# Chapter 1

# The Coordinate Bethe Ansatz for the Heisenberg XXX model

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## 1.1 The model

In this chapter, we will be focussing on the Heisenberg XXX model. This spin chain model consists of Hilbert space $\mathcal{H}$ and a Hamiltonian operator $\hat{H} : \mathcal{H} \to \mathcal{H}$. Physically, this model describes a quantum-mechanical system of $L$ spin-1/2 particles forming the sites of a one-dimensional periodic (i.e. circular) lattice. The Hilbert space can be written as

$$\mathcal{H} = \bigotimes_{n \in \mathbb{Z}/L\mathbb{Z}} V_n, \tag{1.1}$$

where each two-dimensional complex vector space $V_n = \mathrm{span}(|\uparrow\rangle, |\downarrow\rangle) \cong \mathbb{C}^2$ is associated to one particle of the spin chain. Note that due to the periodicity of the lattice, the sum is performed over $\mathbb{Z}/L\mathbb{Z}$.

In terms of spin operators $S^i = \frac{1}{2}\sigma^i$ for $i \in \{x, y, z\}$ ($\hbar = 1$), where the $\sigma^i$ are the Pauli matrices, the Hamiltonian of this model can be written as

$$\hat{H} = -J \sum_{n \in \mathbb{Z}/L\mathbb{Z}} \left( S_n^x S_{n+1}^x + S_n^y S_{n+1}^y + S_n^z S_{n+1}^z \right). \tag{1.2}$$

where $S_n^i = I_{\mathbb{C}^2} \otimes ... \otimes I_{\mathbb{C}^2} \otimes S^i \otimes I_{\mathbb{C}^2} \otimes ... \otimes I_{\mathbb{C}^2}$, i.e. a tensor product of $L-1$ identity operators acting on $\mathbb{C}^2$ and a spin operator $S^i$ at the $n$th position. The parameter $J$ is called the coupling constant. In this chapter, we take $J > 0$, which physically corresponds to a ferromagnetic spin chain. Furthermore, the periodic boundary conditions of the lattice imply that $S_n^i = S_{n+L}^i$, which is also displayed by the sum over $\mathbb{Z}/L\mathbb{Z}$. It will prove useful to introduce the raising operator $S^+$ and the lowering operator $S^-$, which are given by $S^\pm = S^x \pm iS^y$ as usual. In matrix form, this yields

$$S^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad S^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{1.3}$$

With these definitions we rewrite the Hamiltonian as

$$\hat{H} = -\frac{J}{2} \sum_{n \in \mathbb{Z}/L\mathbb{Z}} \left( S_n^+ S_{n+1}^- + S_n^- S_{n+1}^+ + 2 S_n^z S_{n+1}^z \right). \tag{1.4}$$

Since we will frequently make use of it, we write down the action of the spin operators on the basis states $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ below:

$$\begin{aligned} S^+ |\uparrow\rangle &= 0, \;\; S^- |\uparrow\rangle = |\downarrow\rangle, \;\; S^z |\uparrow\rangle = \frac{1}{2} |\uparrow\rangle, \\ S^- |\downarrow\rangle &= 0, \;\; S^+ |\downarrow\rangle = |\uparrow\rangle, \;\; S^z |\downarrow\rangle = -\frac{1}{2} |\downarrow\rangle. \end{aligned} \tag{1.5}$$

The operators $S_n^z$ and $S_n^\pm$ act similarly on the spin at the $n$th lattice site, leaving spins at all other sites invariant.

What are the symmetries of this Hamiltonian? From the definition (1.4), it is clear that the Hamiltonian is translation invariant, corresponding to the symmetry group $\mathbb{Z}/L\mathbb{Z}$. Furthermore, using the commutation relations:

$$[S^x, S^y] = iS^z \text{ and cyclic permutations of this relation,} \tag{1.6}$$

it easily follows that

$$[\hat{H}, S^i] = 0 \tag{1.7}$$

for all $i \in \{x, y, z\}$. Thus, $\hat{H}$ commutes with all the generators of the Lie algebra $\mathfrak{su}(2)$, hence the Hamiltonian is $SU(2)$ invariant. The full symmetry group of the Hamiltonian is therefore $G = \mathbb{Z}/L\mathbb{Z} \times SU(2)$. For our purposes, it will suffice to concentrate on the $\mathbb{Z}/L\mathbb{Z} \times U(1)_z \subset G$, the subscript $z$ denoting that we focus on the $U(1)$ symmetry around the quantization axis.

The goal of this chapter is to find the (point) spectrum of the Hamiltonian of the Heisenerg XXX model together with the corresponding eigenvectors. As $\dim\mathcal{H} = 2^L$, there are $2^L$ of such eigenvectors, which means we must diagonalize the $2^L \times 2^L$ Hamiltonian matrix. This is a less than thrilling job. However, note that due to the symmetry of the Hamiltonian, in particular the total $z$-component of the spin is conserved. Therefore, if $\hat{H}$ acts on a state with a given number $M \in \{0, 1, ..., L\}$ of spins down, the resulting state will have $M$ spins down as well. Hence, for these states we can write

$$S^z = \frac{L}{2} - M. \tag{1.8}$$

and we can decompose the Hilbert space in invariant subspaces of the Hamiltonian:

$$\mathcal{H} = \bigoplus_{M=0}^{L} \mathcal{H}_M, \tag{1.9}$$

where $\mathcal{H}_M$ is the subspace of $\mathcal{H}$ containing states with $M$ spins down. Consequently, we can diagonalize the Hamiltonian by diagonalizing its restriction $\hat{H}|_{\mathcal{H}_M} : \mathcal{H}_M \to \mathcal{H}_M$ to each invariant subspace seperately.

In the following, we will perform this diagonalization for $M = 0$, $M = 1$ and $M = 2$. We will see that finding the spectrum for the first two cases requires little to no effort at all. The case $M = 2$ is less trivial. Here the Coordinate Bethe Ansatz will provide us with a method to find the spectrum. Subsequently, we can generalize the results obtained for $M = 2$ to general $M$. At the end of the chapter we will briefly discuss string solutions, which we can use to describe complexes of bound magnon states in the thermodynamic limit $L \to \infty$.

## 1.2 The case $M = 0$

For $M = 0$, we consider the invariant subspace $\mathcal{H}_0$, which is one dimensional and spanned by the vector $|0\rangle = |\uparrow\uparrow ... \uparrow\rangle$. Acting with the Hamiltonian on this state, we see that the first two terms inside the summation yield 0, because $S_n^-$ kills spin-up states. The last term inside the summation yields a factor $\frac{1}{2}$ for each $n \in \mathbb{Z}/L\mathbb{Z}$. Therefore $\hat{H}\,|0\rangle = E_0\,|0\rangle$, where the eigenvalue is given by

$$E_0 = -\frac{JL}{4}, \tag{1.10}$$

which concludes the spectrum for $M = 0$.

Note that while $|0\rangle$ is indeed the ground state of the ferromagnetic state, for the case $J < 0$ the notation $|0\rangle$ is deceiving! Indeed, for the anti-ferromagnetic state ($J < 0$) the ground state is a state in which nearest neighbours are misaligned. We can however safely ignore this subtlety, as we will only consider $J > 0$.

## 1.3 The case $M = 1$

For $M = 1$, we consider the invariant subspace $\mathcal{H}_1$, which has dimension $L$. It is spanned by the vectors consisting of 1 spin down at some site $n \in \mathbb{Z}/L\mathbb{Z}$ and $L-1$ spins up. We will denote these basis vectors as $|n\rangle = S_n^-\,|0\rangle = |\uparrow ... \uparrow\downarrow\uparrow ... \uparrow\rangle$

where the spin down is at site $n^{th}$ position. For this case, using the translational symmetry $\mathbb{Z}/L\mathbb{Z}$, we can guess that eigenvectors must have the following form

$$|\psi_k\rangle = \frac{1}{\sqrt{L}} \sum_{n\in\mathbb{Z}/L\mathbb{Z}} e^{ikn} |n\rangle \, , \tag{1.11}$$

where $k \in \frac{2\pi}{L}\mathbb{Z}/L\mathbb{Z}$ due to the periodicity of the lattice. Indeed, we see that $|\psi_k\rangle$ is an eigenstate of the left-translation operator defined by $T|n\rangle = |n-1\rangle$:

$$\begin{aligned} T|\psi_k\rangle &= \frac{1}{\sqrt{L}} \sum_{n\in\mathbb{Z}/L\mathbb{Z}} e^{ikn} |n-1\rangle \\ &= \frac{1}{\sqrt{L}} \sum_{n\in\mathbb{Z}/L\mathbb{Z}} e^{ik(n+1)} |n\rangle = e^{ik} |\psi_k\rangle \end{aligned} \tag{1.12}$$

and right-translation is similar. Note that we have set the spacing between adjacent sites to $\Delta x = 1$.

To find the corresponding eigenvalues, we act with the Hamiltonian on (1.11). Let us first write down the action of the terms in $\hat{H}$ on the basis states. When the operator $\sum_n S_n^+ S_{n+1}^-$ acts on a basis state $|m\rangle$, the $S_n^+$ will yield 0, except when $n = m$, as follows from (1.5). For $n = m$, the down spin is flipped to up, after which the up spin one site to the right is flipped down by $S_{n+1}^-$. Hence we have that

$$\sum_{n\in\mathbb{Z}/L\mathbb{Z}} S_n^+ S_{n+1}^- |m\rangle = |m+1\rangle \, . \tag{1.13}$$

We see that this operator just acts like a shift operator to the right on the down spin. Similarly, the operator $\sum_n S_n^- S_{n+1}^+$ acts like a shift operator to the left:

$$\sum_{n\in\mathbb{Z}/L\mathbb{Z}} S_n^+ S_{n+1}^- |m\rangle = |m-1\rangle \, . \tag{1.14}$$

Finally, the the operator $\sum_n S_n^z S_{n+1}^z$ gives a contribution of $-\frac{1}{4}$ for each pair of misaligned adjacent spins and a contribution of $\frac{1}{4}$ for all other adjacent pairs. Since each basis state has two misaligned adjacent pairs, we get

$$\sum_{n\in\mathbb{Z}/L\mathbb{Z}} S_n^z S_{n+1}^z |m\rangle = \frac{L-4}{4} |m\rangle \, . \tag{1.15}$$

With these last three equations, it is easy to write down the action of the Hamiltonian on the vector (1.11). The result is

$$\begin{aligned} -\frac{2}{J}\hat{H}\,|\psi_k\rangle &= \sum_{m\in\mathbb{Z}/L\mathbb{Z}} \left(S_m^+S_{m+1}^- + S_m^-S_{m+1}^+ + 2S_m^zS_{m+1}^z\right)\frac{1}{\sqrt{L}}\sum_{n\in\mathbb{Z}/L\mathbb{Z}} e^{ikn}\,|n\rangle \\ &= \frac{1}{\sqrt{L}}\sum_{n\in\mathbb{Z}/L\mathbb{Z}} e^{ikn}\left(|n+1\rangle + |n-1\rangle + \frac{L-4}{2}\,|n\rangle\right) \\ &= \frac{1}{\sqrt{L}}\sum_{n\in\mathbb{Z}/L\mathbb{Z}} \left(e^{ik(n-1)} + e^{ik(n+1)} + \frac{L}{2} - 2\right)|n\rangle \\ &= 2(\cos k - 1 + \frac{L}{4})\,|\psi_k\rangle =: -\frac{2}{J}E_k\,|\psi_k\rangle \end{aligned} \tag{1.16}$$

from which we find the following eigenvalues $E_k$:

$$E_k - E_0 = J(1-\cos k) := E_1(k), \tag{1.17}$$

where we used (1.10). Note we indeed have that $E_{k=0} = E_0$. This degeneracy (i.e. an eigenvalue with an eigenspace of dimension $> 1$) is a consequence of the full $SU(2)$ symmetry: note that we have only exploited a subgroup $U(1)_z \subset SU(2)$. We have found $L$ eigenvectors and the corresponding eigenvalues, so we are done with the case $M = 1$.

Equation (1.17) is also known as the magnon dispersion relation. Here a magnon is a quasiparticle which is used to describe a collective excitation of the spin structure of the spin chain above the groud state. The eigenstates are quantized spin waves with wave number $k$.

## 1.4 The case $M = 2$

For $M = 2$, we consider the subspace $\mathcal{H}_2$ which has dimension $\binom{L}{2} = \frac{1}{2}L(L-1)$. We will denote the basis states as $|n_1, n_2\rangle := S_{n_1}^- S_{n_2}^- |0\rangle$. Unlike the case $M = 1$, the translational invariance is not sufficient to give an Ansatz for the eigenstates. Let us first write down the expression for a general state:

$$|\psi\rangle = \sum_{n_2>n_1} f(n_1, n_2)\,|n_1, n_2\rangle\,, \tag{1.18}$$

which is just a general linear combination of basis vectors of $\mathcal{H}_2$. The summation is over both $n_1$ and $n_2$. We restrict the sum to $n_2 > n_1$ since $|n_2, n_1\rangle = |n_1, n_2\rangle$ and $|n_1, n_1\rangle = 0$. The periodicity condition then becomes $f(n_1, n_2) = f(n_2, n_1 + L)$. Without making any Ansatz just yet, let us apply the Hamiltonian to this general state and see where this leads us.

Firstly, we write down the action of the terms in the Hamiltonian on the basis states. Since we only have nearest-neighbour interactions, we only act on pairs of adjacent spins. Therefore, if the two down spins are not adjacent, the result will strongly resemble the $M = 1$ case. We will treat the case where the two down spins are adjacent (i.e. $m_2 = m_1 \pm 1 \bmod L$) separately.

If the two down spins are not adjacent, the operator $\sum_n S_n^+ S_{n+1}^-$ acts like a shift operator to the right as in the case $M = 1$, but this time on both $m_1$ and $m_2$. If the two down spins are adjacent, this shift operator can only work on $m_2$. Indeed, the down spin at site $m_1$ cannot be shifted to the right, since the spin at $m_1 + 1 = m_2$ is already down. We get:

$$\sum_{n\in\mathbb{Z}/L\mathbb{Z}} S_n^+ S_{n+1}^- |m_1, m_2\rangle = \begin{cases} |m_1, m_1+2\rangle & \text{if } m_2 = m_1 \pm 1 \bmod L \\ |m_1+1, m_2\rangle + |m_1, m_2+1\rangle & \text{otherwise.} \end{cases} \tag{1.19}$$

The operator $\sum_n S_n^- S_{n+1}^+$ works similarly as a shift operator to the left. If the spins are adjacent, only the spin at site $m_1$ can be shifted. We get:

$$\sum_{n\in\mathbb{Z}/L\mathbb{Z}} S_n^- S_{n+1}^+ |m_1, m_2\rangle = \begin{cases} |m_1-1, m_1+1\rangle & \text{if } m_2 = m_1 \pm 1 \bmod L \\ |m_1-1, m_2\rangle + |m_1, m_2-1\rangle & \text{otherwise.} \end{cases} \tag{1.20}$$

As in the case $M = 1$, the operator $\sum_n S_n^z S_{n+1}^z$ gives a contribution $-\frac{1}{4}$ for each pair of misaligned adjacent spins and a contribution $\frac{1}{4}$ for all other adjacent pairs spins. If the two down spins are not adjacent, there are four pairs of misaligned spins; if the down spins are adjacent there are two such pairs. Therefore we get:

$$\sum_{n\in\mathbb{Z}/L\mathbb{Z}} S_n^z S_{n+1}^z |m_1, m_2\rangle = \begin{cases} \left(\frac{L-2}{2} - \frac{1}{2}\right) |m_1, m_1+1\rangle & \text{if } m_2 = m_1 \pm 1 \bmod L \\ \left(\frac{L-4}{4} - 1\right) |m_1, m_2\rangle & \text{otherwise.} \end{cases} \tag{1.21}$$

With this at hand, it is easy to write down the action of the Hamiltonian on the general state (1.18). If we afterwards demand that $|\psi\rangle$ be an eigenvector, we get equations relating the eigenvalues to the amplitudes $f(n_1, n_2)$. One way to do this, is to write the result in the form

$$\hat{H} |\psi\rangle = \sum_{n_2 > n_1+1} \alpha(n_1, n_2) |n_1, n_2\rangle + \sum_{n\in\mathbb{Z}/L\mathbb{Z}} \beta(n) |n, n+1\rangle . \tag{1.22}$$

Demanding $|\psi\rangle$ to be an eigenvector with eigenvalue $E$ then yields the equations:

$$\begin{cases} \alpha(n_1, n_2) = Ef(n_1, n_2) & \text{for } n_2 > n_1 + 1, \\ \beta(n) = Ef(n, n+1). \end{cases} \tag{1.23}$$

We start with splitting the sum in the case where $m_1$ and $m_2$ are adjacent and the case in which they are not. This yields:

$$
\begin{aligned}
-\frac{2}{J}\hat{H}\left|\psi\right\rangle &= \sum_{n_2>n_1} f(n_1,n_2) \sum_{m\in\mathbb{Z}/L\mathbb{Z}} \left(S_m^+S_{m+1}^- + S_m^-S_{m+1}^+ + 2S_m^zS_{m+1}^z\right)\left|n_1,n_2\right\rangle \\
&= \sum_{n_2>n_1+1} f(n_1,n_2)\Bigg( \left|n_1+1,n_2\right\rangle + \left|n_1,n_2+1\right\rangle \\
&\quad + \left|n_1-1,n_2\right\rangle + \left|n_1,n_2-1\right\rangle + \frac{L-8}{2}\left|n_1,n_2\right\rangle \Bigg) \\
&\quad + \sum_{n\in\mathbb{Z}/L\mathbb{Z}} f(n,n+1)\left(\left|n,n+2\right\rangle + \left|n-1,n+1\right\rangle + \frac{L-4}{2}\left|n,n+1\right\rangle\right) \\
&= \sum_{n_2>n_1} f(n_1-1,n_2)\left|n_1,n_2\right\rangle + \sum_{n_2>n_1+2} f(n_1,n_2-1)\left|n_1,n_2\right\rangle \\
&\quad + \sum_{n_2>n_1+2} f(n_1+1,n_2)\left|n_1,n_2\right\rangle + \sum_{n_2>n_1} f(n_1,n_2+1)\left|n_1,n_2\right\rangle \\
&\quad + \sum_{n_2>n_1+1} \frac{L-8}{2} f(n_1,n_2)\left|n_1,n_2\right\rangle \\
&\quad + \sum_{n\in\mathbb{Z}/L\mathbb{Z}} f(n,n+1)\left(\left|n,n+2\right\rangle + \left|n-1,n+1\right\rangle + \frac{L-4}{2}\left|n,n+1\right\rangle\right)
\end{aligned}
\tag{1.24}
$$

where in the last step we shifted the first sum so as to get $\left|n_1,n_2\right\rangle$ in every term. We now change the sums over $n_1$ and $n_2$ such that all these sums all run over $n_2 > n_1 + 1$. Of course must also subtract the terms that we add in doing so.

This yields:

$$
\begin{aligned}
-\frac{2}{J}\hat{H}\,|\psi\rangle = & \sum_{n_2>n_1+1}\Big(f(n_1-1,n_2)+f(n_1,n_2-1)+f(n_1+1,n_2)\\
&\qquad +f(n_1,n_2+1)+\frac{L-8}{2}f(n_1,n_2)\Big)\,|n_1,n_2\rangle\\
&+\sum_{n\in\mathbb{Z}/L\mathbb{Z}}\Big(f(n-1,n+1)\,|n,n+1\rangle-f(n,n+1)\,|n,n+2\rangle\\
&\qquad -f(n+1,n+2)\,|n,n+2\rangle+f(n,n+2)\,|n,n+1\rangle\Big)\\
&+\sum_{n\in\mathbb{Z}/L\mathbb{Z}}f(n,n+1)\Big(|n,n+2\rangle+|n-1,n+1\rangle+\frac{L-4}{2}\,|n,n+1\rangle\Big)\\
= & \sum_{n_2>n_1+1}\Big(f(n_1-1,n_2)+f(n_1,n_2-1)+f(n_1+1,n_2)\\
&\qquad +f(n_1,n_2+1)+\frac{L-8}{2}f(n_1,n_2)\Big)\,|n_1,n_2\rangle\\
&+\sum_{n\in\mathbb{Z}/L\mathbb{Z}}\Big(f(n-1,n+1)+f(n,n+2)-\frac{L-4}{2}f(n,n+1)\Big)\,|n,n+1\rangle\,.
\end{aligned}
\tag{1.25}
$$

Subsequently we read off the functions $\alpha$ and $\beta$. Equation (1.23) then becomes

$$
\begin{aligned}
(E-E_0)f(n_1,n_2) = & \frac{J}{2}\Big(4f(n_1,n_2)-f(n_1-1,n_2)-f(n_1,n_2-1)\\
& -f(n_1+1,n_2)-f(n_1,n_2+1)\Big) \qquad \text{for } n_2>n_1+1
\end{aligned}
\tag{1.26}
$$

$$
(E-E_0)f(n,n+1) = \frac{J}{2}\left(2f(n,n+1)-f(n-1,n+1)-f(n,n+2)\right) \tag{1.27}
$$

where we used (1.10).

Note that in an alternative approach, one could start from equation (1.24) and change the sums over $n_1$ and $n_2$ such that all these sums all run over $n_2 > n_1$. However the final equations agree with our approach, some subtleties will arise. Details about these subtleties can be found in appendix 1.7.

To continue, we need to know what the coefficients $f(n_1, n_2)$ look like. Considering the case $M = 1$, a naive guess would be to take the product of two free waves:

$$
f_{naive}(n_1,n_2) = Ae^{i(k_1n_1+k_2n_2)}. \tag{1.28}
$$

Naturally, this Ansatz is a very unsatisfactory one, in the sense that it completely ignores the interaction which is clearly present in the Hamiltonian. To include the effect of this interaction, we use the celebrated Ansatz made by Hans Bethe in his 1931 paper:

$$f(n_1,n_2) = Ae^{i(k_1n_1+k_2n_2)} + Be^{i(k_2n_1+k_1n_2)}. \tag{1.29}$$

This Ansatz is known as the Coordinate Bethe Ansatz. Physically, we can see from this Ansatz that the magnons interact and in doing so they exchange momenta. Bethe's brilliant insight was that in the regions where the down spins are separated, the solution should look like a free wave, because we only consider nearest-neighbour interactions. One way to see these free waves is by rewriting the general state as

$$|\psi\rangle = A\sum_{n_2>n_1} e^{i(k_1n_1+k_2n_2)}|n_1,n_2\rangle + B\sum_{n_1>n_2} e^{i(k_1n_1+k_2n_2)}|n_1,n_2\rangle\,, \tag{1.30}$$

where in the second sum we switched the dummy indices $n_1$ and $n_2$. We see that this yields a product of free waves in the regions $n_2 > n_1$ and $n_1 > n_2$, but when passing between these regions the amplitude changes due to the interaction. [1]

We can now determine the spectrum by inserting the Coordinate Bethe Ansatz (1.29) in equation (1.26). This yields

$$\begin{aligned}
(E-E_0)f(n_1,n_2) &= \frac{J}{2}\Big(4\left(Ae^{i(k_1n_1+k_2n_2)} + Be^{i(k_2n_1+k_1n_2)}\right)\\
&\quad -\left(Ae^{i(k_1n_1+k_2n_2)}e^{-ik_1} + Be^{i(k_2n_1+k_1n_2)}e^{-ik_2}\right)\\
&\quad -\left(Ae^{i(k_1n_1+k_2n_2)}e^{ik_1} + Be^{i(k_2n_1+k_1n_2)}e^{ik_2}\right)\\
&\quad -\left(Ae^{i(k_1n_1+k_2n_2)}e^{-ik_2} + Be^{i(k_2n_1+k_1n_2)}e^{-ik_1}\right)\\
&\quad -\left(Ae^{i(k_1n_1+k_2n_2)}e^{ik_2} + Be^{i(k_2n_1+k_1n_2)}e^{ik_1}\right)\\
&= \frac{J}{2}f(n_1,n_2)\left(4 - e^{ik_1} - e^{ik_2} - e^{-ik_1} - e^{-ik_2}\right)\\
\Leftrightarrow E - E_0 &= J(2-\cos k_1 - \cos k_2) = \sum_{i=1}^{2} E_1(k_i).
\end{aligned} \tag{1.31}$$

We see that the energy is just the sum of two free-magnon energies with momenta $k_1$ and $k_2$ respectively. This seems quite remarkable, because the magnons do interact with each other. However, the interaction will result in momenta $k_i$ which are different from the free magnon case. Consequently, the energy will also be different from the non-interacting case.

[1] Note that the regions $n_2 > n_1$ and $n_1 > n_2$ are actually ill-defined due to the periodicity of the lattice. However, we can define these regions by picking one site which we define as $n = 0$, and then let $n_1, n_2 \in \{0, 1, ..., L-1\}$. The periodicity condition $f(n_1,n_2) = f(n_2, n_1 + L)$ makes sure that nothing special happens when a down spin crosses $n = 0$.

Having found the eigenvalues, we note that inserting (1.31) back in (1.26) and putting $n_2 = n_1 + 1$, the equation still holds if we define $f(n_1, n_2)$ by the Coordinate Bethe Ansatz (1.29) for $n_1 = n_2$ as well. Subsequently we subtract (1.26) for $n_2 = n_1 + 1$ from (1.27) to get

$$f(n, n) + f(n + 1, n + 1) - 2f(n, n + 1) = 0. \tag{1.32}$$

Note that the extension of $f(n_1, n_2)$ to $n_1 = n_2$ has no physical meaning. It is merely defined by (1.29). Equation (1.32) therefore depends on the form of the Coordinate Bethe Ansatz.

If we insert the Bethe Ansatz in equation (1.32), we obtain a restriction on the amplitudes $A$ and $B$:

$$\begin{aligned}
&(A + B)e^{i(k_1+k_2)n} + (A + B)e^{i(k_1+k_2)(n+1)} - 2(Ae^{i(k_1n+k_2(n+1)} + Be^{i(k_1(n+1)+k_2n)}) = 0 \\
&\Leftrightarrow (A + B)\left(1 + e^{i(k_1+k_2)}\right) - 2Ae^{ik_2} - 2Be^{ik_1} = 0 \\
&\Leftrightarrow \frac{A}{B} = -\left(\frac{e^{i(k_1+k_2)} + 1 - 2e^{ik_1}}{e^{i(k_1+k_2)} + 1 - 2e^{ik_2}}\right). \qquad (1.33)
\end{aligned}$$

If we suppose that $k_1, k_2 \in \mathbb{R}$, we note that

$$\begin{aligned}
\left|\frac{A}{B}\right|^2 &= \left(\frac{e^{i(k_1+k_2)} + 1 - 2e^{ik_1}}{e^{i(k_1+k_2)} + 1 - 2e^{ik_2}}\right)\left(\frac{e^{-i(k_1+k_2)} + 1 - 2e^{-ik_1}}{e^{-i(k_1+k_2)} + 1 - 2e^{-ik_2}}\right) \\
&= \left(\frac{e^{i(k_1+k_2)} + 1 - 2e^{ik_1}}{e^{i(k_1+k_2)} + 1 - 2e^{ik_2}}\right)\left(\frac{1 + e^{i(k_1+k_2)} - 2e^{ik_2}}{1 + e^{i(k_1+k_2)} - 2e^{ik_1}}\right) = 1, \qquad (1.34)
\end{aligned}$$

hence $\frac{A}{B}$ is a phase and we can write

$$e^{i\theta} := \frac{A}{B} = -\left(\frac{e^{i(k_1+k_2)} + 1 - 2e^{ik_1}}{e^{i(k_1+k_2)} + 1 - 2e^{ik_2}}\right). \tag{1.35}$$

If $k_1$ and $k_2$ are complex, then $\theta$ is complex as well. With this definition we can rewrite the Bethe Ansatz:

$$f(n_1, n_2) = e^{i(k_1n_1+k_2n_2+\frac{\theta_{12}}{2})} + e^{i(k_2n_1+k_1n_2+\frac{\theta_{21}}{2})} \tag{1.36}$$

where $\theta_{12} := \theta =: -\theta_{21}$. Note that we omitted an overall constant. Ignoring an overall phase (which is physically irrelevant), another way of writing this is

$$f(n_1, n_2) = e^{i(k_1n_1+k_2n_2)} + S(k_2, k_1)e^{i(k_2n_1+k_1n_2)} \tag{1.37}$$

where $S(k_2, k_1) = e^{-i\theta}$ is an S-matrix element. Here we see that when the magnons interact, they scatter and exchange momenta.

We still need to check the periodicity condition $f(n_1, n_2) = f(n_2, n_1 + L)$. Substituting the Bethe Ansatz in the form (1.36) in here yields

$$\begin{aligned}
&e^{i(k_1n_1+k_2n_2+\frac{\theta}{2})} + e^{i(k_1n_2+k_2n_1-\frac{\theta}{2})} = e^{i(k_1n_2+k_2n_1+\frac{\theta}{2}+k_2L)} + e^{i(k_1n_1+k_2n_2-\frac{\theta}{2}+k_1L)} \\
&\Leftrightarrow e^{ik_1L} = e^{i\theta} \quad \text{and} \quad e^{ik_2L} = e^{-i\theta}. \qquad (1.38)
\end{aligned}$$

The physical interpretation of this equation is that when magnon 1 goes around the whole spin chain once, it picks up a phase $\theta$ as it scatters off particle 2. Taking the logarithm of (1.38) yields

$$\begin{cases} k_1 L = \theta + 2\pi m_1 \\ k_2 L = -\theta + 2\pi m_2 \end{cases} \tag{1.39}$$

where $m_1, m_2 \in \mathbb{Z}/L\mathbb{Z}$ are called the Bethe quantum numbers. These equations are called the Bethe equations. Note that they differ from the free magnon case, where we found $kL = 2\pi m$, $m \in \mathbb{Z}/L\mathbb{Z}$. Like the energy, the total momentum $K = k_1 + k_2 = \frac{2\pi}{L}(m_1 + m_2)$ looks like a sum of free magnon momenta.

We know have everything we need to determine the spectrum and the corresponding eigenstates of the Hamiltonian restricted to the subspace $\mathcal{H}_2$. The next step is to find pairs $(m_1, m_2)$ such that the Bethe equations (1.39) have a solution for $k_1$ and $k_2$, where $\theta$ should be such that (1.35) holds. The corresponding eigenvalue then follows from (1.31). Without loss of generality we can choose representatives $m_1, m_2 \in \{0, 1, ..., L-1\}$ and pick $m_2 \geq m_1$, since interchanging $m_1$ and $m_2$ will produce the same eigenstate (up to an irrelevant overall phase). There are $\frac{1}{2}L(L+1)$ of such pairs $(m_1, m_2)$, but $L$ of these will not yield a solution to the Bethe equations, so that we acquire exactly the amount of eigenstates we want.

The Bethe equations can be solved using numerical methods. Here, we will only give a brief description of the resulting spectrum.

- When $m_1 = 0$, it is easy to see that $k_1 = 0$, $\theta = 0$ and $k_2 = \frac{2\pi m_2}{L}$ solve the Bethe equations. The corresponding energies are then $E = J(1 - \cos k_2)$. This degeneracy with the $M = 1$ case is an expected result. Since we only exploited $U(1)_z \subset SU(2)$ symmetry, it is a consequence of the full $SU(2)$ symmetry.
- When $|m_2 - m_1| \geq 2$ (for all representatives of $m_1$ and $m_2$ in $\mathbb{Z}/L\mathbb{Z}$) we have scattering states. There are $\frac{1}{2}(L-2)(L-3)$ of these states. They are characterized by having $(k_1, k_2) \in \mathbb{R}^2$, which physically corresponds to 'real' quasiparticles scattering off each other.
- When $(m_2 - m_1) \in \{0, \pm 1\} \bmod L$, we find bound states. There are $2L-3$ of these pairs $(m_1, m_2)$, but only $L-3$ of them yield a solution. The bound states are characterized by $(k_1, k_2) \in \mathbb{C}^2$. Physically this is not so strange, as $k_1$ and $k_2$ are not real momenta but rather pseudomomenta of bound quasiparticles. However, the total momentum $K = k_1 + k_2$ of the bound state must be real. It can be shown that in the thermodynamic limit, in which the number of lattice sites $L$ is sent to infinity, the energy is given by $E - E_0 = \frac{J}{2}(1 - \cos K)$ (see equation (1.59) in section 1.6). This shows that the bound state indeed seems to behave like one entity rather than like two particles. It can also be shown that the amplitudes $f(n_1, n_2)$

decrease exponentially as the distance between the two down spins grows. More about bound states in the thermodynamic limit can be found in section 1.6.

## 1.5 The general case

Here we let $M \in \{0, 1, ..., L\}$ be arbitrary. We consider the invariant subspace $\mathcal{H}_M$, which has dimension $\binom{L}{M}$. It is spanned by the vectors consisting of $M$ spins down at sites $n_i$, $i = 1, ..., M$. We will denote these basis vectors as $|n_1, ..., n_M\rangle = S^-_{n_1}...S^-_{n_M}|0\rangle$. A general state can then be written as

$$|\psi\rangle = \sum_{L \geq n_M > ... > n_1 \geq 1} f(n_1, ..., n_M)\,|n_1, ..., n_M\rangle\,. \tag{1.40}$$

To find the spectrum, we need to generalize the coordinate Bethe Ansatz. We can generalize (1.29) as follows:

$$f(n_1, ..., n_M) = \sum_{P \in S_M} A_P \exp\left(i \sum_{j=1}^{M} k_{P(j)} n_j\right), \tag{1.41}$$

where $S_M$ is the permutation group over the set $\{1, ..., M\}$. However, it will prove convenient to instead generalize the Ansatz (1.36):

$$f(n_1, ..., n_M) = \sum_{P \in S_M} \exp\left(i \sum_{j=1}^{M} k_{P(j)} n_j + \frac{i}{2} \sum_{l<j} \theta_{P(l)P(j)}\right), \tag{1.42}$$

for reasons which will become clear in a moment. For now, just treat $\sum_{l<j} \theta_{P(l)P(j)}$ as any function of the quasimomenta $k_i$ and the permutation $P$.

We can now act with the Hamiltonian on (1.40), demand that $|\psi\rangle$ is an eigenvector and substitute (1.42) in the result, just like we did in the case $M = 2$. We will not do this explicitly, but rather give the result immediately:

$$E - E_0 = J \sum_{i=1}^{M} (1 - \cos k_i) = \sum_{i=1}^{M} E_1(k_i), \tag{1.43}$$

where $E_1(k_i)$ is defined in (1.17). Again, this looks like a sum of free-magnon energies. However, as in the $M = 2$ case, we will find that the interaction between the magnons results in different quasimomenta $k_i$ than one would obtain in the non-interacting case.

The action of $\hat{H}$ on $|\psi\rangle$ also yields the following conditions on the $\theta_{jl}$:

$$e^{i\theta_{jl}} = -\left(\frac{e^{i(k_j+k_l)} + 1 - 2e^{ik_j}}{e^{i(k_j+k_l)} + 1 - 2e^{ik_l}}\right). \tag{1.44}$$

This generalizes equation (1.35); note that this is actually exactly the same equation! So the $\theta_{jl}$ are scattering phases, just like in the case $M = 2$.

Applying the periodicity condition $f(n_1, ..., n_M) = f(n_2, ..., n_M, n_1 + L)$ yields even more similarities to the $M = 2$ case. Let us see what happens. Inserting the Bethe Ansatz (1.42) in the periodicity condition gives

$$\sum_{P \in S_M} \exp\left( i \sum_{j=1}^{M} k_{P(j)} n_j + \frac{i}{2} \sum_{l<j} \theta_{P(l)P(j)} \right)$$
$$= \sum_{P \in S_M} \exp\left( i \sum_{j=1}^{M-1} k_{P(j)} n_{j+1} + i k_{P(M)}(n_1 + L) + \frac{i}{2} \sum_{l<j} \theta_{P(l)P(j)} \right). \quad (1.45)$$

Now note that all permutations appear on both sides of the equation. Let $P \in S_M$ be arbitrary. Then we must have that the coefficient in front of $\exp\left(i \sum_{j=1}^{M} k_{P(j)} n_j\right)$ on the left-hand side must be equal to the coefficient of this same exponent on the right-hand side. Let $P'$ be the permutation such that $P'(j) = P(j + 1)$ and $P'(M) = P(1)$. It follows that we must have

$$\exp\left( i \sum_{j=1}^{M} k_{P(j)} n_j + \frac{i}{2} \sum_{l<j}^{M} \theta_{P(l)P(j)} \right)$$
$$= \exp\left( i \sum_{j=1}^{M-1} k_{P'(j)} n_{j+1} + i k_{P'(M)} n_1 + \frac{i}{2} \sum_{l<j}^{M} \theta_{P'(l)P'(j)} + i k_{P'(M)} L \right)$$
$$= \exp\left( i \sum_{j=1}^{M-1} k_{P(j+1)} n_{j+1} + i k_{P(1)} n_1 + \frac{i}{2} \sum_{l<j}^{M-1} \theta_{P(l+1)P(j+1)} + \frac{i}{2} \sum_{l=2}^{M} \theta_{P(l)P(1)} + i k_{P(1)} L \right) \quad (1.46)$$

and hence

$$\exp\left( \frac{i}{2} \sum_{l<j}^{M} \theta_{P(l)P(j)} \right) = \exp\left( \frac{i}{2} \sum_{l<j}^{M-1} \theta_{P(l+1)P(j+1)} + \frac{i}{2} \sum_{l=2}^{M} \theta_{P(l)P(1)} + i k_{P(1)} L \right). \quad (1.47)$$

We see that the first sum on the right-hand side cancels all terms in the sum on the left-hand side except when $l = 1$. Then taking the second sum on the right-hand side to the left-hand side and using $\theta_{ij} = -\theta_{ji}$ we get

$$\exp\left( i \sum_{j=2}^{M} \theta_{P(1)P(j)} \right) = \exp\left( i k_{P(1)} L \right). \quad (1.48)$$

Since $P \in S_M$ was arbitrary, this must hold for any index $P(1)$ and we can write this as

$$\prod_{j \neq l} \exp\left( i\theta_{lj} \right) = \exp\left( i k_l L \right) \quad (1.49)$$

where the product is over $j$ only. This equation has an interesting interpretation. We see that if some particle $l$ goes around the spin chain once, it picks up a phase $\theta_{lj}$ each time it passes another particle $j$. This is the same result as in the $M = 2$ case. The equation (1.49) shows that the S-matrix for the general $M$ case can be expressed in terms of products of two-body S-matrix elements. This result is also known as *factorized scattering.* Note that we actually anticipated this highly non-trivial result by choosing the Coordinate Bethe Ansatz in the form (1.36), such that the phases $\theta_{ij}$ are exactly the same as in the $M = 2$ case. The results clearly show that solving the $M = 2$ case first has been useful: the general results can be expressed in terms of the $M = 2$ results.

Taking the logarithm of equation (1.49) gives

$$k_l L = 2\pi m_l + \sum_{j\neq l} \theta_{lj}. \tag{1.50}$$

These are the general Bethe equations. Once again, the interaction introduces phase shifts in the quasimomenta with respect to the free magnon quasimomenta. The total momentum is then given by

$$K = \frac{2\pi}{L} \sum_{j=1}^{M} m_j \tag{1.51}$$

which again looks like a sum of free magnon momenta. Similarly to the case $M = 2$, finding the eigenvectors now amounts to finding $(m_1, ..., m_M) \in (\mathbb{Z}/L\mathbb{Z})^M$ such that the equations (1.44) and (1.50) have solutions for some $(k_1, ..., k_M) \in \mathbb{C}^M$.

Factorized scattering is usually a consequence of conserved quantities. A flaw of the Coordinate Bethe Ansatz is that we cannot find these conserved quantities. However, with the Algebraic Bethe Ansatz this is possible. Other advantages of the Algebraic Bethe Ansatz are that it can be used to prove the integrability of the model and that it yields algebraic equations which are easier to solve. More on the Algebraic Bethe Ansatz can be found in a later chapter.

## 1.6 String solutions

We start by introducing the rapidity, which is defined as

$$\lambda_j = \cot\frac{k_j}{2} = i\frac{\exp\left(\frac{ik_j}{2}\right) - \exp\left(-\frac{ik_j}{2}\right)}{\exp\left(\frac{ik_j}{2}\right) + \exp\left(\frac{-ik_j}{2}\right)} \tag{1.52}$$

so that

$$k_j = \frac{1}{i}\ln\frac{\lambda_j + i}{\lambda_j - i}. \tag{1.53}$$

We will now write the Bethe equations in terms of these rapidities. We start with the $M = 2$ case. Here we rewrite the left-hand side of the Bethe equations (1.38) using (1.53), and we rewrite the scattering phases (1.35) in terms of the rapidities. This yields

$$\left(\frac{\lambda_j + i}{\lambda_j - i}\right)^L = \frac{\lambda_j - \lambda_l + 2i}{\lambda_j - \lambda_l - 2i} \tag{1.54}$$

where $j, l \in \{1, 2\}$, $j \neq l$. Note that the right-hand side now depends only on the difference of rapidities, which was one reason introduce rapidities in the first place.

Next, let us consider the case where $\text{Im}\lambda_j \neq 0$. As stated before, these states correspond to bound states. If we now take the thermodynamic limit $L \to \infty$ of (1.54), the left-hand side either goes to 0 or to $\infty$. This must also be true for the right-hand side. We therefore find that

$$\lambda_1 - \lambda_2 = \pm 2i. \tag{1.55}$$

Furthermore, requiring that the total momentum $k_1 + k_2$ be real, which implies $\lambda_1 = \lambda_2^*$, then yields

$$\lambda_{1,2} = \lambda \pm 2i \tag{1.56}$$

where $\lambda \in \mathbb{R}$. For the total momentum $K_{1/2} = k_1 + k_2$ we find using (1.53):

$$e^{iK_{1/2}} = \frac{\lambda + 2i}{\lambda - i}. \tag{1.57}$$

Defining the energy

$$\epsilon = -J\frac{\mathrm{d}K}{\mathrm{d}\lambda} \tag{1.58}$$

we find

$$\epsilon_{1/2} = \frac{4J}{\lambda^2 + 4} = \frac{J}{2}(1 - \cos K_{1/2}). \tag{1.59}$$

Notice that

$$\epsilon_{1/2}(K) < \epsilon_0(K - k) + \epsilon_0(k) \tag{1.60}$$

$\forall K, k \in [0, 2\pi)$, where $\epsilon_0(k) = J(1-\cos k)$ is an individual-magnon energy. This shows that the bound state is energetically favorable in comparison to a state where the two magnons are free. However, the most important observation here is that the bound state behaves like a single entity with momentum $K_{1/2}$ and energy $\epsilon_{1/2}$, rather than as two particles with momenta $k_1$ and $k_2$ respectively. This particular bound state is also called a 1/2-complex (hence the subscripts). We will come back to this terminology below.

We now turn to the case of general $M$. Again, we look at complex rapidities. Now we wish to know if also in this case, we can describe the physics by treating single entities of bound states of magnons. This would certainly be a simplification compared to the case where we describe $M$ individual interacting magnons! The answer to this question is the content of the *string hypothesis*. This hypothesis states that we can partition the $M$ rapidities of an eigenvector of the Hamiltonian into so-called $r$-complexes, also known as strings. Such an $r$-complex, where $r \in \frac{1}{2}\mathbb{N} \cup \{0\}$, consists of $2r+1$ rapidities of with equal real parts $\lambda_r$ and imaginary parts equidistantly distributed around the real line, in the following way:

$$\lambda_{r,m} = \lambda_r + 2im. \tag{1.61}$$

Here, $m \in \{-r, -r+1, ..., r-1, r\}$, so the solution looks like a 'string' of equidistant points (rapidities) in the complex plane. A 0-complex is just an individual magnon.

Let us see how these complexes behave. We can compute the quantity $K_r := k_{-r} + k_{-r+1} + ... + k_r$ where the quasimomenta $k_i$ correspond to the rapidities in the $r$-complex under consideration. Using (1.53), the result is

$$\begin{aligned} K_r &= \frac{1}{i} \ln \left( \frac{\lambda_r + i(2r+1)}{\lambda_r + i(2r-1)} \frac{\lambda_r + i(2r-1)}{\lambda_r + i(2r-3)} \cdots \frac{\lambda_r + i(-2r+1)}{\lambda_r + i(-2r-1)} \right) \\ &= \frac{1}{i} \ln \left( \frac{\lambda_r + i(2r+1)}{\lambda_r - i(2r+1)} \right). \end{aligned} \tag{1.62}$$

Note that this result only depends on $\lambda_r$, which is a quantity all rapidities in the complex have in common! Therefore we can treat $K_r$ as the momentum of the $r$-complex. The corresponding energy of the complex follows from (1.58), which yields

$$\epsilon_r = \frac{J}{2r+1} \left(1 - \cos K_r\right). \tag{1.63}$$

Therefore, validity of the string hypothesis implies that we can indeed treat the complexes of bound states as single entities. Note that we saw this in the thermodynamic limit of the case $M = 2$, where we had a 1/2-complex which was described as a single entity with energy $\epsilon_{1/2}$. The string hypothesis generalizes this to the case of general $M$. Other physical processes can be described in terms of the complexes as well. For example, we can consider the process where a magnon scatters off an $r$-complex. The S-matrix element can the be found by just taking the product of all scattering phases with the magnons in the complex. Using (1.54), this yields

$$S_{0,r}(\lambda_0 - \lambda_r) = \frac{\lambda_0 - \lambda_r + 2ir}{\lambda_0 - \lambda_r - 2ir} \frac{\lambda_0 - \lambda_r + 2i(r+1)}{\lambda_0 - \lambda_r - 2i(r+1)} \tag{1.64}$$

where all other factors of the product cancel (in a similar way to the calculation of $K_r$).

Being able to treat these complexes like single entities is of course tremendously convenient. However, when is the string hypothesis actually valid? It has been shown that outside the thermodynamic limit, the string hypothesis fails to hold. Counterexamples to the string hypothesis have also been provided for quite large systems. However, in the thermodynamic limit, it is believed that the string solutions provide a good description of the thermodynamics of the system. This is despite the fact that the hypothesis has never been proven to exhaust the whole Hilbert space.

## 1.7 Appendix 1: Alternative derivation of (1.26) and (1.32)

Here we provide an alternative derivation of the energy for the case $M = 2$ and the condition (1.32). This alternative derivation is for example followed by Arutyunov (see reference 3). In this approach, rather than writing the action of the Hamiltonian on a general state as (1.22), we write

$$\hat{H}\left|\psi\right\rangle = \sum_{n_2>n_1} \tilde{\alpha}(n_1, n_2)\left|n_1, n_2\right\rangle + \sum_{n\in\mathbb{Z}/L\mathbb{Z}} \tilde{\beta}(n)\left|n, n+1\right\rangle. \tag{1.65}$$

Demanding $|\psi\rangle$ to be an eigenvector with eigenvalue $E$ then yields the equations:

$$\begin{cases} \tilde{\alpha}(n_1, n_2) = Ef(n_1, n_2), \\ \tilde{\beta}(n) = 0. \end{cases} \tag{1.66}$$

We start from (1.24). This time, we extend the sums over $n_1$ and $n_2$ by adding terms such that all these sums all run over $n_2 > n_1$. Of course we also have to subtract the terms we added. This yields:

$$\begin{aligned}
-\frac{2}{J}\hat{H}\left|\psi\right\rangle = &\sum_{n_2>n_1} \Big( f(n_1-1, n_2) + f(n_1, n_2-1) + f(n_1+1, n_2) \\
&\quad + f(n_1, n_2+1) + \frac{L-8}{2} f(n_1, n_2) \Big) \left|n_1, n_2\right\rangle \\
&- \sum_{n\in\mathbb{Z}/L\mathbb{Z}} \Big( f(n,n)\left|n, n+1\right\rangle + f(n, n+1)\left|n, n+2\right\rangle + f(n+1, n+1)\left|n, n+1\right\rangle \\
&\quad + f(n+1, n+2)\left|n, n+2\right\rangle + \frac{L-8}{2} f(n, n+1)\left|n, n+1\right\rangle \Big) \\
&+ \sum_{n\in\mathbb{Z}/L\mathbb{Z}} f(n, n+1)(\left|n, n+2\right\rangle + \left|n-1, n+1\right\rangle + \frac{L-4}{2}\left|n, n+1\right\rangle)\Big) \\
= &\sum_{n_2>n_1} \Big( f(n_1-1, n_2) + f(n_1, n_2-1) + f(n_1+1, n_2) \\
&\quad + f(n_1, n_2+1) + \frac{L-8}{2} f(n_1, n_2) \Big) \left|n_1, n_2\right\rangle \\
&- \sum_{n\in\mathbb{Z}/L\mathbb{Z}} \Big( f(n,n) + f(n+1, n+1) - 2f(n, n+1) \Big) \left|n, n+1\right\rangle.
\end{aligned} \tag{1.67}$$

You should now be shocked and very confused: the above procedure immediately yields terms $f(n_1, n_2)$ with $n_1 = n_2$, whereas these terms are not defined in the expansion (1.18)! Note that at this point, we did not define the Coordinate Bethe Ansatz yet either. In other words, we have assumed here that whatever form the $f(n_1, n_2)$ take, we can extend this form to $n_1 = n_2$. But why this is the case is not clear at this point. Despite this oddity, this assumption does yield the correct result: reading off the functions $\tilde{\alpha}$ and $\tilde{\beta}$, (1.66) becomes

$$\begin{aligned}(E - E_0)f(n_1, n_2) &= \frac{J}{2}(4f(n_1, n_2) - f(n_1 - 1, n_2) - f(n_1, n_2 - 1) \\ &\quad -f(n_1 + 1, n_2) - f(n_1, n_2 + 1)) \qquad (1.68) \\ 0 &= f(n, n) + f(n + 1, n + 1) - 2f(n, n + 1) \qquad (1.69)\end{aligned}$$

where we used (1.10). But these are exactly the results we are supposed to get, i.e. (1.26) and (1.32)! Of course, the reason is that in our other approach, we found that (1.26) also holds for $n_1 = n_2$. This justified the extension of $f(n_1, n_2)$, after which equation (1.69) followed from equation (1.27). However, to conclude this, we made use of that fact that we had already defined the Coordinate Bethe Ansatz.

We conclude that we need introduce the Bethe Ansatz in order to justify the extension to $f(n, n)$ and hence also to justify equation (1.32). Thus, when using this approach, we either need to define the Bethe Ansatz at an earlier stage (which is done in Arutyunov's notes), or use our original approach to justify the validity of equation (1.69).

## 1.8 Appendix 2: Homework: comparison to the article bu Langlands and Saint-Aubin

In the article by Robert P. Langlands and Yvan Saint-Aubin (see reference 4), the same quantum-mechanical model is treated. Here we compare the first two pages of this article to the conventions in our approach.

- In the first equation $\mathcal{X}$ is just the Hilbert space, defined in the same way as we did.
- $u^+ = |\uparrow\rangle, u^- = |\downarrow\rangle$ or vice versa.
- $u_{m_1,...,m_r}$ is the state with $r$ spins up at sites $m_1, ..., m_r$. The article characterizes states by the sites at which the spins are up rather than down, contrary to our approach. Both approaches are equivalent. This also follows from their decomposition of the Hilbert spaces in invariant subspaces with $r$ spins up rather than $M$ spins down.
- The action of $H_r$ yields $a'_{m_1,...,m_r}$. The first term inside the sum corresponds to the action of $\sum_{n\in\mathbb{Z}/L\mathbb{Z}} S^z_n S^z_{n+1}$, the other terms correspond to the other terms in the Hamiltonian. Indeed: we have seen that these other terms shift the misaligned spin by one site.
- The Bethe Ansatz put forward in the article is indeed a sum of permutations as in our case, $w$ corresponding to the scattering phases and $z$ corresponding to the pseudomomenta.

## 1.9 References


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