Higher spin chain and XXXs model Hamiltonian

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Abstract

Up until now we have studied algebraic bethe ansatz for 1/2-spin models like $XXX_{\frac{1}{2}}$ or 6-vertex, where the various operators and R-matrices satisfied the relation $TTR = RTT$. Now we will take a look on higher spin models, where a more general relation is satisfied. This relation reads as $LLR = RLL$ where L stands for the Lax operator. Furthemore we will construct the hamiltonian for the XXX_s model, where the index s denotes the higher spin.

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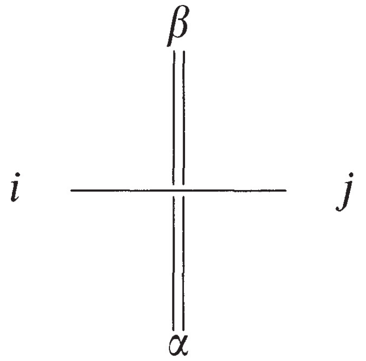

Figure 1: Diagramatic form of the lax operator. The single line stands for the $\frac{1}{2}$ representation and the auxilary space, while the double line stands for the ρ representation and the local quantume space

1 The relation $LLR = RLL$ and the lax operators

In this section we will generalise the relation $TTR = RTT$. We will do this by using the lax operators. The lax operator has the form:

$$L = \left(L^{\frac{1}{2},\rho}\right)_{i\,\alpha}^{j\,\beta} \tag{1}$$

where $\frac{1}{2}$ stands for the $\frac{1}{2}$ represantation and the auxilari space, while the ρ stands for the ρ representation and the local quantum space. In the special case that $\rho = \frac{1}{2}$ then $L \equiv T$. We can also take a look in the diagrammatic form of the lax operator in fig.1.

By using (1) we can write down the explicit form of the relation $LLR = RLL$ which is:

$$\sum_{j_1 j_2 \beta} \left(L^{\frac{1}{2},\rho}(\mu)\right)_{j_2\beta}^{k_2\gamma} \left(L^{\frac{1}{2},\rho}(\lambda)\right)_{j_1\alpha}^{k_1\beta} \left(R^{\frac{1}{2},\frac{1}{2}}(\lambda-\mu)\right)_{i_1 i_2}^{j_1 j_2}$$
$$= \sum_{j_1 j_2 \beta} \left(R^{\frac{1}{2},\frac{1}{2}}(\lambda-\mu)\right)_{j_1 j_2}^{k_1 k_2} \left(L^{\frac{1}{2},\rho}(\lambda)\right)_{i_1\beta}^{j_1\gamma} \left(L^{\frac{1}{2},\rho}(\mu)\right)_{i_2\alpha}^{j_2\beta} \tag{2}$$

This is not the most general relation among lax operators and R-matrices, because the auxilary space is restricted to be 2-dimensional. However by knowing the solutions $L^{\frac{1}{2},\rho_1}$,$L^{\frac{1}{2},\rho_1}$ of (2) we can find another R-matrix ,R^{ρ_1,ρ_2} (diagrammatic form in Fig.2),such that:

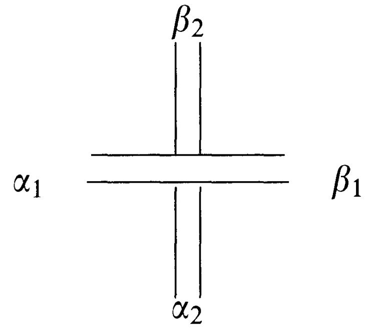

Figure 2: Diagramatic form of the R-matrix R^{ρ_1,ρ_2} . As we can see both horizontal and vertical are double lines, this denotes that both auxilary and quantum space are higher dimensional for this operator

$$\sum_{j\beta_1\beta_2} \left(L^{\frac{1}{2},\rho_1}(\mu)\right)^{k\gamma_1}_{j\beta_1} \left(L^{\frac{1}{2},\rho_2}(\lambda)\right)^{j\gamma_2}_{i\beta_2} \left(R^{\rho_1,\rho_2}(\lambda-\mu)\right)^{\beta_1\beta_2}_{\alpha_1\alpha_2} = \sum_{j\beta_1\beta_2} \left(R^{\rho_1,\rho_2}(\lambda-\mu)\right)^{\gamma_1\gamma_2}_{\beta_1\beta_2} \left(L^{\frac{1}{2},\rho_2}(\lambda)\right)^{k\beta_2}_{j\alpha_2} \left(L^{\frac{1}{2},\rho_1}(\mu)\right)^{j\beta_1}_{i\alpha_1} \quad (3)$$

Finally by solving (3) for every pair of solutions of (2) we get a collection of R-matrices which satisfy the most general Yang-Baxter equation:

$$R^{\rho_1\rho_2}_{12}(\lambda)R^{\rho_1\rho_3}_{13}(\lambda+\mu)R^{\rho_2\rho_3}_{23}(\mu) = R^{\rho_2\rho_3}_{23}(\mu)R^{\rho_1\rho_3}_{13}(\lambda+\mu)R^{\rho_1\rho_2}_{12}(\lambda) \quad (4)$$

We can also write all the above equations in diagrammatic form. We do this in the fingures 3,4 and 5.

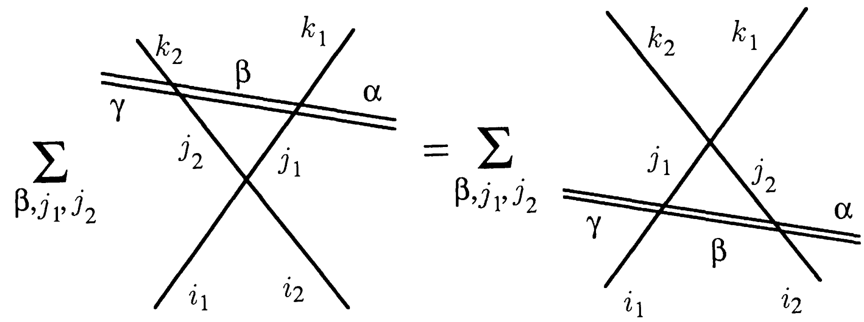

Figure 3: Diagrammatic form of (2). The intersection between single lines goes for the R-matrix while between single lines and the double line goes for the lax operators

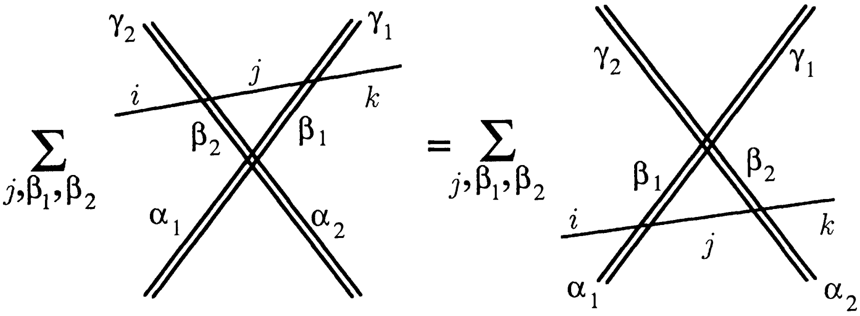

Figure 4: Diagrammatic form of (3). The intersection between double lines goes for the R-matrix while between double lines and single line goes for the lax operators

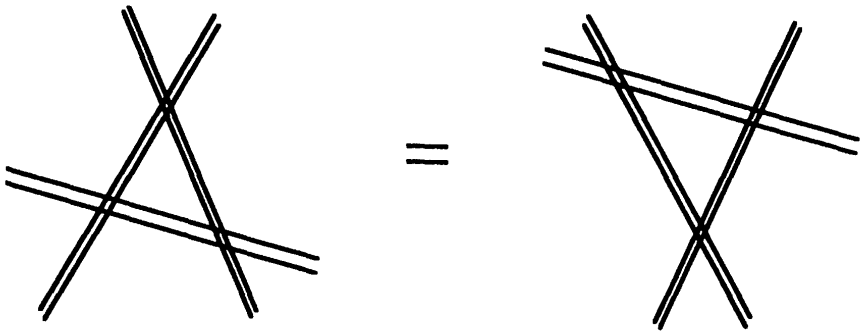

Figure 5: Diagrammatic form of (4). All the intersections stands for R-matrices acting on higher dimensional auxilary and quantum spaces

We can achieve a better mathematical understanding of the above equations by considering the R-matrices as intertwiners between either $\frac{1}{2}-\frac{1}{2}$, $\frac{1}{2}-\rho$ or $\rho-\rho$ representations. In this sense, all the above equations are nothing more than the one basic equation which reflects the quasi-triangularity of the affine hopf algebra $U_q\left(A_1^{(1)}\right)$ projected in different representations.

2 Construction of XXX_s model Hamiltonian

In this section we will try to construct the Hamiltonian for the higher spin case of the XXX model. We will start by introducing a slightly different notation for the lax operators and R-matrices. In the next step we will introduce a new monodromy and transfer matrix. In order to proceed further, we will need to find another operator, the fundamental lax operator. Finally, we will introduce another monodromy and transfer matrix, and we will write down the Hamiltonian of the model.

2.1 Lax operators, monodromy and transfer matrices

As I have alreay mentioned we will use slightly different notation. We will write the lax operators as:

$$L = L_{n,\alpha}(\lambda) = \lambda I_n \otimes I_\alpha + i\sum_a S_n^a \otimes \sigma_\alpha^a = \begin{pmatrix} \lambda I_n + iS_n^3 & iS_n^- \\ iS_n^+ & \lambda I_n - iS_n^3 \end{pmatrix}, \quad (5)$$

where the α goes for the auxilary space and n goes for the quantum space. Moreover, we will write the R-matrix as $R = R_{\alpha_1,\alpha_2}$. Hence, the comutation relation takes the form:

$$L_{n,\alpha_1}(\lambda)L_{n,\alpha_2}(\mu)R_{\alpha_1,\alpha_2}(\lambda-\mu) = R_{\alpha_1,\alpha_2}(\lambda-\mu)L_{n,\alpha_2}(\mu)L_{n,\alpha_1}(\lambda) \quad (6)$$

By using the above lax operators we can define a new monodromy matrix

$$T_\alpha(\lambda) = \prod_{n=1}^{L} L_{n,\alpha} = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \quad (7)$$

from which, and because the (6) is satisfied, we get the commutative family of transfer matrices:

$$t(\lambda) = tr_\alpha T_\alpha = A(\lambda) + D(\lambda) \quad (8)$$

with:

$$[t(\lambda), t(\mu)] = 0 \quad (9)$$

2.2 Diagonalization of the transfer matrix

In this subsection we will try to diagonalize tha transfer matrix following the same procedure we have seen in the Algebraic Bethe Ansatz. We start by introducing our refernce state $|\Omega>$:

$$|\Omega\rangle = |\omega_1\rangle\,|\omega_2\rangle \cdots |\omega_L\rangle \quad (10)$$

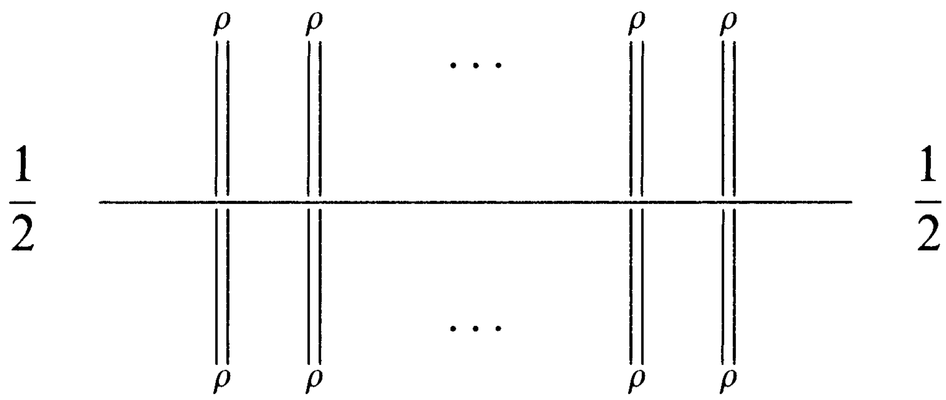

Figure 6: Diagrammatic form of Monodromy matrix. The vertical double lines stand for the higher dimensional local quantum spaces while the single horizontal line for the two dimensional auxilary space

where $|\omega_i\rangle$ is the local vacuum. $|\Omega\rangle$ is a highest weight state, this means that the creation operator S^+ kills $|\Omega\rangle$. We also introduce the state with M spin lower than the highest:

$$|\Psi_M(\lambda)\rangle = B(\lambda_1)\cdots B(\lambda_M)\,|\Omega\rangle \tag{11}$$

Next we will act with the transfer matrix to this second state, but before we need to check how does the various elements of the transfer matrix act on our refernce state, hence we have that:

$$A(\lambda)\,|\Omega\rangle = (\lambda + is)^L \tag{12}$$

$$D(\lambda)\,|\Omega\rangle = (\lambda - is)^L \tag{13}$$

$$C(\lambda)\,|\Omega\rangle = 0 \tag{14}$$

We get the first two results by considering the matrix multiplication of L lax operators. If we do the calculations we can find that each element of A and D kills the vacuum apart from the term $(\lambda \pm is)^L$. After that, we should find the eigenvalue of the transfer matrix for the state $|\Psi_M\rangle$. In order to do this, we will also need the commutators between A, D and B, which are:

$$A(\lambda)B(\mu) = \frac{\lambda-\mu-i}{\lambda-\mu}B(\mu)A(\lambda) + \frac{i}{\lambda-\mu}B(\lambda)A(\mu) \tag{15}$$

$$D(\lambda)B(\mu) = \frac{\lambda-\mu+i}{\lambda-\mu}B(\mu)D(\lambda) - \frac{i}{\lambda-\mu}B(\lambda)D(\mu) \tag{16}$$

By using the relations (12)-(16) we can find the eigenvalue of the transfer matrix for the state $|\Psi_M\rangle$ which is:

$$t(\lambda)\,|\Psi_M\rangle = (\lambda + is)^L \prod_{j=1}^{M} \frac{\lambda - \lambda_j - i}{\lambda - \lambda_j} + (\lambda - is)^L \prod_{j=1}^{M} \frac{\lambda - \lambda_j + i}{\lambda - \lambda_j} \tag{17}$$

from where we get the bethe ansatz equation:

$$\left(\frac{\lambda_k + is}{\lambda_k - is}\right)^L = \prod_{j\neq k, j=1}^{M} \frac{\lambda_k - \lambda_j + i}{\lambda_k - \lambda_j - i} \tag{18}$$

2.3 The fundamental lax operator

We are not ready yet to construct the Hamiltonian, and thats because our lax operator $L_{n,\alpha}$ acts on two essentially different spaces, the auxilary and the quantum which are 2-d and 2s+1-d Hilbert spaces respectively. Therefor there is no point λ for which $L_{n,\alpha}(\lambda)$ can act as a permutation. We can solve this problem if we find another lax operator for which the auxilary space will be the same as the quantum space. This operaror is called the fundamental lax operator and we will denote it as $L_{n,f}$.

We will find this operator by using the abstract Yang-Baxter equation in a slightly different form than before:

$$R_{12}R_{32}R_{31} = R_{31}R_{32}R_{12} \tag{19}$$

we can achieve easily this form starting from the common yang-baxter equation by using permutations. The next step is to apply the represantation $\rho(s_1,\lambda)\otimes\rho(s_2,\mu)\otimes\rho(\frac{1}{2},0)$ on the above equation. If we do this, we get:

$$R^{s_1,s_2}(\lambda-\mu)R^{\frac{1}{2},s_2}(-\mu)R^{\frac{1}{2},s_1}(-\lambda) = R^{\frac{1}{2},s_1}(-\lambda)R^{\frac{1}{2},s_2}(-\mu)R^{s_1,s_2}(\lambda-\mu) \tag{20}$$

We can identify the $R^{\frac{1}{2},s}$ with the lax operators we 've already had, but the R^{s_1,s_2} gives a new operator. In the case that $s_1 = s_2$, this is the operator we are searching for. We will try to calculate this new operator. In orde to make simpler the calculations we will use two different spin variables (S,T) and so our lax operators take the for:

$$L_T(\lambda) = \lambda + i(T,\sigma) \tag{21}$$

$$L_S(\lambda) = \lambda + i(S,\sigma) \tag{22}$$

We will also search for R^{s_1,s_2} in the form:

$$R^{s_1,s_2}(\lambda) = P^{s_1,s_2} r((T,S),\lambda) \quad , \tag{23}$$

where P^{s_1,s_2} is the permutation between s_1 and s_2, $(T,S) = \sum_a T^a \otimes S^a = C$ is the casimir element, and $r(C,\lambda)$ is just a function. By substituting (21)-(23) to (20) we get:

$$\begin{aligned} P^{s_1,s_2} r(C,\lambda-\mu)\,(\mu - i(T,\sigma))\,(\lambda - i(S,\sigma)) \\ = (\lambda - i(S,\sigma))\,(\mu - i(T,\sigma))\,P^{s_1,s_2} r(C,\lambda-\mu) \end{aligned} \tag{24}$$

in this point we will use the fact that:

$$(T,\sigma)(S,\sigma) = (T,S) + i\,((S\times T),\sigma) \tag{25}$$

and that:

$$P(T,\sigma)P = (S,\sigma) \tag{26}$$

By considering the above we end up with the equation:

$$\begin{aligned} r(C,\lambda)\,(\lambda(S,\sigma) + ((T\times S),\sigma) = (\lambda(T,\sigma) + ((T\times S),\sigma))\,r(C,\lambda) \Rightarrow \\ r(C,\lambda)\,(\lambda S^a + (T\times S)^a) = (\lambda T^a + (T\times S)^a)\,r(C,\lambda) \quad , \end{aligned} \tag{27}$$

where we have made the substitution $\lambda-\mu \to \lambda$. (27) consist actually of three equations, one for each axis x,y and z. However, because of the symmetry of XXX_s model we can consider just one of the equations. For simplicity we will consider the combination

$$\begin{aligned} r(C,\lambda)\left[\lambda(S^1 + iS^2) + (T\times S)^1 + i(T\times S)^2\right] = \\ \left[\lambda(T^1 + iT^2) + (T\times S)^1 + i(T\times S)^2\right] r(C,\lambda) \Rightarrow \end{aligned}$$

$$r(C,\lambda)\left(\lambda S^+ + i[T^3S^+ - S^3T^+]\right) = \left(\lambda T^+ + i[T^3S^+ - S^3T^+]\right) r(C,\lambda) \tag{28}$$

in a highest weight subspace. In such a subspace $T^+ + S^+ = 0$, and also the element $J = S^3 + T^3$ is constant and it can play the role of the casimir element. Therefore we will use J, insead of C, from now on. By taking all these into account the last equation takes the form:

$$r(J,\lambda)\big(\lambda S^+ + iJS^+\big) = \left(-\lambda S^+ + iJS^+\right) r(J,\lambda) \tag{29}$$

in addition to the previous, we also consider that:

$$S^+J = (J-1)S^+ \tag{30}$$

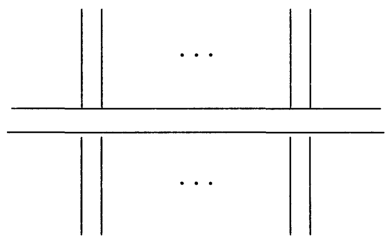

Figure 7: Diagramatic form of the monodromy matrix T_f. The horizontal double line stands for the higher dimensional auxilary space while vertical double lines stand for the local quantum spaces

hence,

$$r(J,\lambda)\,(\lambda+iJ) = (-\lambda+i(J-1))\,r((J-1),\lambda) \tag{31}$$

which has as solution:

$$r(J,\lambda) = \frac{\Gamma(J+1+i\lambda)}{\Gamma(J+1-i\lambda)} \tag{32}$$

and so the fundamental lax operator has the form:

$$L_{n,f} = P\frac{\Gamma(J+1+i\lambda)}{\Gamma(J+1-i\lambda)} \tag{33}$$

2.4 Another Monodromy and Transfer matrix

Now that we have in hand the fundamental lax operator we can define one more monodromy matrix, which has the form:

$$T_f(\lambda) = \prod_{i=1}^{L} L_{i,f} \tag{34}$$

from which we get the transfer matrix:

$$t_f(\lambda) = trT_f \tag{35}$$

Furthermore, because the fundamental lax operators satisfy the more general commutator relation:

$$R_{f_1,f_2}(\lambda-\mu)L_{n,f_1}(\lambda)L_{n,f_2}(\mu) = L_{n,f_2}(\mu)L_{n,f_1}(\lambda)R_{f_1,f_2}(\lambda-\mu) \tag{36}$$

we can show that t_f consist a commutative family of operators, which means that:

$$[t_f(\lambda), t_f(\mu)] = 0 \quad . \tag{37}$$

On top of that, because in (20) we can identify the $R^{\frac{1}{2},s}$ with the $L_{n,\alpha}$ and the $R^{s,s}$ with the $L_{n,f}$, it is possible to prove the commutativity between that families of transfer matrices $t(\lambda)$ and $t_f(\lambda)$:

$$[t(\lambda), t_f(\mu)] = 0 \tag{38}$$

The last relation is really important because it shows that the two families of transfer matrices have the same eigenvectors, and so we can use $t_f(\lambda)$ to get observables while we can use $t(\lambda)$ to make the construct the Bethe Ansatz Equation.

2.5 Calculating the Hamiltonian of XXX_s model

Finally we are ready to calculate the Hamiltonian. We start from the formula:

$$\begin{aligned}
H &= i\frac{d}{d\lambda}ln(t_f(\lambda))|_{\lambda=0} = i\sum_{i=1}^{L}\frac{d}{d\lambda}r(J,\lambda)|_{\lambda=0} \Rightarrow \\
H &= i\sum_{i=1}^{L}\frac{\Gamma'(J+1+i\lambda)\Gamma(J+1) - \Gamma(J+1)\Gamma'(J+1-i\lambda)}{\Gamma^2(J+1)} \Rightarrow \\
H &= i\sum_{i=1}^{L}\frac{\Gamma'(J+1+i\lambda) - \Gamma'(J+1-i\lambda)}{\Gamma(J+1)} \Rightarrow \\
H &= \sum_{i=1}^{L}(-2\sum_{j=1}^{2s}\frac{1}{j} - \gamma) \\
H &= \sum_{i=1}^{L}\left(\sum_{j=1}^{2s}C_j(\sum_a S_n^a S_{n+1}^a)^j)\right)
\end{aligned} \tag{39}$$

In the case of spin-1 particles, $C_1 = -C_2$, hence:

$$H = \sum_{a,n}\left(S_n^a S_{n+1}^a - (S_n^a S_{n+1}^a)^2\right) \tag{40}$$

As we can see, this is not just a naive generalisation of the Hamiltonian of the spin-$\frac{1}{2}$ case, but it has one more term. However it preserve all the symmetries of the spin-$\frac{1}{2}$ hamiltonian.

3 Conclusion

In this notes, first, we introduce a new set of operators, the lax operators, which we use to generalise the relation $TTR = RTT$. We did this in three steps. Our final relation was among R-matrices for which both auxilary and quantums space were $2s+1$ dimensional spaces. In the next part we worked on the XXX_s model. We gave a more specific form to the lax operators, and the generalised relations $LLR = RLL$, and we used them to find the fundamental lax operator, a very important entity for the construction of the Hamiltonian. Furthemore, we introduce two new pairs of monodromy and transfer matrices, and we saw that the two families of transfer matrices have the same base of eigenvectors. Finally, we used the transfer matrix t_f to construct the Hamiltonian.The construction of the integrable hamiltonians for spin s magnetic chains is one of real achievements of the Algebraic Bethe Ansatz.

4 Appendix

First, let's take a more detailed look to the calculations between (24) and (27), we start from (24)

$$
\begin{aligned}
&P^{s_1,s_2} r(C,\lambda-\mu)\,(\mu - i(T,\sigma))\,(\lambda - i(S,\sigma)) = \\
&(\lambda - i(S,\sigma))\,(\mu - i(T,\sigma))\,P^{s_1,s_2} r(C,\lambda-\mu) \Rightarrow \\
&P^{s_1,s_2} r(C,\lambda-\mu)\,(\mu - i(T,\sigma))\,(\lambda - i(S,\sigma)) = \\
&P^{s_1,s_2}P^{s_1,s_2}\,(\lambda - i(S,\sigma))\,P^{s_1,s_2}P^{s_1,s_2}\,(\mu - i(T,\sigma))\,P^{s_1,s_2} r(C,\lambda-\mu) \Rightarrow
\end{aligned}
$$

using that $P(T,\sigma)P = (S,\sigma)$ we get:

$$r(C,\lambda-\mu)\,(\mu - i(T,\sigma))\,(\lambda - i(S,\sigma)) = (\lambda - i(S,\sigma))\,(\mu - i(T,\sigma))\,r(C,\lambda-\mu)$$

considering the Pauli matrix property $(T,\sigma)(S,\sigma) = (T,S) + i((S\times T),\sigma)$ we get

$$
\begin{aligned}
&r(C,\lambda-\mu)\,(-i\mu(S,\sigma) - i\lambda(T,\sigma) - (T,S) + i((T\times S),\sigma)) = \\
&\qquad (-i\mu(T,\sigma) - i\lambda(S,\sigma) - (T,S) + i((T\times S),\sigma))\,r(C,\lambda-\mu) \Rightarrow \\
&((\lambda-\mu)(T,\sigma) + i((T\times S),\sigma))\,r(C,\lambda-\mu) = r(C,\lambda-\mu)\,((\lambda-\mu)(S,\sigma) + i((T\times S),\sigma))
\end{aligned}
$$

finally by setting $\lambda - \mu \to \lambda$ we get:

$$(\lambda T^a + i(T\times S)^a)\,r(C,\lambda) = r(C,\lambda)\,(\lambda S^a + i(T\times S)^a)$$

Next we will see how did we get to (28). Considering (27) in one direction we get:

$$r(C,\lambda)(\lambda S^1 + (T\times S)^1) = (\lambda T^1 + (T\times S)^1)r(C,\lambda)$$
$$ir(C,\lambda)(\lambda S^2 + (T\times S)^2) = i(\lambda T^2 + (T\times S)^2)r(C,\lambda)$$

by adding the two equations and use the fact that $S^+ = S^1 + iS^2$ we get:

$$r(C,\lambda)(\lambda S^+ + (T\times S)^1 + i(T\times S)^2) =$$
$$(\lambda T^+ + (T\times S)^1 + i(T\times S)^2)r(C,\lambda) \Rightarrow$$
$$r(C,\lambda)(\lambda S^+ + \frac{1}{2}[(T\times S)^1 - (S\times T)^1] + \frac{i}{2}[[(T\times S)^2 - (S\times T)^2]) =$$
$$(\lambda T^+ + \frac{1}{2}[(T\times S)^1 - (S\times T)^1] + \frac{i}{2}[[(T\times S)^2 - (S\times T)^2])r(C,\lambda) \Rightarrow$$
$$r(C,\lambda)(\lambda S^+ + \frac{i}{2}[-T^+S^3 + T^3S^+ + S^+T^3 - S^3T^+]) =$$
$$(\lambda T^+ + \frac{i}{2}[-T^+S^3 + T^3S^+ + S^+T^3 - S^3T^+])r(C,\lambda) \Rightarrow$$
$$r(C,\lambda)(\lambda S^+ + i(T^3S^+ - S^3T^+)) = (\lambda T^+ + i(T^3S^+ - S^3T^+))r(C,\lambda)$$