L^p(\mathcal{A}, \mathrm{tr})$ for $p \in [1, \infty]$ with norms induced by $\mathrm{tr}(| - |^p)^{1/p}$. If $p = \infty$, one obtain $\mathcal{A}$, for $p = 1$, one gets the predual $\mathcal{A}_*$. So, one should think of $L^2(\mathcal{A}, \mathrm{tr})$ as the Hilbert-Schmidt operators associated to the predual $\mathcal{A}_*$.