Quasi-Triangular Hopf Algebras, and the Quantum Double Construction

WISM545 Quantum Integrable Models - Presentation

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Abstract

In this paper, we will address quasi-triangular Hopf algebras and the quantum double construction. First, we will recapture definitions concerning Hopf algebras. Next, quasi-triangularity is introduced. We will start with some axioms descriping the algebra, after which we will take a physical approach and derive a key equation. Finally, we will see that any quasi-triangular bialgebra produces solutions to the Yang-Baxter equation. The last chapter deals with the quantum double construction. We will discover that any quantum double construction allows us to find universal $\mathcal{R}$-matrices that satisfy the Yang-Baxter equation.

1 Recapturing Hopf Algebras

Before we start exploring quasi-triangularity, we will recapture some definitions that we have seen in the previous class (14 November) where they were introduced by Alexandros Aerakis. We will define the following:

i. we have an algebra ($\mathcal{A}$, m, η; Δ, ϵ), where $\mathcal{A}$ is the associative algebra on the field $\mathbb{C}$ with:

  - the multiplication m: $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$,
  - the unit η: $\mathbb{C} \to \mathcal{A}$,
  - the co-multiplication Δ: $\mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$

- and the co-unit ϵ: $\mathcal{A} \to \mathbb{C}$.

ii. An algebra is called a bialgebra if the compatibility conditions hold, i.e.:

  - $\Delta(ab) = \Delta(a)\Delta(b)$,
  - and $\epsilon(ab) = \epsilon(a)\epsilon(b)$.

iii. A Hopf algebra is a bialgebra with an antipode γ: $\mathcal{A} \to \mathcal{A}$ which is an antihomomorphism, i.e. $\gamma(ab) = \gamma(b)\gamma(a)$ satisfying:

$$m\left(\gamma \otimes \mathbb{I}\right)\Delta(a) = m\left(\mathbb{I} \otimes \gamma\right)\Delta(a) = \epsilon(a)\eta, \forall a \in \mathcal{A}.$$

The previous conditions tell us how a Hopf algebra is defined. Next, we will look at commutativity and co-commutativity.

iv. If the multiplication m is commutative, we call the algebra commutative. If m is not commutative, the agebra is also non-commutative. Commutativity means that the following diagram holds:

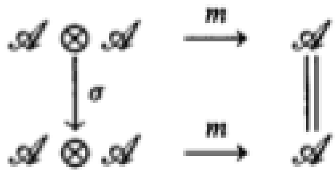

Diagram 1: Commutativity

with the permutation map:

$$\sigma_{\mathcal{A},\mathcal{A}}: \quad \begin{array}{l} \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \\ a \otimes b \to b \otimes a. \end{array}$$

So commutativity means that we have:

$$ab \equiv m\left(a \otimes b\right) = m\left(\sigma_{\mathcal{A},\mathcal{A}}\left(a \otimes b\right)\right) = m\left(b \otimes a\right) \equiv ba, \forall a, b \in \mathcal{A}.$$

v. Just like the multiplication, the co-multiplication Δ may or may not be commutative. An algebra is co-commutative if the following holds:

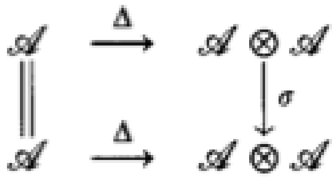

Diagram 2: Co-commutativity

which is equal to saying:

$$\Delta(a) = \sigma_{\mathcal{A},\mathcal{A}}\Delta(a) \equiv \Delta^{op}(a), \forall a \in \mathcal{A}. \tag{1}$$

We are primarily interested in Hopf algebras that are neither commutive nor co-commutative. Such algebras are referred to as quantum groups. This means for the just listed defitions, that we want i-iii to hold, and iv and v not to hold.

Considering the diagrams shown here, and the ones we have seen in previous classes, we observe that if we reverse the arrows and interchange m with Δ and η with ϵ we retrieve the same diagrams. The antipode γ remains unchanged under such an operation and therefore, its existence is higly non-trivial. This is why we are more interested in Hopf algebras than in bi-algebras.[1]

2 Quasi-Triangular Hopf Algebras

In this chapter, we will explore the concept of quasi-triangular Hopf algebras, also referred to as braided Hopf algebras. We will see that there are some extra properties, in addition to what we have discussed in the previous chapter, that define such algebras. These properties will be explored in the first section. In the second section, we will take a physical point of view and actually derive one of these properties from the Yang-Baxter equation of the six vertex model. Finally, we will address the question of why we are interested in such algebras. We will prove that all quasi-triangular bialgebras produce solutions to the Yang-Baxter equation.

2.1 Defininining Quasi-Triangularity

In this section we will define quasi-triangularity and we will show that it satisfies the Yang-Baxter equation. We start with the following axioms.

Axiom 1 Let $\mathcal{A}$ be a Hopf algebra. We call it quasi-cocommutative if there exists an invertible element $\mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$, with $\mathcal{R}$ being the universal $\mathcal{R}$-matrix, such that:

$$\Delta^{op}(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1}, \qquad \forall a \in \mathcal{A}, \tag{2}$$

where $\Delta^{op} = \sigma_{\mathcal{A},\mathcal{A}} \circ \Delta$, and $\sigma_{\mathcal{A},\mathcal{A}}$ are defined in the first chapter.

[1] For this section I have used [1], pp. 184-186.

Axiom 2 We call a quasi-cocommutative Hopf algebra a quasi-triangular Hopf algebra, if the following is satisfied:

$$\begin{aligned} (\mathbb{I} \otimes \Delta)\,\mathcal{R} &= \mathcal{R}_{13}\mathcal{R}_{12}, &(3)\\ (\Delta \otimes \mathbb{I})\,\mathcal{R} &= \mathcal{R}_{13}\mathcal{R}_{23}, &(4)\end{aligned}$$

with $\mathcal{R}$ being the universal $\mathcal{R}$-matrix.

We write $\mathcal{R} = \sum_i A_i \otimes B_i$ such that:

$$\begin{aligned} \mathcal{R}_{12} &= \sum_i A_i \otimes B_i \otimes \mathbb{I}, \\ \mathcal{R}_{13} &= \sum_i A_i \otimes \mathbb{I} \otimes B_i, \\ \mathcal{R}_{23} &= \sum_i \mathbb{I} \otimes A_i \otimes B_i. \end{aligned}$$

These two axioms have the following consequence.

Consequence Let $\mathcal{A}$ be a quasi-triangular Hopf algebra, then the universal $\mathcal{R}$-matrix $\mathcal{R}$ satisfies the Yang-Baxter equation:

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12} \tag{5}$$

and we have:

$$(\epsilon \otimes \mathbb{I})\,\mathcal{R} = 1 = (\mathbb{I} \otimes \epsilon)\,\mathcal{R}. \tag{6}$$

Moreover, if $\mathcal{A}$ has an invertible antipode γ^{-1}, we also have:

$$(\gamma \otimes \mathbb{I})\,\mathcal{R} = \left(\mathbb{I} \otimes \gamma^{-1}\right)\mathcal{R} = \mathcal{R}^{-1}, \tag{7}$$

and

$$(\gamma \otimes \gamma)\,\mathcal{R} = \mathcal{R}. \tag{8}$$

Equation(5) is the main equation in this axiom. Equations(6-8) are listed for the purpose of completion. Equation(5) can be derived from equations(2-4). We start with the following:

$$
\begin{aligned}
\left[(\sigma \circ \Delta) \otimes \mathbb{I}\right] \mathcal{R} &= \sum_i \left(\Delta^{op} \otimes \mathbb{I}\right) \left(A_i \otimes B_i\right) \\
&= \sum_i \Delta^{op} \left(A_i\right) \otimes B_i \\
&= \sum_i \mathcal{R}_{12} \Delta \left(A_i\right) \mathcal{R}_{12}^{-1} \otimes B_i \\
&= \mathcal{R}_{12} \left(\sum_i \Delta \left(A_i\right) \otimes B_i \right) \mathcal{R}_{12}^{-1} \\
&= \mathcal{R}_{12} \left[\left(\Delta \otimes \mathbb{I}\right) \mathcal{R}\right] \mathcal{R}_{12}^{-1} \\
&= \mathcal{R}_{12} \mathcal{R}_{13} \mathcal{R}_{23} \mathcal{R}_{12}^{-1}. \qquad (9)
\end{aligned}
$$

This equation also gives:

$$
\begin{aligned}
\left[(\sigma \circ \Delta) \otimes \mathbb{I}\right] \mathcal{R} &= \sigma_{12} \left(\Delta \otimes \mathbb{I}\right) \mathcal{R} \\
&= \sigma_{12} \left(\mathcal{R}_{13} \mathcal{R}_{23}\right) \\
&= \mathcal{R}_{23} \mathcal{R}_{13}. \qquad (10)
\end{aligned}
$$

Combining this with our previous result we find:

$$
\begin{aligned}
\mathcal{R}_{12} \mathcal{R}_{13} \mathcal{R}_{23} \mathcal{R}_{12}^{-1} &= \mathcal{R}_{23} \mathcal{R}_{13} \\
\mathcal{R}_{12} \mathcal{R}_{13} \mathcal{R}_{23} \mathcal{R}_{12}^{-1} \mathcal{R}_{12} &= \mathcal{R}_{23} \mathcal{R}_{13} \mathcal{R}_{12} \\
\mathcal{R}_{12} \mathcal{R}_{13} \mathcal{R}_{23} &= \mathcal{R}_{23} \mathcal{R}_{13} \mathcal{R}_{12}. \qquad (11)
\end{aligned}
$$

We have thus shown that quasi-triangular Hopf algebras as defined in axioms 1 and 2 indeed satisfy the Yang-Baxter equation.[2]

2.2 Derivation of $\Delta^{op}(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1}$

In the previous section we have seen that quasi-triangular Hopf algebras produce $\mathcal{R}$-matrices that satisfy the Yang-Baxter equation. In this section, we will take a closer look by investigating quasi-triangularity through the means of a familiar example. We will show that equation(2) can be derived by going back to the Yang-Baxter equation satisfied by the $\mathcal{R}$-matrix of the Yang-Baxter algebra $\mathcal{A}$ of the six-vertex model we have seen before.

[2]For this section, I have used [1], pp. 186-187, and [2], pp. 173-175.

We start with the following familiar Yang-Baxter equation:

$$\sum_{j_1,j_2,j_3} \mathcal{R}^{k_1k_2}_{j_1j_2}(u-v)\mathcal{R}^{j_1k_3}_{i_1j_3}(u)\mathcal{R}^{j_2j_3}_{i_2i_3}(v) = \sum_{j_1,j_2,j_3} \mathcal{R}^{k_2k_3}_{j_2j_3}(v)\mathcal{R}^{k_1j_3}_{j_1i_3}(u)\mathcal{R}^{j_1j_2}_{i_1i_2}(u-v). \quad (12)$$

Remembering the relation between the monodromy matrices T and the universal $\mathcal{R}$-matrices, we have the adjoint representation:

$$\left(T^j_i(u)\right)^\beta_\alpha = \mathcal{R}^{j\beta}_{i\alpha}(u) \quad (13)$$

where i, j are the indices of auxiliary space so they label the elements of our six-vertex algebra $\mathcal{A}$, and α, β are the indices of the quantum space which indicate the representation of our algebra $\mathcal{A}$. Applying equation(13) to equation(12) and surpressing the indices in quantum space allows us to write the Yang-Baxter equation as:

$$\sum_{j_1,j_2} \mathcal{R}^{k_1k_2}_{j_1j_2}(u-v)\left(T^{j_1}_{i_1}(u)\right)^{k_3}_{j_3}\left(T^{j_2}_{i_2}(v)\right)^{j_3}_{i_3} = \sum_{j_1,j_2}\left(T^{k_2}_{j_2}(v)\right)^{k_3}_{j_3}\left(T^{k_1}_{j_1}(u)\right)^{j_3}_{i_3}\mathcal{R}^{j_1j_2}_{i_1i_2}(u-v). \quad (14)$$

The $\mathcal{R}$-matrix in this $\mathcal{R}TT$-equation plays an auxiliary role. This can be seen by looking at the monodromy matrices T, which carry only auxiliary indices. We see that in equation(14) the T's only carry indices with 1 and 2 just as our $\mathcal{R}$-matrix. Therefore, we can indeed say that the $\mathcal{R}$-matrix plays an auxiliary role. This also means that $V_1 \otimes V_2$ is the auxiliary space and V_3 is the quantum space.

We are interested in reversing the roles of these spaces, because we are then able to retrieve the equation we want to find, i.e. $\Delta^{op}(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1}$. In order to do so, we use that the $\mathcal{R}$-matrix of the six-vertex model satisfies the following symmetry:

$$\mathcal{R}^{k_1k_2}_{j_1j_2}(u) = \mathcal{R}^{k_2k_1}_{j_2j_1}(u), \quad (15)$$

or $P\mathcal{R}(u)P = \mathcal{R}(u)$ where P is the permutation operator we considered before:

$$P : V_1 \otimes V_2 \to V_2 \otimes V_1.$$

Equation(15) allows us to interchange the auxiliary and the quantum spaces which, in a physical sense, boils down to the arbitrariness of the choice of the vertical direction in the euclidean plane to be time.

Using equation(15) we can rewrite equation(12) to find:

$$\sum_{j_1,j_2,j_3} \mathcal{R}^{k_2k_1}_{j_2j_1}(u-v)\mathcal{R}^{k_3j_1}_{j_3i_1}(u)\mathcal{R}^{j_3j_2}_{i_3i_2}(v) = \sum_{j_1,j_2,j_3} \mathcal{R}^{j_3k_1}_{i_3j_1}(u)\mathcal{R}^{k_3k_2}_{j_3j_2}(v)\mathcal{R}^{j_2j_1}_{i_2i_1}(u-v). \quad (16)$$

The attentive reader will notice that we have interchanged the first two $\mathcal{R}$-matrices on the right hand side of the equal sign. We do this, because we want to rewrite this equation in a certain way later on. It is not clear why this is allowed. It might be a mistake in the source text. We will however continue with this equation as stated here such that we can find what we desire.

We find the following $\mathcal{R}TT = TT\mathcal{R}$-equation:

$$\sum_{j_1,j_2,j_3} \mathcal{R}^{k_2k_1}_{j_2j_1}(u-v)\left(T^{k_3}_{j_3}(u)\right)^{j_1}_{i_1}\left(T^{j_3}_{i_3}(v)\right)^{j_2}_{i_2} = \sum_{j_1,j_2,j_3}\left(T^{j_3}_{i_3}(u)\right)^{k_1}_{j_1}\left(T^{k_3}_{j_3}(v)\right)^{k_2}_{j_2}\mathcal{R}^{j_2j_1}_{i_2i_1}(u-v). \tag{17}$$

In this equation, the monodromy matrices T only carry indices with 3. So, whereas we first had the space $V_1 \otimes V_2$ auxiliary and V_3 quantum, we now have that V_3 is auxiliary and $V_1 \otimes V_2$ is quantum. We have thus succeeded in interchanging the auxiliary and the quantum space. In our new $\mathcal{R}TT = TT\mathcal{R}$-equation(17), $\mathcal{R}$ plays a quantum role.

Equation(17) can be written in a simpler form:

$$\mathcal{R}(u-v)\left(T^k_j(u)\otimes T^j_i(v)\right) = \left(T^j_i(u)\otimes T^k_j(v)\right)\mathcal{R}(u-v) \tag{18}$$

where the tensor product takes place in $V_1 \otimes V_2$ and $\mathcal{R}(u-v) \in \mathrm{End}(V_1 \otimes V_2)$, where an endomorphism is a morphism, i.e. a structure preserving map, from a mathematical object to itself.

Remembering that the co-product Δ is defined as:

$$\begin{aligned}\Delta: \quad & \mathcal{A} \to \mathcal{A}\otimes\mathcal{A} \\ & T^j_i(u) \to \sum_k T^k_i(u)\otimes T^j_k(u),\end{aligned}$$

we define the following two relations:

$$\Delta_{u,v}\left(T^k_i(w)\right) = T^j_i(u)\otimes T^k_j(v), \tag{19}$$
$$\Delta^{op}_{u,v}\left(T^k_i(w)\right) = T^k_j(u)\otimes T^j_i(v), \tag{20}$$

with w some dummy variable. One should realize that, as opposed to just swapping the monodromy matrices with their variables, taking the opposed product means reversing the monodromy matrices with respect to each other and then putting the variables u and v in. We can now rewrite equation(18) such that we find:

$$\Delta_{u,v}\left(T^k_i\right) = \mathcal{R}(u-v)\Delta^{op}\left(T^k_i\right)\mathcal{R}^{-1}(u-v). \tag{21}$$

We have thus retrieved what we set out to find.

To get a better understanding of how this works, let us consider the case where we take the braid limit, i.e. $u \to \pm\infty$, of equation(18), which will bring us to the spin-$\frac{1}{2}$ representation of $U_q(sl(2))$. We remember that the braid limit $u \to +\infty$

was taken before in the lecture given on October 24, by Daniel Medina. The $\mathcal{R}$-matrix has the following form:

$$\mathcal{R} = \begin{pmatrix} a(u) & 0 & 0 & 0 \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ 0 & 0 & 0 & a(u) \end{pmatrix}$$

with

$$a(u) = \sinh(u + i\gamma) = \frac{1}{2}\left(e^{u+i\gamma} - e^{-u-i\gamma}\right), \tag{22}$$

$$b(u) = \sinh(u) = \frac{1}{2}\left(e^{u} - e^{-u}\right), \tag{23}$$

$$c(u) = i\sin(\gamma). \tag{24}$$

Daniel showed us that the limit $u \to +\infty$ of $\mathcal{R}$ gives us the following result:[3]:

$$\mathcal{R} = \begin{pmatrix} q^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & q^{-\frac{1}{2}} & 0 & 0 \\ 0 & q^{-\frac{1}{2}}\left(q - q^{-1}\right) & q^{-\frac{1}{2}} & 0 \\ 0 & 0 & 0 & q^{\frac{1}{2}} \end{pmatrix}$$

where $q = e^{i\gamma}$ with γ some parameter related to the anisotropy. The limit $u \to -\infty$ then gives us the inverse $\mathcal{R}^{-1}$:

$$\mathcal{R}^{-1} = \begin{pmatrix} q^{-\frac{1}{2}} & 0 & 0 & 0 \\ 0 & q^{\frac{1}{2}} & -q^{\frac{1}{2}}\left(q - q^{-1}\right) & 0 \\ 0 & 0 & q^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & q^{-\frac{1}{2}} \end{pmatrix}$$

Next, we need to find the braid limit for the monodromy matrices. Remembering that they look like this:

$$T = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

with:

$$A(u) = \begin{pmatrix} a(u) & 0 \\ 0 & b(u) \end{pmatrix}$$

[3]Remember that the answer stated by Daniel was for R whereas we are considering $\mathcal{R}$. Therefore, we applied $R_{ij}^{kl} = \mathcal{R}_{ij}^{lk}$ to find the desired answer.

$$B(u) = \begin{pmatrix} 0 & 0 \\ c(u) & 0 \end{pmatrix}$$

$$C(u) = \begin{pmatrix} 0 & c(u) \\ 0 & 0 \end{pmatrix}$$

and lastly:

$$D(u) = \begin{pmatrix} b(u) & 0 \\ 0 & a(u) \end{pmatrix}$$

with $a(u)$, $b(u)$ and $c(u)$ as defined before. If we now take the limit $u \to +\infty$, we find for $A(u)$:

$$\begin{aligned} \lim_{u\to+\infty} A(u) &= \lim_{u\to+\infty} \frac{1}{2} \begin{pmatrix} e^{u+i\gamma} - e^{-u-i\gamma} & 0 \\ 0 & e^{u} - e^{-u} \end{pmatrix} & (25) \\ &= \frac{1}{2} e^{u} e^{i\gamma/2} \begin{pmatrix} e^{i\gamma/2} & 0 \\ 0 & e^{-i\gamma/2} \end{pmatrix} & (26) \\ &= \frac{1}{2} e^{u} q^{\frac{1}{2}} q^{S^z}, & (27) \end{aligned}$$

where $q = e^{i\gamma}$ as before, and $S^z = \frac{1}{2}\sigma^z$ with σ^z the familiar Pauli matrix, is the Cartan generator in the spin-$\frac{1}{2}$ representation of $SU(2)$. Similarly, for the limit $u \to -\infty$, we find:

$$\begin{aligned} \lim_{u\to-\infty} A(u) &= \lim_{u\to-\infty} \frac{1}{2} \begin{pmatrix} e^{u+i\gamma} - e^{-u-i\gamma} & 0 \\ 0 & e^{u} - e^{-u} \end{pmatrix} & (28) \\ &= -\frac{1}{2} e^{-u} e^{-i\gamma/2} \begin{pmatrix} e^{-i\gamma/2} & 0 \\ 0 & e^{+i\gamma/2} \end{pmatrix} & (29) \\ &= -\frac{1}{2} e^{-u} q^{-\frac{1}{2}} q^{-S^z}. & (30) \end{aligned}$$

The limits of $B(u)$, $C(u)$, and $D(u)$ can be found in a like fashion. Before we can take the limit of our monodromy matrices T, we need to mention a subtlety that we did not address for the braid limit of $\mathcal{R}$ as we simply stated the result. In the process of taking the limit $u \to +\infty$ of $\mathcal{R}$, we rescaled by $2e^{-i\gamma/2}e^{-u}$, so if we now take the limit $u \to +\infty$, we find:

$$T_+ = \begin{pmatrix} q^{S^z} & 0 \\ \sqrt{q}^{-1}\left(q - q^{-1}\right) S^- & q^{-S^z} \end{pmatrix}$$

Likewise, for $u \to -\infty$, the $\mathcal{R}$-matrix was rescaled by $-2e^{i\gamma/2}e^{u}$ such that we find:

$$T_- = \begin{pmatrix} q^{-S^z} & -\sqrt{q}^{-1}\left(q - q^{-1}\right) S^+ \\ 0 & q^{S^z} \end{pmatrix}$$

where q and S^z where mentioned before, and $S^{\pm} = \frac{1}{2}\left(\sigma^x \pm i\sigma^y\right)$ are the off-diagonal generators of $SU(2)$ in the spin-$\frac{1}{2}$ irreducible reptresentatation. We have that:

$$\begin{aligned}\left[S^z, S^{\pm}\right] &= \pm S^{\pm}, \\ \left[S^+, S^-\right] &= \frac{q^{2S^z} - q^{-2S^z}}{q - q^{-1}},\end{aligned}$$

which are the defining relations for the quantum group $U_q\left(sl(2)\right)$ which is some sort of deformation of the Lie algebra $sl(2)$ with $q = e^{i\gamma}$ acting as a deformation parameter.

Taking the braid limit of our co-product gives:

$$\Delta(q^{S^z}) = q^{S^z} \otimes q^{S^z}, \tag{31}$$

$$\Delta(S^{\pm}) = S^{\pm} \otimes q^{S^z} + q^{-Sz} \otimes S^{\pm}, \tag{32}$$

and the opposed co-product is:

$$\Delta^{op}(q^{S^z}) = q^{S^z} \otimes q^{S^z}, \tag{33}$$

$$\Delta^{op}(S^{\pm}) = S^{\pm} \otimes q^{-S^z} + q^{Sz} \otimes S^{\pm}. \tag{34}$$

Looking at the previous equations, one can see that these relations hold.

Putting all of this back into equation(21), we see that we find:

$$\Delta^{op}(g) = \mathcal{R}\Delta(g)\mathcal{R}^{-1} \tag{35}$$

with $g \in \{q^{\pm S^z}, S^{\pm}\}$. That this equation is satisfied can be seen by looking at a specific example. We take $g = q^{S^z}$ and write equation(35) as:

$$\Delta^{op}(q^{S^z})\mathcal{R} = \mathcal{R}\Delta(q^{S^z}). \tag{36}$$

From equation(31) and equation(33), we see that $\Delta^{op}(q^{S^z}) = \Delta(q^{S^z})$. If we now plug in our result for $\mathcal{R}$ in the braid limit, we see that this indeed yields the same on both sides as $q^{\frac{1}{2}}$ clearly commutes with q^{S^z}. The same can be done for the other cases, such that we find that equation(35) is indeed what we find.

However, equation(21) can also be viewed in a more general sense at the level of the quantum group $U_q\left(sl(2)\right)$ before we construct its specific representations. In other words, instead of perceiving $\mathcal{R}$ as a numerical matrix as in equation(35), it is now understood to be an element of $U_q\left(sl(2)\right) \otimes U_q\left(sl(2)\right)$. Therefore, we call the matrix $\mathcal{R}$ the universal $\mathcal{R}$-matrix, and its existence guarantees that the co-multiplications Δ and Δ^{op} of $U_q\left(sl(2)\right)$ are equivalent at a purely algebraic level.[4]

[4]For this section I used [1], pp. 40, 41, 45, 47, 49, 51-55.

2.3 Relevance

Finally, we need to address why quasi-triangular Hopf algebras are so interesting to us. The reason was already alluded to in the previous section, namely that quasi-triangular Hopf algebras generate solutions to the Yang-Baxter equation naturally.

We are going to prove that there exists a solution of the Yang-Baxter equation on every module of a braided bialgebra $(\mathcal{A},\ m,\ \eta;\ \Delta,\ \epsilon)$. We will define the following.

Definition Let V be a vector space over a field k. A linear automorphism c of $V \otimes V$ is said to be an $\mathcal{R}$-matrix if it is a solution of the Yang-Baxter equation

$$(c \otimes \mathbb{I}_V)\,(\mathbb{I}_V \otimes c)\,(c \otimes \mathbb{I}_V) = (\mathbb{I}_V \otimes c)\,(c \otimes \mathbb{I}_V)\,(\mathbb{I}_V \otimes c) \tag{37}$$

that holds in the automorphism group of $V \otimes V \otimes V$.

To relate this general definition to what we have seen before, we realize that the field k in the definition is what we before called field $\mathbb{C}$. Moreover, equation(37) is a general way of writing down the Yang-Baxter equation. Equation(12) can also be given the same shape:

$$(\mathcal{R}(v) \otimes \mathbb{I})\,(\mathbb{I} \otimes \mathcal{R}(u))\,(\mathcal{R}(u-v) \otimes \mathbb{I}) = (\mathbb{I} \otimes \mathcal{R}(u-v))\,(\mathcal{R}(u) \otimes \mathbb{I})\,(\mathbb{I} \otimes \mathcal{R}(v))\,.$$

We will use this theorem to prove that quasi-triangular bialgebras lead to solutions of the Yang-Baxter equation. Let V and W be two $\mathcal{A}$-modules. Due to the presence of the universal $\mathcal{R}$-matrix $\mathcal{R}$ in $\mathcal{A} \otimes \mathcal{A}$, we can build a natural isomorphism $c^{\mathcal{R}}_{V,W}$ of $\mathcal{A}$-modules between $V \otimes W$ and $W \otimes V$. An isomorphism is map such that both the map and its inverse are mophisms. This isomorpism defines the flip $\sigma_{V,W}$ between the factors V and W which is defined as before:

$$c^{\mathcal{R}}_{V,W}\,(v \otimes w) = \sigma_{V,W}\,(\mathcal{R}\,(v \otimes w)) = \sum_i B_i w \otimes A_i v, \tag{38}$$

where $v \in V$ and $w \in W$ and $\mathcal{R} = \sum_i A_i \otimes B_i$. We use equation(7) to show that $c^{\mathcal{R}}_{V,W}$ is an isomorphism with its inverse given by:

$$\left(c^{\mathcal{R}}_{V,W}\right)^{-1}(w \otimes v) = \mathcal{R}^{-1}\,(v \otimes w) = \sum_i \gamma\,(A_i)\,v \otimes B_i w = \sum_i A_i v \otimes \gamma^{-1}\,(B_i)\,w. \tag{39}$$

Notice here that the inverse of $\mathcal{R}$ is defined in the same way as we saw before in equation(7). Now these hypotheses allow us to make the following proposition.

Proposition For any triple (U, V, W) of $\mathcal{A}$-modules, we have the following relations:

$$\begin{aligned} c^{\mathcal{R}}_{U\otimes V,W} &= \left(c^{\mathcal{R}}_{U,W} \otimes \mathbb{I}_V\right)\left(\mathbb{I}_U \otimes c^{\mathcal{R}}_{V,W}\right), \\ c^{\mathcal{R}}_{U,V\otimes W} &= \left(\mathbb{I}_V \otimes c^{\mathcal{R}}_{U,W}\right)\left(c^{\mathcal{R}}_{U,V} \otimes \mathbb{I}_W\right), \\ \left(c^{\mathcal{R}}_{V,W} \otimes \mathbb{I}_U\right)\left(\mathbb{I}_V \otimes c^{\mathcal{R}}_{U,W}\right)\left(c^{\mathcal{R}}_{U,V} \otimes \mathbb{I}_W\right) &= \left(\mathbb{I}_W \otimes c^{\mathcal{R}}_{U,V}\right)\left(c^{\mathcal{R}}_{U,W} \otimes \mathbb{I}_V\right)\left(\mathbb{I}_U \otimes c^{\mathcal{R}}_{V,W}\right). \end{aligned}$$

The last equation is identified as the Yang-Baxter equation. These relations can be proven by using equation(38). We have thus shown our previous assumption that any quasi-triangular bialgebra generates solutions to the Yang-Baxter equation. [5]

3 Quantum Double Construction

We will see in this chapter that the quantum double construction is a very powerful tool to produce $\mathcal{R}$-matrices that satisfy the Yang-Baxter equation. In this section, I will follow what I have presented in class. Due to time constraints, this is not very extensive as it will only provide the reader with a rough understanding of how the construction works. Nevertheless, as my primary intentions were to delve deeper into this topic, I have prepared a very elaborate evaluation of the quantum double construction including its application to $U(sl(2))$ which can be found in appendix A.

3.1 Construction of Quantum Double

To construct the quantum double $\mathcal{D}(\mathcal{A}, \mathcal{A}^*)$, we start with a finite-dimensional algebra $\mathcal{A}$ and its dual $\mathcal{A}^*$ were a dual is defined such that:

$$\begin{array}{rccccc} \mathcal{A}: & m & \eta & \Delta & \epsilon & \gamma \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \mathcal{A}^*: & \Delta & \epsilon & m & \eta & \gamma, \end{array}$$

where the arrows mean that the various operations coincide. We will define the following.

Definition We define the quantum double $\mathcal{D}(\mathcal{A}, \mathcal{A}^*)$ of the Hopf algebra $\mathcal{A}$ as the bicrossed product of $\mathcal{A}$ and of $\mathcal{A}^*$:

$$\mathcal{D}(\mathcal{A}, \mathcal{A}^*) = \mathcal{A}^* \bowtie \mathcal{A} = \mathcal{A}^* \bowtie \mathcal{A}. \tag{40}$$

[5]For this section I closely followed [2], pp. 178-179. I also used [1], p. 48.

The bicrossed product is a combination of the semidirect product $\ltimes$ and $\rtimes$ which are operations often encountered in group theory. For an extensive explanation of this bicrossed product, I refer the reader to Appendix A.

We let the set $\{a_i\}$ be the basis vectors for $\mathcal{A}$ and $\{a^i\}$ be those for $\mathcal{A}^*$. Physically, we can interpret $\mathcal{A}$ as being an algebra made up of only creation operators, and $\mathcal{A}^*$ as being made up of only annihilation operators. To be able to put them together in the quantum double construction, we need to normal order these bases. This familiar operation consists of pulling all annihilation operators to the right of the creation operators, such that the basis for our quantum double construction $\mathcal{D}(\mathcal{A}, \mathcal{A}^*)$ is $\{a_i a^j\}$.

If we now set our universal $\mathcal{R}$-matrix $\mathcal{R}$ as:

$$\mathcal{R} = \sum_i (\mathbb{I}_{\mathcal{A}} \otimes a_i) \otimes \left(a^i \otimes \mathbb{I}_{\mathcal{A}^*}\right) \in \mathcal{D}(\mathcal{A}, \mathcal{A}^*) \otimes \mathcal{D}(\mathcal{A}, \mathcal{A}^*), \tag{41}$$

we can prove that the Hopf algebra $\mathcal{D}(\mathcal{A}, \mathcal{A}^*)$ with this $\mathcal{R}$ is quasi-triangular. This means that any quantum double construction produces $\mathcal{R}$-matrices that satisfy the Yang-Baxter equation.[6]

[6]For this section I have used [1], pp. 187-189, [2], p. 213, and I have used some knowledge that André Henriques provided me with.

References

[1] Gómez, C., M. Ruiz-Altaba, and G. Sierra, *Quantum Groups in Two-Dimensional Physics*. Cambridge University Press, 1996, pp. 40, 41, 45, 47-49, 51-55, 184-189.

[2] Kassel, C., *Quantum Groups*. Springer Science+Business Media New York, 1995, pp. 109, 110, 134-138, 173-175, 178, 179, 202-207, 213, 214, 223-235.

Appendix A

A.1 Duality

Before we will go into the concept of the quantum double construction, I will first introduce the concept of duality. Duality is needed to prove many theorems that are stated in the following sections. Therefore, an understanding of what this means is quite useful. We will define the following.

Definition Given bialgebras $(U, m, \eta, \Delta, \epsilon)$ and $(H, m, \eta, \Delta, \epsilon)$ and a bilinear form $<,>$ on $U \times H$, we say that this bilinear form realizes a duality between U and H, or that the bialgebras U and H are in duality, if we have:

$$< uv, x > \;=\; \sum_{(x)} < u, x' >< v, x'' >, \tag{42}$$

$$< u, xy > \;=\; \sum_{(u)} < u', x >< u'', y >, \tag{43}$$

$$< \mathbb{I}, x > \;=\; \epsilon(x), \tag{44}$$

$$< u, \mathbb{I} > \;=\; \epsilon(u), \tag{45}$$

for all $u, v \in U$ and $x, y \in H$. Moreover, if U and H are Hopf algebras with antipodes γ, then they are in duality if the underlying bialgebras are in duality and if, moreover, we have $< \gamma(u), x >=< u, \gamma(x) >$ for all $u \in U$ and $x \in H$. [7]

A.2 Quantum Double Construction

In this section we will construct something that is known as Drinfeld's quantum double construction. First, the concept will be introduced and we will see that the quantum double is a braided algebra. Next, we will apply the construction on the quantum group $U_q\,(sl(2))$.

A.2.1 Construction of Quantum Double

We start with two finite-dimensional Hopf algebras $(\mathcal{A}, m, \eta; \Delta, \epsilon; \gamma, \gamma^{-1})$, with the antipode γ and with an existing invertible antipode γ^{-1} and $X = (\mathcal{A}^{op})^* = \left(\mathcal{A}^*, \Delta^*, \epsilon; (m^{op})^*, \eta; (\gamma^{-1})^*, \gamma^*\right)$ with $\mathcal{A}^* = Hom(V, k)$ the dual of $\mathcal{A}$. We then define the quantum double $\mathcal{D}\,(\mathcal{A}, \mathcal{A}^*)$ of the Hopf algebra $\mathcal{A}$ as follows.

[7] I have used [2], pp. 109, 110.

Definition We define the quantum double $\mathcal{D}(\mathcal{A}, \mathcal{A}^*)$ of the Hopf algebra $\mathcal{A}$ as the bicrossed product of $\mathcal{A}$ and of $\mathcal{A}^*$:

$$\mathcal{D}(\mathcal{A}, \mathcal{A}^*) = \mathcal{A}^* \bowtie \mathcal{A} = \mathcal{A}^* \bowtie \mathcal{A}. \tag{46}$$

Now one will probably wonder what this bicrossed product "$\bowtie$" is. I will attempt to give a short description here, for more details see [2], pp. 202-207. We let G be a group with subgroups H and K. We assume that for any element $x \in G$ there exists a unique pair $(y, z) \in H \times K$ satisfying $x = yz$. We can now attach to any pair $(y, z) \in H \times K$ a unique element $z \cdot y \in H$ and a unique element $z^y \in K$ such that $zy = (z \cdot y)z^y$.

Definition We say that a pair (H, K) of groups is matched if there exists a left action α of the group K on the set H and a right action β of the group H on the set K, such that for all $y, y' \in H$ and $z, z' \in K$ we have

$$\begin{aligned}
(zz')^y &= z^{z' \cdot y} z'^y, & (47)\\
z \cdot (yy') &= (z \cdot y)(z^y \cdot y'), & (48)\\
z \cdot 1 &= 1, & (49)\\
1 \cdot y &= y, & (50)
\end{aligned}$$

where $\alpha(z, y) = z \cdot y$ and $\beta(z, y) = z^y$.

We are now ready to define the bicrossed product.

Definition Let (H, K) be a matched pair of groups. Then there exists a unique group structure on the set-theoretic product $H \times K$ with unit $1 \otimes 1$ such that:

$$(y, z)(y', z') = \left(y(z \cdot y'), z^{y'} z'\right) \tag{51}$$

for all $y, y' \in H$ and $z, z' \in K$. This structure is denoted by $H \bowtie K$.

We will now return to the construction of the quantum double $\mathcal{D}(\mathcal{A}, \mathcal{A}^*)$. We let e_a and e^a be the basis vectors for $\mathcal{A}$ and $X = (\mathcal{A}^{op})^*$ respectively. We require that:

$$\begin{aligned}
e_a e_b &= m^c_{ab} e_c, & \Delta(e^a) &= m^a_{cb} e^b \otimes e^c, & (52)\\
e^a e^b &= \mu^{ab}_c e^c, & \Delta(e_a) &= \mu^{bc}_a e_b \otimes e_c, & (53)
\end{aligned}$$

where the "matrices" m and μ are chosen such that both $\mathcal{A}$ and $\mathcal{A}^*$ are co-algebras. The antipode for $\mathcal{A}$ is $\gamma(e_a) = \gamma^b_a e_b$ and the antipode for $\mathcal{A}^*$ is then found from a matrix inversion: $\gamma(e^a) = (\gamma^{-1})^a_b e^b$.

Physically, we can imagine $\mathcal{A}$ as containing creation operaters, and $\mathcal{A}^*$ as containing annihilation operators. Therefore, to put them together into our quantum double $\mathcal{D}(\mathcal{A},\mathcal{A}^*) \subset \mathcal{A}\otimes\mathcal{A}^*$ we need to be able to normal order them. As we have seen numerous times before, we will put the annihiliation operators to the right of the creation operators such that the basis of $\mathcal{D}(\mathcal{A},\mathcal{A}^*)$ is $\left\{e_a e^b\right\}$.

The co-product and the opposed co-product follow directly from our previous relations:

$$\begin{aligned}\Delta\left(e_a e^b\right) &= \mu_a^{cd} m_{vu}^b e_c e^u \otimes e_d e^v, &(54)\\ \Delta^{op}\left(e_a e^b\right) &= \mu_a^{cd} m_{vu}^b e_d e^v \otimes e_c e^u. &(55)\end{aligned}$$

Unfortunately, the product is not as straighforwardly derived. We need a normal ordering description to rewrite $e^b e_a$ in the basis form $e_c e^d$ such that the product and the co-product act like an algebra homomorphisms. We follow Drinfeld's normal ordering:

$$e^a e_b = m_{kd}^x m_{xu}^a \mu_b^{vy} \mu_y^{ck} \left(\gamma^{-1}\right)_v^u e_c e^d \tag{56}$$

where we use Einstein's convention, i.e. we sum over all repeated indices.

If we now write the universal $\mathcal{R}$-matrix of the quantum double as the element

$$\mathcal{R} = \sum_a e_a \otimes \mathbb{I} \otimes \mathbb{I} \otimes e^a \in \mathcal{D}(\mathcal{A},\mathcal{A}^*) \otimes \mathcal{D}(\mathcal{A},\mathcal{A}^*) \tag{57}$$

then upon using the normal ordering as defined in equation(56) we find that:

$$\mathcal{R}\Delta(x) = \Delta^{op}(x)\mathcal{R}, \qquad \forall x \in \mathcal{D}(\mathcal{A},\mathcal{A}^*). \tag{58}$$

We have thus shown that the quantum double is indeed quasi-triangular.[8]

A.2.2 Quantum Double Construction on $U_q(sl(2))$

We will now apply the quantum double construction to the group $U_q(sl(2))$. In the previous section, we have defined the construction for finite-dimensional Hopf algebras, whereas $U_q(sl(2))$ is an infinite-dimensional Hopf algebra. Therefore, we will not do the construction of the universal $\mathcal{R}$-matrix for U_q, but for the finite-dimensional quotient $\bar{U}_q$. This algebra is defined as the quotient of U_q by the twosided ideal generated by the three elements $E^d, F^d, K^d - 1$. In fact, the finite set $\left\{E^i F^j K^l\right\}_{0\leq i,j,l\leq d-1}$ is a basis of the underlying vector space of $\bar{U}_q$. For more on the structure of $\bar{U}_q$ see [2], pp. 134-138. We give the algebra $\bar{U}_q$ a Hopf algebra structure. We want to show that $\bar{U}_q$ is a quasi-triangular Hopf algebra.

[8]For this section I have used [1], pp. 187-189, [2], pp. 202-207, 213.

Firstly, we have seen last week in Alexandros' presentation that we define $U_q = U_q\,(sl(2))$ as the algebra generated by the four variables E, F, K, K^{-1} with:

$$KK^{-1} = K^{-1}K = 1, \qquad KEK^{-1} = q^2E, \tag{59}$$
$$KFK^{-1} = q^{-2}F, \qquad [E,F] = \frac{K-K^{-1}}{q-q^{-1}}, \tag{60}$$
$$\Delta(E) = \mathbb{I}\otimes E + E\otimes K, \qquad \Delta(F) = K^{-1}\otimes F + F\otimes \mathbb{I}, \tag{61}$$
$$\Delta(K) = K\otimes K, \qquad \Delta(K^{-1}) = K^{-1}\otimes K^{-1}, \tag{62}$$
$$\epsilon(E) = \epsilon(F) = 0, \qquad \epsilon(K) = \epsilon(K^{-1}) = 1, \tag{63}$$
$$\gamma(E) = -EK^{-1}, \quad \gamma(F) = -KF, \qquad \gamma(K) = K^{-1}, \quad \gamma(K^{-1}) = K. \tag{64}$$

We propose that the albegra $\bar{U}_q$ has a unique Hopf algebra structure such that the canonical projection from U_q to $\bar{U}_q$ is a morphism of Hopf algebras. This means that the co-multiplication, the co-unit and the antipode of $\bar{U}_q$ are determined by equations(61-64).

To be able to show that $\bar{U}_q$ is a quasi-triangular Hofp algebra, we present $\bar{U}_q$ as the quotient of the quantum double of a Hopf algebra B_q of $\bar{U}_q$. B_q is defined as the subspace of $\bar{U}_q$ linearly generated by the set $\{E^mK^n\}_{0\leq m,n\leq d-1}$. Equations(61-64) show that B_q is a Hopf subalgebra of $\bar{U}_q$. B_q is generated by E and K and

$$KEK^{-1} = q^2E, \qquad E^d = 0, \qquad \text{and} \qquad K^d = 1. \tag{65}$$

Now we will apply the quantum double construction to the Hofp algebra $\mathcal{A} = B_q$. First, $X = \left(B_q^{op}\right)^*$ is determined to be a Hopf algebra. Consider the linear forms α and η on B_q defined on the basis $\{E^mK^n\}_{0\leq m,n\leq d-1}$ by:

$$<\alpha, E^mK^n> = \delta_{m0}q^{2n} \qquad \text{and} \qquad <\eta, E^mK^n> = \delta_{m1}. \tag{66}$$

We will propose that the following relations hold in the Hopf algebra X:

$$\begin{aligned} \alpha^d = 1, \quad \eta^d = 0, \qquad & \alpha\eta\alpha^{-1} = q^{-2}\eta, \\ \Delta(\alpha) = \alpha\otimes\alpha, \qquad & \Delta(\eta) = \mathbb{I}\otimes\eta + \eta\otimes\alpha, \\ \epsilon(\alpha) = 1, \qquad & \epsilon(\eta) = 0, \\ \gamma(\alpha) = \alpha^{d-1}, \qquad & \gamma(\eta) = -\eta\alpha^{d-1}. \end{aligned}$$

On top of that, the set $\{\eta^i\alpha^i\}_{0\leq i,j\leq d-1}$ forms a basis of X.

We can now construct the quantum double $\mathcal{D} = \mathcal{D}\,(B_q)$. By definition, we know that the set $\left\{\eta^i\alpha^j\otimes E^kK^l\right\}_{0\leq i,j,k,l\leq d-1}$ is a basis of $\mathcal{D}$. To make this notation simpler, we identify an element x of $\mathcal{A} = B_q$ with its image $\mathbb{I}\otimes x$ in $\mathcal{D}$ and an element α of the dual X with its image $\alpha\otimes\mathbb{I}$, such that we can write $\eta^i\alpha^j\otimes E^kK^l = \eta^i\alpha^jE^kK^l$.

To determine the multiplication of $\mathcal{D}$, we need to know how the generators α, η, E, K in $\mathcal{D}$ multiply. The following relations hold in $\mathcal{D} = \mathcal{D}(B_q)$:

$$\begin{aligned} K\alpha &= \alpha K, & K\eta &= q^{-2}\eta K, \\ E\alpha &= q^{-2}\alpha E, & E\eta &= -q^{-2}(1 - \eta E - \alpha K) \end{aligned}$$

We have the following relations:

$$\begin{aligned} \alpha\left(K^{-1}?K\right) &= \alpha, & \alpha\left(K^{-1}E?\right) &= 0, \\ \alpha\left(K^{-1}?\right) &= q^{-2}\alpha, & \alpha\left(K^{-1}?E\right) &= 0, \\ \eta\left(K^{-1}?K\right) &= q^{-2}\eta, & \eta\left(K^{-1}E?\right) &= q^{-2}\epsilon, \\ \eta\left(K^{-1}?K\right) &= q^{-2}\eta, & \eta\left(K^{-1}?E\right) &= q^{-2}\alpha, \end{aligned}$$

where $g\left(\gamma^{-1}\left(a'''\right)?a'\right)$ means the map $x \to g\left(\gamma^{-1}\left(a'''\right)xa'\right)$.

Next, we will need the following theorem to show that $\bar{U}_q$ is quasi-triangular.

Theorem Let $\chi : \mathcal{D}(B_q) \to \bar{U}_q$ be the linear map determined by

$$\chi\left(\eta^i\alpha^j E^k K^l\right) = \left(\frac{q - q^{-1}}{q^2}\right) q^{2(i+j)k-i(i-1)} F^i E^k K^{i+j+l} \tag{67}$$

where $0 \leq i, j, k, l \leq d-1$. Then χ is a surjective Hopf algebra morpishm.

This means that every element in χ has a corresponding element in $\mathcal{A}$. We can now prove that the Hopf algebra $\bar{U}_q$ is quasi-triangular.

Proof We know that the Hopf algebra $\mathcal{D} = \mathcal{D}(B_q)$ is quasi-triangular already. We let $\mathcal{R}_{\mathcal{D}} \in \mathcal{D} \otimes \mathcal{D}$ be its universal $\mathcal{R}$-matrix. We define the invertible element $\bar{\mathcal{R}}$ of $\bar{U}_q \otimes \bar{U}_q$ by $\bar{\mathcal{R}} = (\chi \otimes \chi)(\mathcal{R}_{\mathcal{D}})$. As χ is a surjective homomorphism of Hopf algebras, it is evident that $\bar{\mathcal{R}}$ satisfies equations(2-4) and is thus a quasi-triangular algebra.

Our last step is to determine the $\mathcal{R}$-matrix for $\bar{U}_q$. We will simply state the answer here without any proof. The proof can be found in [2], pp. 230-235. The universal $\mathcal{R}$-matrix of $\bar{U}_q$ is given by:

$$\bar{\mathcal{R}} = \frac{1}{d} \sum_{0 \leq i,j,k \leq d-1} \frac{q - q^{-1}}{[k]!} q^{k(k-1)/2 + 2k(i-j) - 2ij} E^k K^i \otimes F^k K^j, \tag{68}$$

where in the tensor product we see the basis of the Hopf algebras $\mathcal{A}$ and $\mathcal{A}^*$ reflected.[9]

[9]For this section I have used [2], pp. 134-138, 214, 223-235.