The Yangian $Y(\mathfrak{gl}_2)$, the quantum determinant and back to the XXX model

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1 Introduction

In the previous lectures we defined the abstract notions of Hopf algebra, enveloping algebra, and quasi-triangularity. They were introduced as mathematical structures, but we found out that they were intimately linked with the Yang Baxter equation and the notion of Yang Baxter algebra. In this lecture we briefly review those concepts and work out a practical example, namely the Yangian of the Lie algebra $\mathfrak{gl}_n$, to see how those abstract mathematical notions connect and can be used from physicists to better understand spin models.
The construction we are going to define provides a complete definition of the Yang Baxter algebra of the XXX model and can help in contextualizing our theory so far. By explicitly building the Yangian, one can achieve a deeper understanding of the XXX model, and open the door to generalizations to less trivial examples. As it turns out, Yangian symmetries appear also in the Hubbard model, 1+1D relativistic quantum field theory and supersymmetric Yang-Mills theory in 4D. In mathematics the study of $Y(\mathfrak{gl}_n)$ is linked with the representation theory of $\mathfrak{gl}_n$.
The key to understand the dynamic and the solutions to quantum spin models rely in the Yang Baxter equation, and in the RTT relation, which together define the so-called Algebraic Bethe Ansatz. A lot of work has been done to prove that the Bethe Ansatz provides a tool to solve spin chains models, and the completeness of the method has been proven for a class of them. Constructing explicitly the Yangian provides a tool to prove the completeness of the Algebraic Bethe Ansatz in the case of the XXX model, a non trivial result. Although this last part is not exposed here, the interested reader is referred to [5].

2 Defining the Yangian

Consider the Lie Algebra $\mathfrak{gl}_n$, generated by the elements $\{E_{ij}\}$, with $0 \leq i, j \leq n$, with abstract commutation relations:

$$[E_{ij}, E_{kl}] = \delta_{kj} E_{il} - \delta_{il} E_{kj} \tag{1}$$

The generators can be pictured as matrices composed of zeros and one in position ij:

$$E_{ij} = \begin{pmatrix} 0 & \cdots & 0 & \cdots & \\ \vdots & \ddots & \vdots & & \\ 0 & \cdots & 1 & \cdots & \\ \vdots & & \vdots & \ddots & \end{pmatrix}$$

Recall the construction of the enveloping algebra, as:

$$U(\mathfrak{gl}_n) \equiv \bigoplus_{k=0}^{\infty} \mathfrak{gl}_n^{\otimes k} / _{[X,Y]=XY-YX}$$

with $\mathfrak{gl}_n^{\otimes 0} \equiv \mathbb{C}$. In the enveloping algebra we can construct the following matrix:

$$E = \sum_{i,j=1}^{n} E_{ij} \otimes E_{ij} = \begin{pmatrix} E_{11} & E_{12} & \cdots \\ E_{21} & E_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Which has as matrix elements the generators of $\mathfrak{gl}_n$. We will then look at (tensor) powers of the matrix E, namely E^s, which we can write as:

$$(E^s)_{ij} = \sum_{a,b,\cdots,l=1}^{n} \overbrace{E_{ia} \otimes E_{ab} \otimes \cdots \otimes E_{lj}}^{s \text{ times}}$$

For which we can prove, inductively, that:

$$[(E^{s+1})_{ij}, (E^r)_{kl}] - [(E^s)_{ij}, (E^{r+1})_{kl}] = (E^s)_{kj}(E^r)_{il} - (E^r)_{kj}(E^s)_{il}$$

Inspired by this construction we define the Yangian $Y(\mathfrak{gl}_n)$ as the unital associative algebra, generated by $\{t_{ij}^{(n)}\}$, with $n \in \mathbb{N}$, $0 \leq i, j \leq n$, such that:

$$[t_{ij}^{(s+1)}, t_{kl}^{(r)}] - [t_{ij}^{(s)}, t_{kl}^{(r+1)}] = t_{kj}^{(s)} t_{il}^{(r)} - t_{kj}^{(r)} t_{il}^{(s)} \tag{2}$$

and we define $t_{ij}^{(0)} = \delta_{ij}$.
It may seem that we have done nothing, just renamed the powers of the matrix E^s and called them $t^{(s)}$. However, it turns out that the Yangian can be defined for any algebra $\mathfrak{a}$ and, quite interestingly, the case of $\mathfrak{gl}_n$ has a natural homomorphism, $Y(\mathfrak{gl}_n) \to U(\mathfrak{gl}_n)$, which is actually what we wrote above. This would also be a representation of the Yangian.

3 The Yangian of $\mathfrak{gl}_2$

From now on we will work with setting $n = 2$, to have a more intuitive picture of what we are doing and because this is enough to study the Yang Baxter algebra of the XXX model. However the generalization of what follows to the case of $\mathfrak{gl}_n$ is straightforward.
We will first define the formal serie:

$$t_{ij}(u) \equiv \sum_{n=0}^{\infty} t_{ij}^{(n)} u^{-n} \in Y(\mathfrak{gl}_2)[[u]] \tag{3}$$

The word “formal” means that we write $t_{ij}(u)$ as a serie, but we do not require it to converge. It is a useful tool that will allow us to pack the defining relations (2) as[1]:

$$(u-v)[t_{ij}(u), t_{kl}(v)] = t_{kj}(u)t_{il}(v) - t_{kj}(v)t_{il}(u) \tag{4}$$

Proof (4) $\iff$ (2): Expanding the left hand side of (4) yields to:

$$\begin{aligned}(u-v)[t_{ij}(u), t_{kl}(v)] &= \sum_{n,m=0}^{\infty}\left([t_{ij}^{(n)}, t_{kl}^{(m)}]u^{-n+1}v^{-m} - [t_{ij}^{(n)}, t_{kl}^{(m)}]u^{-n}v^{-m+1}\right) \\ &= \sum_{n,m=0}^{\infty}\left([t_{ij}^{(n+1)}, t_{kl}^{(m)}] - [t_{ij}^{(n)}, t_{kl}^{(m+1)}]\right)u^{-n}v^{-m} \\ &= \sum_{n,m=0}^{\infty}\left(t_{kj}^{(n)}t_{il}^{(m)} - t_{kj}^{(m)}t_{il}^{(n)}\right)u^{-n}v^{-m}\end{aligned} \tag{5}$$

Where in the second step we just renamed the indices in the sum. □

With the formal series $t_{ij}(u)$ we can now define the monodromy matrices, $T_i(u)$:

$$\begin{aligned}T_1(u) &= \sum_{i,j=1}^{2} t_{ij}(u)\otimes E_{ij}\otimes \mathbb{1} \in Y(\mathfrak{gl}_2)[[u]]\otimes \mathrm{End}(\mathbb{C}^2)\otimes \mathrm{End}(\mathbb{C}^2) \\ T_2(u) &= \sum_{i,j=1}^{2} t_{ij}(u)\otimes \mathbb{1}\otimes E_{ij} \in Y(\mathfrak{gl}_2)[[u]]\otimes \mathrm{End}(\mathbb{C}^2)\otimes \mathrm{End}(\mathbb{C}^2)\end{aligned} \tag{6}$$

Really what the above expressions mean is that the monodromy matrices are 4×4 matrices:

$$T_1(u) = \begin{pmatrix} t_{11}(u) & t_{12}(u) \\ t_{21}(u) & t_{22}(u)\end{pmatrix}\otimes \mathbb{1} = \begin{pmatrix} t_{11}(u) & t_{12}(u) & & \\ t_{21}(u) & t_{22}(u) & & \\ & & t_{11}(u) & t_{12}(u) \\ & & t_{21}(u) & t_{22}(u)\end{pmatrix}$$

$$T_2(u) = \begin{pmatrix} t_{11}(u)\otimes\mathbb{1} & t_{12}(u)\otimes\mathbb{1} \\ t_{21}(u)\otimes\mathbb{1} & t_{22}(u)\otimes\mathbb{1}\end{pmatrix} = \begin{pmatrix} t_{11}(u) & & t_{12}(u) & \\ & t_{11}(u) & & t_{12}(u) \\ t_{21}(u) & & t_{22}(u) & \\ & t_{21}(u) & & t_{22}(u)\end{pmatrix}$$

Having constructed the monodromy matrices we need, for completing the Yang Baxter Algebra structure, the R matrix. We will define it in terms of the permutation operator:

$$P = \sum_{i,j=1}^{2} E_{ij}\otimes E_{ji} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1\end{pmatrix} \in \mathrm{End}(\mathbb{C}^2)\otimes\mathrm{End}(\mathbb{C}^2)$$

[1]later on we will identify the formal variable u as the rapidities of the model, although this has to be done after fixing a representation.

And our R matrix will be:

$$R = \mathbb{1} - P \cdot u^{-1} = \begin{pmatrix} 1-u^{-1} & 0 & 0 & 0 \\ 0 & 1 & -u^{-1} & 0 \\ 0 & -u^{-1} & 1 & 0 \\ 0 & 0 & 0 & 1-u^{-1} \end{pmatrix} \tag{7}$$

The defining relations (4) can be rewritten as:

$$R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v) \tag{8}$$

Proof (8) $\iff$ (4): We start by calculating the action of the permutation operator on the basis elements of $\mathrm{End}(\mathbb{C}^2)\otimes\mathrm{End}(\mathbb{C}^2)$:

$$\begin{aligned} P \cdot (E_{ij} \otimes E_{kl}) &= \sum_{i',j'} \sum_a E_{i'a}E_{aj} \otimes E_{j'a}E_{al} \\ &= E_{kj} \otimes E_{il} \end{aligned} \tag{9}$$

And when we multiply from the right:

$$\begin{aligned} (E_{ij} \otimes E_{kl}) \cdot P &= \sum_{i',j'} \sum_a E_{ia}E_{aj'} \otimes E_{ka}E_{ai'} \\ &= E_{il} \otimes E_{kj} \end{aligned} \tag{10}$$

Then (8) is:

$$\begin{aligned} (\mathbb{1} - (u-v)^{-1}P)T_1(u)T_2(v) &= T_2(v)T_1(u)(\mathbb{1} - (u-v)^{-1}P) \\ (u-v)\left(T_1(u)T_2(v) - T_2(v)T_1(u)\right) &= PT_1(u)T_2(v) - T_2(v)T_1(u)P \end{aligned}$$

By expanding the latter we obtain, on the left hand side:

$$\begin{aligned} &\sum_{i,j,k,l} (u-v)(t_{ij}(u)t_{kl}(v) - t_{kl}(v)t_{ij}(u)) \otimes E_{ij} \otimes E_{kl} = \\ &= \sum_{i,j,k,l} (u-v)[t_{ij}(u), t_{kl}(v)] \otimes E_{ij} \otimes E_{kl} \end{aligned} \tag{11}$$

While the right hand side will be expanded as:

$$\begin{aligned} &\sum_{i,j,k,l} t_{ij}(u)t_{kl}(v) \otimes P \cdot (E_{ij} \otimes E_{kl}) - t_{kl}(v)t_{ij}(u) \otimes (E_{ij} \otimes E_{kl}) \cdot P \\ &= \sum_{i,j,k,l} t_{ij}(u)t_{kl}(v) \otimes E_{kj} \otimes E_{il} - t_{kl}(v)t_{ij}(u) \otimes E_{il} \otimes E_{kj} \\ &= \sum_{i',j',k',l'} (t_{k'j'}(u)t_{i'l'}(v) - t_{k'j'}(v)t_{i'l'}(u)) \otimes E_{i'j'} \otimes E_{k'l'} \end{aligned} \tag{12}$$

Then, by comparing (11) with (12) we obtain (4) again. □

With this we have proven that the Yangian, defined by (2), or more precisely $Y(\mathfrak{gl}_2)[[u]]$ is a Yang Baxter algebra. Furthermore, one can show by using the permutation properties that the R matrix defined above satisfies the quasi-triangularity axiom, thus $Y(\mathfrak{gl}_2)[[u]]$ is also a quasi triangular algebra.

4 Hopf algebra structure, automorphisms and representations

Recall that, in order to build a Hopf algebra structure, we need the co-multiplication, the co-unit and the antipode maps, which we will define as:

$$\begin{aligned}\Delta(t_{ij}(u)) &= \sum_a t_{ia}(u) \otimes t_{aj}(u) \\ \epsilon(T(u)) &= 1 \\ S(T(u)) &= T^{-1}(u)\end{aligned}$$

Note that those definition, given on the monodromy matrices, really specify the action on all generators. For example we have:

$$\Delta(t_{ij}^{(n)}) = \sum_{s+r=n} \sum_a t_{ia}^{(s)} \otimes t_{aj}^{(r)}, \text{with } s, r \in \mathbb{N}$$

And the antipode, on the first coefficients:

$$\begin{aligned}S(t_{ij}^{(0)}) &= t_{ij}^{(0)} \\ S(t_{ij}^{(1)}) &= -t_{ij}^{(1)} \\ S(t_{ij}^{(2)}) &= -t_{ij}^{(2)} + \sum_a t_{ia}^{(1)} t_{aj}^{(1)}\end{aligned}$$

With these definitions the Yangian is indeed turned into an Hopf algebra.

We now look at automorphisms, which turn out to be useful later:

$$\begin{aligned}T(u) &\longmapsto BT(u)B^{-1}, \quad B \in \mathrm{End}(\mathbb{C}^2) \otimes \mathrm{End}(\mathbb{C}^2) \\ T(u) &\longmapsto T(u-z), \quad \text{with } z \in \mathbb{C} \\ T(u) &\longmapsto f(u)T(u), \quad \text{with} f(u) = 1 + f_1 u^{-1} + f_2 u^{-2} + \cdots\end{aligned} \tag{13}$$

For the following we will need the anti-automorphisms:

$$\begin{aligned}T(u) &\longmapsto T(-u) \\ T(u) &\longmapsto T^t(u) \equiv \sum_{i,j=1}^{2} t_{ij}(u) \otimes E_{ji}\end{aligned} \tag{14}$$

We can now find the representation:

$$\begin{aligned}Y(\mathfrak{gl}_2) &\longrightarrow U(\mathfrak{gl}_2) \\ t_{ij}(u) &\longmapsto \mathbb{1} + E_{ij} u^{-1}\end{aligned} \tag{15}$$

Proof: To prove that (15) is a representation of $Y(\mathfrak{gl}_2)$ we first use the automorphism[2]:

$$\begin{aligned}T(u) &\longmapsto T^t(-u) \\ T(u) &\longmapsto \mathbb{1} - \sum_{i,j} E_{ij} \otimes E_{ji} \cdot u^{-1} = \mathbb{1} - u^{-1} P\end{aligned}$$

[2]Since it is a composition of two anti automorphisms.

The RTT relation then becomes:

$$(\mathbb{1}-(u-v)^{-1}P_{23})(\mathbb{1}-u^{-1}P_{12})(\mathbb{1}-v^{-1}P_{13}) =$$
$$(\mathbb{1}-v^{-1}P_{13})(\mathbb{1}-u^{-1}P_{12})(\mathbb{1}-(u-v)^{-1}P_{23})$$

And it holds, as can easily be verified by using the permutation relations. □

We end this section by mentioning the inclusion map of the enveloping algebra in the Yangian. We have:

$$U(\mathfrak{gl}_2) \hookrightarrow Y(\mathfrak{gl}_2)$$
$$E_{ij} \mapsto t_{ij}^{(1)}$$

5 The quantum determinant and the center of $Y(\mathfrak{gl}_2)$

The Hamiltonian of integrable spin chain systems can be thought as an element of a commuting family of operators, the transfer matrices, which also generate all the conserved charges of the system. We derived in the previous lectures that:

$$t(u) = \mathrm{Tr}_a(T_a(u))$$

Where V_a was the auxiliary space. In analogy with what we have seen for the 6-vertex model, the auxiliary space we are going to use is $\mathbb{C}^2$. Inspired by this analogy we define the transfer matrix as:

$$t(u) = t_{11}(u) + t_{22}(u)$$

Each coefficient of the above serie is commuting with all the others. However, they are not in the center of $Y(\mathfrak{gl}_2)$.
The actual center is generated by the coefficients of the serie[3]:

$$\begin{aligned} \mathrm{qdet}(T(u)) &= \sum_{\sigma\in\mathfrak{S}_n} \mathrm{sign}(\sigma)\cdot t_{\sigma(1)1}(u)\cdots t_{\sigma(n)n}(u-n+1), \text{ for the general } \mathfrak{gl}_n \text{ case} \\ \mathrm{qdet}(T(u)) &= t_{11}(u)t_{22}(u-1) - t_{12}(u)t_{21}(u-1), \qquad \text{for the } \mathfrak{gl}_2 \text{ case} \end{aligned} \tag{16}$$

Which is called the *quantum determinant*. Note that, in order to find all the coefficients in the *quantum determinant* serie one has first to expand $(u-1)$ in power serie of u, and then match each coefficient with the corresponding powers of u. The first coefficients are given by:

$$q^{(0)} = t_{11}^{(0)} + t_{22}^{(0)}$$
$$q^{(1)} = t_{11}^{(1)} + t_{22}^{(1)} - t_{12}^{(1)} - t_{21}^{(1)}$$
$$q^{(2)} = t_{11}^{(1)}t_{22}^{(1)} + t_{11}^{(2)} + t_{22}^{(2)} - t_{12}^{(1)}t_{21}^{(1)} - t_{12}^{(2)} - t_{21}^{(2)}$$

It is possible to prove [3] that $Y(\mathfrak{gl}_2) \cong Z \otimes Y(\mathfrak{sl}_2)$, where Z denotes the center of $Y(\mathfrak{gl}_2)$. This also means that, if we want to build the Yangian for $\mathfrak{sl}_2$, we can just take the quotient:

$$Y(\mathfrak{sl}_2) = Y(\mathfrak{gl}_2)/_{\mathrm{qdet}(T(u))=1}$$

[3]Here $\mathfrak{S}_n$ is the permutation group of n elements.

6 The Yang Baxter algebra of the XXX model

Recall from the previous lectures that we established a bridge between the lattices models in statistical physics, and the spin chains models: they share the same R matrix. This can be interpreted as identifying the time line in spin chains model with one of the spatial directions in the statistical lattice models. In physics we call a similar procedure a Wick rotation.
We are then justified to consider the R matrix of the $6v$ model, as we derived it:

$$R = \begin{pmatrix} a(u) & & & \\ & b(u) & c(u) & \\ & c(u) & b(u) & \\ & & & a(u) \end{pmatrix}$$

The XXZ Heisemberg model share the same R matrix, and has asymmetry parameter:

$$\Delta = \frac{a^2 + b^2 - c^2}{2ab}$$

We can easily find the XXX R matrix, by imposing that $\Delta = 1$, and noting that what actually defines the model are the relative values $a(u)/b(u)$ and $c(u)/b(u)$. We can thus use the parametrization:

$$\begin{aligned} a(u) &= 1 - u^{-1} \\ b(u) &= 1 \\ c(u) &= -u^{-1} \end{aligned}$$

Which will give us:

$$\Delta = \frac{1 + (1 - u^{-1})^2 - u^{-2}}{2(1 - u^{-1})} = 1$$

Exactly what we wanted. Furthermore, the R matrix will be:

$$R = \begin{pmatrix} 1 - u^{-1} & & & \\ & 1 & -u^{-1} & \\ & -u^{-1} & 1 & \\ & & & 1 - u^{-1} \end{pmatrix} = \mathbb{1} - Pu^{-1} \in \mathrm{End}(\mathbb{C}^2) \otimes \mathrm{End}(\mathbb{C}^2)$$

Which is exactly the same R matrix of $Y(\mathfrak{gl}_2)$. This is enough to claim that the Yangian we have defined before is indeed the Yang Baxter algebra of the XXX model. We also note that, thanks to the automorphisms (13) it does not actually matter which particular parametrization we choose, as we would expect in a physical model.
What one does at this point to solve the model is fix a representation of the Yangian, namely the one defined by (15), and use the commutation relations to diagonalize the hamiltonian, finding energy eigenstates and eigenvectors. This is equivalent to the procedure we analyzed

in talk 6, where we used the adjoint representation:

$$T_{11}(u) = \begin{pmatrix} 1-u^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$
$$T_{22}(u) = \begin{pmatrix} 1 & 0 \\ 0 & 1-u^{-1} \end{pmatrix}$$
$$T_{12}(u) = \begin{pmatrix} 0 & 0 \\ -u^{-1} & 0 \end{pmatrix}$$
$$T_{21}(u) = \begin{pmatrix} 0 & -u^{-1} \\ 0 & 0 \end{pmatrix}$$

Note also that to retrieve the Yangian of $\mathfrak{sl}_2$ we should take the limit $u \to +\infty$. In this case we find that the *quantum determinant*:

$$\text{qdet}(T(u)) = \frac{u-2}{u-1}\mathbb{1} \stackrel{u\to\infty}{\longrightarrow} \mathbb{1}$$

Studying other representations other than the ones we have analyzed yields to a connection between the Bethe equations and the existence of irreducible representation of the Yangian. With this procedure, in [5], the completeness of the Bethe Ansatz is proven for the XXX model.

7 The Yangian as a symmetry group

The hamiltonian for the XXX model can be written as:

$$H_{XXX} = \sum_k \sum_{i,j=1}^{2} S_k^{ij} S_{k+1}^{ji}$$

Where k denotes the lattice site and S^{ij} are the generators of $\mathfrak{sl}_2$. Those follows the same commutations as the generators of $\mathfrak{gl}_2$:

$$[S^{ij}, S^{kl}] = \delta^{kj} S^{il} - \delta^{il} S^{kj}$$

and are related to the usual Pauli matrices as:

$$\sigma^z = S^{11} - S^{22}$$
$$\sigma^+ = S^{12}$$
$$\sigma^- = S^{21}$$

Because the model is isotropic in spatial direction, we expect it to be invariant under the full rotation group. This yields to the conserved charge:

$$Q^1_{ij} = \sum_k S_k^{ij}$$

However, the symmetry group is much larger, and includes an infinite tower of conserved charges, progressively less local, involving thus $2, 3, \cdots$ sites:

$$Q^2_{ij} = \frac{1}{2}\sum_{l<k}\sum_{a} S^{ia}_l S^{aj}_k - S^{ia}_k S^{aj}_l$$

The two charges we just defined are generators of the Yangian of $\mathfrak{sl}_2$. If we impose the condition $\text{qdet}(T(u)) \longrightarrow \mathbb{1}$, all the charges and their commutation relations can be expressed as function of Q^1_{ij}, Q^2_{ij}.
The relations between the conserved charges and the abstract generators of $Y(\mathfrak{gl}_2)$ are given by:

$$Q^1_{ij} = t^{(1)}_{ij}$$
$$Q^2_{ij} = t^{(2)}_{ij} - \frac{1}{2}\sum_k t^{(1)}_{ik} t^{(1)}_{kj}$$

8 The Yangian of any Lie algebra $\mathfrak{a}$

Let $\mathfrak{a}$ be generated by the set $\{I_a\}_{a=1,\cdots,n}$, with Lie brackets:

$$[I_a, I_b] = f_{abc} I_c$$

And add a second set of generators $\{J_a\}$ in the adjoint representation of $\mathfrak{a}$:

$$[I_a, J_b] = f_{abc} J_c$$

With the coproduct:

$$\Delta(I_a) = \mathbb{1} \otimes I_a + I_a \otimes \mathbb{1}$$
$$\Delta(J_a) = \mathbb{1} \otimes J_a + J_a \otimes \mathbb{1} + \frac{h}{2} f_{abc} I_b \otimes I_c$$

The algebra generated by $\{I_a, J_a\}$ is called the Yangian of the algebra $\mathfrak{a}$, $Y(\mathfrak{a})$.
Here the parameter h is the deformation parameter that makes the Yangian a quantum group, and has the same role of the "Planck constant". Note also that, for $h \to 0$, we retrieve the enveloping algebra. In this sense the Yangian is a deformation of the enveloping algebra.
For the case of $Y(\mathfrak{sl}_2)$, we find the explicit relations between the generators $\{S^i, J(S^i)\}_{i=z,+,-}$, giving an Hopf algebra isomorphism [3]:

$$S^z = t^{(1)}_{11} - t^{(1)}_{22}$$
$$S^+ = t^{(1)}_{12}$$
$$S^- = t^{(1)}_{21}$$
$$J(S^+) = t^{(2)}_{12} - \frac{h}{2}(t^{(1)}_{11} + t^{(1)}_{22} - 1)t^{(1)}_{12}$$
$$J(S^-) = t^{(2)}_{21} - \frac{h}{2}(t^{(1)}_{11} + t^{(1)}_{22} - 1)t^{(1)}_{21}$$
$$J(S^z) = t^{(2)}_{11} - t^{(2)}_{22} - \frac{h}{2}(t^{(1)}_{11} + t^{(1)}_{22} - 1)(t^{(2)}_{11} - t^{(2)}_{22})$$

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