A Quantum Group in the Yang-Baxter Algebra

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Abstract

In these notes we mainly present theory on abstract algebra and how it emerges in the Yang-Baxter equation. First we review what an associative algebra is and then introduce further structures such as coalgebra, bialgebra and Hopf algebra. Then we discuss the construction of an universal enveloping algebra and how by deforming $U[\mathfrak{sl}(2)]$ we obtain the quantum group $U_q[\mathfrak{sl}(2)]$. Finally, we discover that the latter is actually the Braid limit of the Yang-Baxter algebra and we use our algebraic knowledge to obtain the elements of its representation.

Contents

1 Algebras

1.1 Bialgebra and Hopf algebra

To start with, the most familiar algebraic structure is surely that of an associative algebra.

Definition 1 *An associative algebra $\mathcal{A}$ over a field $\mathcal{C}$ is a linear vector space V equipped with*

- *Multiplication* $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$
  which is
  - ***bilinear***
  - ***associative*** $m(1 \otimes m) = m(m \otimes 1)$ *that pictorially corresponds to the commutative diagram*

$$
\begin{array}{ccc}
\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} & \xrightarrow{1 \otimes m} & \mathcal{A} \otimes \mathcal{A} \\
\Big\downarrow{\scriptstyle m \otimes 1} & & \Big\downarrow{\scriptstyle m} \\
\mathcal{A} \otimes \mathcal{A} & \xrightarrow{m} & \mathcal{A}
\end{array}
$$

- *Unit* $\eta : \mathbb{C} \to \mathcal{A}$
  which satisfies the axiom $m(\eta \otimes 1) = m(1 \otimes \eta) = 1$.

$$
\begin{tikzcd}
\mathcal{A} \otimes \mathbb{C} \arrow[r, dash, "\cong"] \arrow[dr, "1 \otimes \eta"'] & \mathcal{A} \arrow[r, dash, "\cong"] & \mathbb{C} \otimes \mathcal{A} \arrow[dl, "\eta \otimes 1"] \\
& \mathcal{A} \otimes \mathcal{A} \arrow[u, "m"] &
\end{tikzcd}
$$

By reversing the arrows we get another structure which is called coalgebra.

Definition 1 *A coalgebra* $\mathcal{A}$ *over a field* $\mathcal{C}$ *is a linear vector space* V *equipped with*

- *Co-multiplication* $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$
  which is
  - ***bilinear***
  - ***co-associative*** $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$

$$
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\Delta} & \mathcal{A} \otimes \mathcal{A} \\
\Big\downarrow{\scriptstyle \Delta} & & \Big\downarrow{\scriptstyle 1 \otimes \Delta} \\
\mathcal{A} \otimes \mathcal{A} & \xrightarrow{\Delta \otimes 1} & \mathcal{A}
\end{array}
$$

- *Co-unit* $\epsilon : \mathcal{A} \to \mathbb{C}$
  satisfying the axiom $(1 \otimes \epsilon)\Delta = (\epsilon \otimes 1)\Delta = I_{\mathbb{C}}$.

$$
\begin{tikzcd}
\mathcal{A} \otimes \mathbb{C} \arrow[r, dash, "\cong"] & \mathcal{A} \arrow[r, dash, "\cong"] \arrow[d, "\Delta"'] & \mathbb{C} \otimes \mathcal{A} \\
& \mathcal{A} \otimes \mathcal{A} \arrow[ul, "1 \otimes \epsilon"] \arrow[ur, "\epsilon \otimes 1"'] &
\end{tikzcd}
$$

Those two algebras can be combined together to give a structure $\mathcal{A}(m, \eta, \Delta, \epsilon)$ called **bialgebra**. That is feasible with two compatibility conditions which demand Δ and ϵ to be homomorphisms of the algebra side of the bialgebra, that is

$$\Delta(ab) = \Delta(a)\Delta(b)$$

$$\epsilon(ab) = \epsilon(a)\epsilon(b)$$

or in diagrammatic language

$$\begin{tikzcd}
\mathcal{A}\otimes\mathcal{A} \arrow[r, "m"] \arrow[d, "\Delta\otimes\Delta"'] & \mathcal{A} \arrow[r, "\Delta"] & \mathcal{A}\otimes\mathcal{A} \\
\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} \arrow[rr, "1\otimes\tau\otimes1"] & & \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} \arrow[u, "m\otimes m"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\epsilon"] & \mathbb{C} \arrow[r, "\eta"] \arrow[dr, "\eta\otimes\eta"'] & \mathcal{A} \arrow[d, "\Delta"] \\
\mathcal{A}\otimes\mathcal{A} \arrow[u, "m"] \arrow[ur, "\epsilon\otimes\epsilon"'] & & \mathcal{A}\otimes\mathcal{A}
\end{tikzcd}$$

where τ is the permutation map $\tau(h \otimes g) = g \otimes h$.
The notion of a commutative or co-commutative bialgebra also exists.

- **Commutative**

$$m = m\tau$$

$$\begin{tikzcd}
\mathcal{A}\otimes\mathcal{A} \arrow[r, "m"] \arrow[d, "\tau"'] & \mathcal{A} \\
\mathcal{A}\otimes\mathcal{A} \arrow[ur, "m"'] &
\end{tikzcd}$$

- **Co-commutative**

$$\Delta = \tau\Delta$$

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & \mathcal{A}\otimes\mathcal{A} \\
\mathcal{A}\otimes\mathcal{A} \arrow[ur, "\tau"'] &
\end{tikzcd}$$

In the spirit of adding more structure, we can construct the so-called **Hopf Algebra**.

Definition 1 *A Hopf algebra is a bialgebra with the additional operation of an antipode* $\gamma : \mathcal{A} \to \mathcal{A}$
which

- *is an* ***antihomomorphism***

$$\gamma(ab) = \gamma(b)\gamma(a)$$

- *satisfies*

$$m(\gamma \otimes 1)\Delta = m(1 \otimes \gamma)\Delta = \eta\epsilon$$

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\epsilon"] \arrow[d, "\Delta"'] & \mathbb{C} \arrow[r, "\eta"] & \mathcal{A} \\
\mathcal{A} \otimes \mathcal{A} \arrow[rr, "1\otimes\gamma,\ \gamma\otimes 1"] & & \mathcal{A} \otimes \mathcal{A} \arrow[u, "m"']
\end{tikzcd}$$

The antipode can be seen as the algebraic analogue of the inverse in a group. Its embedding in the bialgebra also guarantees its uniqueness.

1.2 Universal enveloping algebra

The discussion starts with a definition of a Lie algebra is in order.

Definition 1 *A Lie algebra L over a field $\mathbb{C}$ is a linear vector space V with the operation of Lie bracket*

$$[\ .\ ,.\ ] : L \otimes L \to L$$

which respects three properties:

- ***bilinearity***
  $[\alpha u + \beta u', v] = \alpha[u, v] + \beta[u', v]$ *and* $[u, \alpha v + \beta v'] = \alpha[u, v] + \beta[u, v']$, $\forall u, u, v, v' \in L$ *and* $\alpha, \beta \in \mathbb{C}$.

- ***antiymmetry***
  $[u, v] = -[v, u]$, $\forall u, v \in L$.

- ***Jacobi identity***
  $[u, [v, w]] + [w, [u, v]] + [v, [w, u]] = 0$, $\forall u, v, w \in L$.

Obviously, a Lie algebra is not equipped with an associative operation. Nevertheless, we can embed it in a larger associative algebra by indentifying the Lie bracket of two elements with the commutator. Namely, if . denotes the multiplication of the associative algebra then we demand that

$$[a, b] \equiv a.b - b.a\ ,\ \forall a, b \in L.$$

This condition should also be respected by the representations of the Lie algebra and the desired associative one. Explicitly, the representation ρ should satisfy

$$\rho([a,b]) = \rho(a).\rho(b) - \rho(b).\rho(a).$$

In order to construct this new algebra from L, we need to choose it from the *Tensor algebra* of V, denoted as $T(V)$, that basically is the algebra of all tensors in V.

$$T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n} = \mathbb{C} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \dots$$

With the multiplication operation being the tensor product this algebra is associative. An arbitrary element of $T(V)$ is $X_1 \otimes X_2 \otimes \cdots \otimes x_r = X_1 X_2 \dots X_r$, basically a string of letters, that is a word or monomial in the mathematical terminilogy, of arbitrary lenght.
Now by imposing inside $T(V)$ that $[a,b] = ab - ba$ yields the **Universal enveloping algebra** $U[L]$. A bit more formally,

$$U(L) = T(L)/_{ab-ba=[a,b]}$$

In this vector space the elements are words that can be related using the commutator. For example, $ANAGRAM = ANAGRMA + ANAGR[A,M]$. This also allowes us to "normal order" words.
More specifically, if $\{x_i\}_{1 \leq i \leq n}$ is a basis of L then, according to the **Poincaré-Birkhoff-Witt Theorem**, the monomials $\{X_1^{r_1} X_2^{r_2} \dots X_n^{r_n}\}$ with $r_i \in \mathcal{Z}^+$ is a basis of $U[L]$.
A familiar example of a universal enveloping algebra is $U[\mathfrak{sl}(2)]$. As we know $\mathfrak{sl}(2)$ is generated by the Cartan subalgebra σ^z and its roots σ^+, σ^- which are represented in the 2-dimensional module of $U[\mathfrak{sl}(2)]$ as

$$\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

What is remarkable for the enveloping algebras is that in $U[L]$ resides an element that commutes with any other element. It is called **Casimir element** and is written

$$C \equiv \sum_a X_a X^a.$$

where $\{X_a\}$ constitute the basis of L where $[X_a, X_b] = C_{abc} X_c$ with C_{abc} totally antisymmetric. X^a is defined on the dual space to give the following inner product $< X_a X^b >= \delta_a^b$.

For our proposition that the Casimir element commutes with everything in $U[L]$ suffices to prove that is commutes with the basis of L. This is easily shown,

$$\begin{aligned}[C, X_d] = \sum_a [X_a X^a, X_d] &= \sum_a X_a [X^a, X_d] + [X_a, X_d] X^a = \\ &= \sum_a C_{adc} X_a X_c + C_{adc} X_c X_a = \\ &= 0\end{aligned}$$

where we used the antisymmetry of structure constants and that $X_a = X^a$. The existence of a Casimir element is very important for the finite dimensional representation of an algebra since Schur's lemma ensures that in an invariant subspace it will be proportional to the unit operator. The Casimir acts as a scalar and therefore this eigenvalue characterizes uniquely the specific module.
In physics, one often encounters the casimir element of the rotation group $SO(3)$ which is nothing else than the total angular momentum operator,

$$\mathbf{S}^2 = \vec{S}.\vec{S} = \sigma^x \sigma^x + \sigma^y \sigma^y + \sigma^z \sigma^z = 2\sigma^+ \sigma^- + 2\sigma^- \sigma^+ + \sigma^z \sigma^z$$

which categorizes the irreducible representation with its quantum number $s(s+1)$.
Surprisingly, it turns out that any enveloping algebra is a special case of a Hopf algebra.

Proposition 1 *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $U[\mathfrak{g}]$ its universal enveloping algebra. Introducing the operations of co-multiplication, co-unit and antipode*

$$\Delta(X) = 1 \otimes X + X \otimes 1, \ \epsilon(X) = 0, \ S(X) = -X, \ \forall X \in \mathfrak{g}$$

$U[\mathfrak{g}]$ turns to a Hopf algebra.

To prove this one first has to extend Δ, ϵ to algebra homomorphisms, while S to an antihomomorphism and check if this is consistent with the algebra properties. Afterwards, verifying that all the previous diagrams hold we prove the proposition. Remarkably, $U[\mathfrak{g}]$ is co-commutative and consequently $S^2 = 1$[1].

[1]Note that we changed the notation for the antipode from γ to S.

1.3 The quantum $U_q[\mathfrak{sl}(2)]$

The idea of this section is that a Hopf algebra, $U[\mathfrak{sl}(2)]$ in specific, can be continuously changed by introducing a deformation parameter that when set to zero we get the algebra we start with.
$U[\mathfrak{sl}(2)]$ constitutes a Hopf algebra according to the previous section and is freely generated[2] by $\{X, Y, H\}$ obeying the commutation relations

$$[X, Y] = H \tag{1}$$
$$[H, X] = 2X \tag{2}$$
$$[H, Y] = -2Y \tag{3}$$

Of course, we recognise X as the raising operator, Y as the lowering operator and H as the Cartan element.
Consider now $q \in \mathbb{C}$ which is not a root of unity and the algebra freely generated by $\{E, F, K, K^{-1}\}$ with the following rules

$$KK^{-1} = K^{-1}K = 1 \tag{4}$$
$$KEK^{-1} = q^2 E \tag{5}$$
$$KFK^{-1} = q^{-2} F \tag{6}$$
$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}}. \tag{7}$$

This algebra we will name $U_q[\mathfrak{sl}(2)]$ and think of it as a "deformation" of $U[\mathfrak{sl}(2)]$. This statement is actually valid even though $U_q[\mathfrak{sl}(2)]$ has one more generator. The relation is made clear by the following parametrization

$$\begin{aligned} q &= e^{\hbar} \\ K &= q^H = e^{\hbar H} \\ E &= X \\ F &= Y \end{aligned}$$

Plugging those in the above relations yields conditions on X, Y and H. Obviously, equation (4) is identically satisfied. Substitution in (5) gives

$$e^{\hbar H} X e^{-\hbar H} = e^{2\hbar} X$$

Using the Baker-Campbell-Hausdorff formula on the left hand side and the standard exponential expansion on the other we obtain

$$X + \hbar[H, X] + \frac{\hbar^2}{2!}[H, [H, X]] + \ldots = \left(1 + 2\hbar + \frac{(2\hbar)^2}{2!} + \frac{(2\hbar)^3}{3!} + \ldots\right) X$$

[2]This means that the monomials of the enveloping algebra contain only the generators of the Lie algebra.

which is solved in all orders of $\hbar$ by equation (2). In the same way equation (3) is satisfied by our parametrization.
Equation (7) does not have an analogue in $U[\mathfrak{sl}(2)]$ but in the "classical" limit $\hbar \to 0$ or $q \to 1$ we do restore it! In particular,

$$\begin{aligned}
[E, F] = [X, Y] = \frac{e^{\hbar H} - e^{-\hbar H}}{e^{\hbar} - e^{-\hbar}} &= \frac{\big(1 + \hbar H + \mathcal{O}(\hbar^2 H^2)\big) - \big(1 - \hbar H + \mathcal{O}(\hbar^2 H^2)\big)}{\big(1 + \hbar + \mathcal{O}(\hbar^2)\big) - \big(1 - \hbar H + \mathcal{O}(\hbar^2)\big)} \\
&= \frac{2\hbar H + \mathcal{O}(\hbar^3 H^3)}{2\hbar + \mathcal{O}(\hbar^3)}
\end{aligned}$$

where in the limit $\hbar \to 0$ we get back to equation (1).

Now it is made clear that the parameter q signifies the departure from the "classical" algebra $U[\mathfrak{sl}(2)]$ to a deformed or "quantum" version. The algebraic structure is also deformed with the Hopf algebra structure of $U_q[\mathfrak{sl}(2)]$ being

$$\begin{aligned}
\Delta(E) &= 1 \otimes E + E \otimes K & \epsilon(E) &= 0 \\
\Delta(F) &= K^{-1} \otimes F + F \otimes 1 & \epsilon(F) &= 0 \\
\Delta(K^{\pm 1}) &= K^{\pm 1} \otimes K^{\pm 1} & \epsilon(K^{\pm 1}) &= 1
\end{aligned}$$

and

$$\begin{aligned}
S[E] &= -EK^{-1} \\
S(F) &= -KF \\
S(K^{\pm 1}) &= K^{\mp 1}
\end{aligned}$$

Note here also that the co-commutativity gives $S^2 \neq 1$ and is restored only when $q \to 1$.
The morale of this discussion is that starting from an enveloping algebra, we can insert an auxiliary parameter to deform it accordingly so that it flows continuously to another Hopf algebra which is not necessarily an enveloping or Lie algebra. The deformed Hopf algebra is conventionally called **Quantum Group** because it was discovered in the context of quantum integrable models. In physics terms, deforming an algebra corresponds to breaking some symmetry of the system by turning on an anisotropy parameter q just as it happens with the XXX and XXZ models.

The representation theory on the quantum group $U_q[\mathfrak{sl}(2)]$ does not deviate from that of $\mathfrak{sl}(2)$. The discussion follows the same lines apart from some differences due to the noncommutativity of the algebra[3].

[3] When for two variables y and x the property $xy = yx$ is replaced by $xy = q(yx)$ with $q \in \mathbb{C}$ then the variables are said to live on the *quantum plane* and all the usual algebraic calculations get modified.

Since we are interested in finding finite-dimensional U_q-modules, the Cartan subalgebra and its roots we mentioned earlier will be our basic tool. To be specific, we need to introduce the notions of highest weight and its corresponding vector.

Definition 1 *Let V be a U_q-module and λ a scalar. An element $u \neq 0$ of V is a **highest weight vector** of weight λ if*

$$Ku = \lambda u$$

$$Eu = 0$$

. This vector generates the entirety of V.

Then a most important theorem establishes our further analysis.

Theorem 1 *Any non-zero n-dimensional U_q-module contains a unique highest weight vector u_0 of heighest weight $\lambda = \epsilon q^n$ where $\epsilon = \pm 1$.*

Now, with the vaccuum state guaranteed one can start exploring the Verma module built by acting with the raising operator F. With a normalization of the eigenstates u_p of K, the action of the elements on them is as follows

$$u_p = \frac{1}{[p]!} F^p u_0, \; p > 0 \text{ and } [p]_q := \frac{q^p - q^{-p}}{q - q^{-1}}$$

$$Ku_p = \epsilon q^{n-2p} u_p$$

$$Eu_p = \epsilon [n - p + 1] u_{p-1}$$

$$Fu_p = [p] u_{p+1}$$

Obviously, in the basis of $\{u_0, u_1, \dots, u_n\}$ of a $(n+1)$-dimensional module the operators are represented as

$$\rho(E) = \epsilon \begin{pmatrix} 0 & [n] & 0 & \cdots & 0 \\ 0 & 0 & [n-1] & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \; \rho(F) = \begin{pmatrix} 0 & 0 \cdots & 0 & 0 \\ 1 & 0 \cdots & 0 & 0 \\ 0 & [2] & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots [n] & 0 \end{pmatrix}$$

$$\rho(K) = \epsilon \begin{pmatrix} q^n & 0 & \cdots & 0 & 0 \\ 0 & q^{n-2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & q^{-n+2} & 0 \\ 0 & 0 & \cdots & 0 & q^{-n} \end{pmatrix}.$$

2 The Yang-Baxter algebra and the quantum $U_q[\mathfrak{sl}(2)]$

With the mathematical wisdom gained from the previous sections now we are able to recognise and translate the Yang-Baxter equation in algebraic terms. The *Yang-Baxter algebra* consists of a couple $(\mathcal{R}, T)$, where $\mathcal{R}$ is a family of invertible matrices parametrized by a spectral parameter u while the monodromy matrices $T^i_j(u)$ are the $n \times n$ generators of the algebra. The linking relation between those two is the so-called $RTT = TTR$ equation

$$\sum_{j_1=1}^{n}\sum_{j_2=1}^{n} \mathcal{R}^{k_1 k_2}_{j_1 j_2}(u-v) T^{j_1}_{i_1}(u) T^{j_2}_{i_2}(v) = \sum_{j_1=1}^{n}\sum_{j_2=1}^{n} T^{k_2}_{j_2}(v) T^{k_1}_{j_1}(u) \mathcal{R}^{j_1 j_2}_{i_1 i_2}(u-v) \quad (8)$$

Interestingly enough, this associative algebra admits a co-multiplication and a co-unit operation that preserve the above equation and promote it to a bialgebra:

$$\Delta\big(T^i_j(u)\big) = \sum_{k=1}^{n} T^k_j(u) \otimes T^i_k(u)$$
$$\epsilon\big(T^i_j(u)\big) = \delta^i_j$$

From now on we will confine ourselves to the Yang-Baxter algebra of the 6-vertex model. The $\mathcal{R}^{(6v)}$-matrix is

$$\mathcal{R}^{(6v)} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix}$$

with Boltzmann weights parametrized by

$$a(u) = \sinh(u + i\gamma),\ b(u) = \sinh u,\ c(u) = i \sin\gamma$$

with u being the rapidity and $\Delta = \cos\gamma$ the anisotropy parameter.
We are also interested in the “Braid limit” representation, $u \to \pm\infty$, where the spectral parameter dependence dissappears and ones has to properly rescale the basis. In this basis the $\mathcal{R}$-matrix and the monodromy matrices are rescaled to

$$\tilde{\mathcal{R}}^{j_1 j_2}_{i_1 i_2}(u) = e^{u(i_1 - j_1)} \mathcal{R}^{j_1 j_2}_{i_1 i_2}(u)$$
$$\tilde{T}^j_i(u) = e^{u(i-j)} T^j_i(u).$$

We would like to investigate (8) in the limit $u \to \pm\infty$, so we need to find the limits of the monodromy matrices. Choosing the adjoint spin-$\frac{1}{2}$ representation where $\left(T_i^j(u)\right)_l^k = \mathcal{R}_{il}^{jk}(u)$, we have

$$T_0^0(u) = \begin{pmatrix} a(u) & 0 \\ 0 & b(u) \end{pmatrix}$$

$$T_1^0(u) = \begin{pmatrix} 0 & 0 \\ c(u) & 0 \end{pmatrix}$$

$$T_0^1(u) = \begin{pmatrix} 0 & c(u) \\ 0 & 0 \end{pmatrix}$$

$$T_1^1(u) = \begin{pmatrix} b(u) & 0 \\ 0 & a(u) \end{pmatrix}$$

Taking the limits

$$\lim_{u\to\pm\infty} \tilde{T}_0^0(u) = \pm\frac{1}{2} e^{\pm u} q^{\pm\frac{1}{2}} q^{\pm S^z}$$

$$\lim_{u\to\pm\infty} \tilde{T}_1^1(u) = \pm\frac{1}{2} e^{\pm u} q^{\pm\frac{1}{2}} q^{\mp S^z}$$

$$\lim_{u\to+\infty} \tilde{T}_1^0(u) = 0, \quad \lim_{u\to-\infty} \tilde{T}_1^0(u) = \frac{1}{2} e^{-u} (q - q^{-1}) S^-$$

$$\lim_{u\to+\infty} \tilde{T}_0^1(u) = \frac{1}{2} e^{u} (q - q^{-1}) S^+, \quad \lim_{u\to-\infty} \tilde{T}_0^1(u) = 0$$

where $S^\pm$, S^z are the $SU(2)$ generators in the spin-$\frac{1}{2}$ representation and $q = e^{i\gamma}$.

The entire T and R matrices at the braid limits are

$$T_+ \equiv \lim_{u\to+\infty} 2q^{-\frac{1}{2}} e^{-u} \begin{pmatrix} \tilde{T}_0^0 & \tilde{T}_0^1 \\ \tilde{T}_1^0 & \tilde{T}_1^1 \end{pmatrix} = \begin{pmatrix} q^{S^z} & 0 \\ q^{-\frac{1}{2}}(q - q^{-1}) S^- & q^{-S^z} \end{pmatrix}$$

$$T_- \equiv \lim_{u\to-\infty} (-2) q^{\frac{1}{2}} e^{u} \begin{pmatrix} \tilde{T}_0^0 & \tilde{T}_0^1 \\ \tilde{T}_1^0 & \tilde{T}_1^1 \end{pmatrix} = \begin{pmatrix} q^{-S^z} & -q^{\frac{1}{2}}(q - q^{-1}) S^+ \\ o & q^{S^z} \end{pmatrix}$$

$$R_+ \equiv \lim_{u\to+\infty} 2q^{-\frac{1}{2}} e^{-u} \tilde{R}(u) = \begin{pmatrix} q^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & 0 & q^{-\frac{1}{2}} & 0 \\ 0 & q^{-\frac{1}{2}} & q^{-\frac{1}{2}}(q - q^{-1}) & 0 \\ 0 & 0 & 0 & q^{\frac{1}{2}} \end{pmatrix}$$

$$R_- \equiv \lim_{u\to-\infty} (-2) q^{\frac{1}{2}} e^{u} \tilde{R}(u) = \begin{pmatrix} q^{-\frac{1}{2}} & 0 & 0 & 0 \\ 0 & q^{\frac{1}{2}}(q - q^{-1}) & q^{\frac{1}{2}} & 0 \\ 0 & q^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 & q^{-\frac{1}{2}} \end{pmatrix}$$

Now taking the various limits $u \to \pm\infty$, $v \to \pm\infty$ in (8) we obtain a system of entagled equations that luckily reduces to the following algebraic system:

$$[S^z, S^\pm] = \pm S^z \tag{9}$$

$$[S^+, S^-] = \frac{q^{2S^z} - q^{-2S^z}}{q - q^{-1}} \tag{10}$$

This temptingly resembles with the $U_q[\mathfrak{sl}(2)]$ we encountered before. Basically a small step is needed to make the indentification and that is the following transformation from $U_q[\mathfrak{sl}(2)]$ to (9) and (10)

$$\begin{aligned} q &= e^\hbar \\ K &= e^{2\hbar S^z} \\ E &= S^+ e^{+\hbar S^z} \\ F &= S^- e^{-\hbar S^z} \end{aligned}$$

which take us from (4), (5), (6), (7) to (9), (10).
We discovered that the in the braid limit the Yang-Baxter algebra coincides with the quantum group $U_q[\mathfrak{sl}(2)]$ generated by $\{S^\pm, S^z\}$. The comultiplication preserves the algebraic relations as well

$$\begin{aligned} \Delta\big(q^{S^z}\big) &= q^{S^z} \otimes q^{S^z} \\ \Delta\big(S^\pm\big) &= S^\pm \otimes q^{S^z} + q^{-S^z} \otimes S^\pm. \end{aligned}$$

Our result is very important because we can immediately apply our knowledge of $U_q[\mathfrak{sl}(2)]$ to find representations of the Yang-Baxter algebra in the braid limit and q is not a root of unity. Thus, a finite-dimensional U_q is spanned by states characterized by two numbers $|j, m>$. The first quantum number j takes integer or half-integer values and corresponds to the eigenvalue of the Casimir operator, which in our case is

$$C = X_- X_+ + \left(\frac{q^{\frac{2S^z+1}{2}} - q^{-\frac{2S^z+1}{2}}}{q - q^{-1}} \right)^2$$

with $C|j, m> = \left[\frac{2j+1}{2}\right]_q^2 |j, m>$.
The second one is the eigenvalues of $H \equiv S^z$ with

$$H|j, m> = 2m|j, m>, \ m \in \{-j, -j+1, \dots, j-1, j\}.$$

Just as for the z-spin projection in $SU(2)$, m can be lowered or raised with $S^\pm$,

$$S^\pm|j, m> = \sqrt{[j \mp m]_q [j \pm m + 1]_q} |j, m \pm 1> .$$

Finally, the ultimate question is how to represent tensor products of representations. It turns out that the tensor product is fully reducible to the direct sum of smaller-dimensional irreps. For the tensor product of two spaces V^{j_1}, V^{j_2} the block-diagonalisation is the same as in ordinary $SU(2)$

$$V^{j_1} \otimes V^{j_2} = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} V^j.$$

Even the group theoretic $3j$ and $6j$ symbols can be used here in order to relate the basis of the two different sides of the equation. However, things differ due to the deformation parameter q so we are dealing with the quantum version of $3j$ symbols or *quantum Clebsch-Gordan coefficients*

$$|j_1, m_1 > \otimes |j_2, m_2 > = \sum_j \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q |j, m > .$$

A general expression for them, that completes our discussion, is

$$\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q = \delta^m_{m_1+m_2} \Delta(j_1, j_2, j) q^{[(j_1+j_2-j)(j_1+j_2+j+1)+j_1m_2-j_2m_1]/2}$$
$$\sqrt{[j_1+m_1]![j_1-m_1]![j_2+m_2]!}\sqrt{[j_2-m_2]![j-m]![j+m]![2j+1]}$$
$$\sum_{r\geq 0} \left[\frac{(-1)^r q^{-r(j_1+j_2+j+1)}}{[r]![j_1+j_2-j-r]![j_1-m_1-r]![j_2+m_2-r]![j-j_2+m_1+r]![j-j_1+r-m_2]!} \right]$$

where

$$\Delta(a, b, c) = \sqrt{\frac{[-a+b+c]![a-b+c]![a+b-c]!}{[a+b+c+1]!}}.$$

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