part one
solution II:
proof of
(2.163)
(2.164)
(2.165)
(2.166)

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We want to compute H_2: essentially this amounts to calculating

$$\partial_u^2|_{u=0} \log t(u) = \partial_u\big|_{u=0} t(u)^{-1}\, t'(u) = t(0)^{-1} t''(0) - t(0)^{-2} t'(0)^2 .$$

Let's write $R_{aj}(u) := \frac{\mathcal{R}_{aj}(u)}{a(u)}$, so that $R_{aj}(0) = P_{aj}$. Also write $\check{R}_{aj} := P_{aj}\, R_{aj}$.
Then

$$t'(u) = \sum_j \mathrm{tr}_a\left[R_{aL}\cdots R_{a\,j+1}\, R'_{aj}\, R_{a\,j-1}\cdots R_{a1}\right],$$

from which

$$t(0)^{-1}\, t'(0) = e^{+iP} \sum_j \mathrm{tr}_a\left[P_{aL}\cdots P_{a\,j+1}\, R'_{aj}\, P_{a\,j-1}\cdots P_{a1}\right]$$
$$= e^{+iP} \sum_j \mathrm{tr}_a\left[P_{aL}\cdots P_{a\,j+1}\, P_{aj}\, \check{R}'_{aj}\, P_{a\,j-1}\cdots P_{a1}\right]$$
$$= e^{+iP} \sum_j \mathrm{tr}_a\left[P_{a\,j-1}\cdots P_{a1}\, P_{aL}\cdots P_{a\,j+1}\, P_{aj}\, \check{R}'_{aj}\right].$$

cyclic property of trace and permutations on the right of $\check{R}'_{aj}$ don't have subscript j

But, using cycle notation,

notice how easy this is using cycle notation instead of $P_{aj}P_{ai} = P_{ij}P_{aj}$ etc (!)

$$P_{a\,j-1}\cdots P_{a1} P_{aL}\cdots P_{aj} = (a\;j{-}1)\cdots(a\;1)(a\;L)\cdots(a\;j) = (1\;2\cdots j{-}1\;\;a\;\;j\;\;j{+}1\cdots L)$$
$$= (1\;2\cdots j{-}1\;\;j\;\;j{+}1\cdots L)(j{-}1\;\;a) = e^{-iP}\, P_{a\,j-1},$$

so we further find

$$t(0)^{-1} t'(0) = \sum_j \mathrm{tr}_a\left[P_{a\,j-1}\, \check{R}'_{aj}\right] = \sum_j \mathrm{tr}_a\left[\check{R}'_{j-1\,j}\; P_{a\,j-1}\right] = \sum_j \check{R}'_{j-1\,j}\; \mathrm{tr}_a P_{a\,j-1} = \sum_j \underbrace{\check{R}'_{j-1\,j}}_{\boxed{=-ih_{j-1\,j}}\ (!)}$$

and therefore

$$t(0)^{-2} t'(0)^2 = \sum_{i,j} \check{R}'_{i\,i+1}\, \check{R}'_{j\,j+1} = \sum_j \left(\check{R}'^{\,2}_{j\,j+1} + \check{R}'_{j\,j+1}\check{R}'_{j+1\,j+2} + \check{R}'_{j+1\,j+2}\check{R}'_{j\,j+1}\right) + 2\sum_{i+1<j} \check{R}'_{i\,i+1}\, \check{R}'_{j\,j+1}$$

these can be interchanged whenever $i+1 \neq j$, $i \neq j+1$

$$= \sum_j \left(\check{R}'^{\,2}_{j\,j+1} - h_{j\,j+1}\, h_{j+1\,j+2} - h_{j+1\,j+2}\, h_{j\,j+1}\right) + 2\sum_{i+1<j} \check{R}'_{i\,i+1}\, \check{R}'_{j\,j+1}.$$

Next:

$$t''(u) = 2\sum_{i<j} \mathrm{tr}_a\left[R_{aL}\cdots R_{a\,j+1}\, R'_{aj}\, R_{a\,j-1}\cdots R_{a\,i+1}\, R'_{ai}\, R_{a\,i-1}\cdots R_{a1}\right] + \sum_j \mathrm{tr}_a\left[R_{aL}\cdots R_{a\,j+1}\, R''_{aj}\, R_{a\,j-1}\cdots R_{a1}\right]$$

To compute $t(0)^{-1} t''(0)$ we work out these two terms separately.
The second one is easy: just as above we now find

$$t(0)^{-1} \sum_j \mathrm{tr}_a\left[P_{aL}\cdots P_{a\,j+1}\, P_{aj}\, \check{R}''_{aj}\, P_{a\,j-1}\cdots P_{a1}\right] = \sum_j \check{R}''_{j-1\,j} = \sum_j \check{R}''_{j\,j+1}.$$

The first term can be massaged as follows:

$$\sum_{i<j} \mathrm{tr}_a\left[P_{aL}\cdots P_{a\,j+1}\, P_{aj}\, \check{R}'_{aj}\, P_{a\,j-1}\cdots P_{a\,i+1}\, P_{ai}\, \check{R}'_{ai}\, P_{a\,i-1}\cdots P_{a1}\right]$$
$$= \sum_{i<j} \mathrm{tr}_a\left[P_{aL}\cdots P_{aj}\, P_{a\,j-1}\cdots P_{ai}\;\; \check{R}'_{j-1\,j}\, \check{R}'_{ai}\, P_{a\,i-1}\cdots P_{a1}\right]$$
$$= \sum_{i<j} \mathrm{tr}_a\left[P_{a\,i-1}\cdots P_{a1}\, P_{aL}\cdots P_{ai}\;\; \check{R}'_{j-1\,j}\, \check{R}'_{ai}\right]$$

like above: cyclicity et cetera

$$= e^{-iP} \sum_{i<j} \mathrm{tr}_a\left[P_{a\,i-1}\, \check{R}'_{j-1\,j}\, \check{R}'_{ai}\right]$$

like above: use cycle notation

use $P_{a\,j-1}\cdots P_{a\,i+1}\, P_{ai}$
$= (a\;j{-}1)\cdots(a\;i{+}1)\,(a\;i)$
$= (i\;\;i{+}1\cdots j{-}1\;\;a)$
and
$\check{R}'_{aj}\,(i\;\;i{+}1\cdots j{-}1\;\;a)$
$= (i\;\;i{+}1\cdots j{-}1\;\;a)\, \check{R}'_{j-1\,j}$
$= P_{a\,j-1}\cdots P_{a\,i+1}\, P_{ai}\, \check{R}'_{j-1\,j}$

$$= e^{-iP} \sum_{i<j} \check{R}'_{j-1\,j}\; \mathrm{tr}_a\left[P_{a\,i-1}\, \check{R}'_{ai}\right] = e^{-iP} \sum_{i<j} \check{R}'_{j-1\,j}\, \check{R}'_{i-1\,i}\; \mathrm{tr}_a\left[P_{a\,i-1}\right] = e^{-iP} \sum_{i<j} \check{R}'_{j-1\,j}\, \check{R}'_{i-1\,i}$$

we can only interchange these when $i<j-1$ i.e. when $i+1<j$

$$= e^{-iP} \sum_j \check{R}'_{j-1\,j}\, \check{R}'_{j-2\,j-1} + e^{-iP} \sum_{i+1<j} \check{R}'_{j-1\,j}\, \check{R}'_{i-1\,i}$$
$$= e^{-iP} \sum_j \check{R}'_{j+1\,j+2}\, \check{R}'_{j\,j+1} + e^{-iP} \sum_{i+1<j} \check{R}'_{i-1\,i}\, \check{R}'_{j-1\,j}$$
$$= -e^{-iP} \sum_j h_{j+1\,j+2}\, h_{j\,j+1} + e^{-iP} \sum_{i+1<j} \check{R}'_{i\,i+1}\, \check{R}'_{j\,j+1}.$$

Combining the two yields

$$t(0)^{-1} t''(0) = -2\sum_j h_{j+1\,j+2}\, h_{j\,j+1} + 2\sum_{i+1<j} \check{R}'_{i\,i+1}\, \check{R}'_{j\,j+1} + \sum_j \check{R}''_{j\,j+1}.$$

Finally we find

$$\partial_u^2|_{u=0} \log t(u) = \Big(\sum_j \check{R}''_{j\,j+1} + 2\sum_{i+1<j} \cancel{\check{R}'_{i\,i+1}\, \check{R}'_{j\,j+1}} - \cancel{2} \sum_j h_{j+1\,j+2}\, h_{j\,j+1}\Big)$$
$$- \Big(\sum_j \big(\check{R}'^{\,2}_{j\,j+1} - h_{j\,j+1}\, h_{j+1\,j+2} - \cancel{h_{j+1\,j+2}\, h_{j\,j+1}}\big) + 2\sum_{i+1<j} \cancel{\check{R}'_{i\,i+1}\, \check{R}'_{j\,j+1}}\Big)$$
$$= \sum \left(\check{R}''_{j\,j+1} - \check{R}'^{\,2}_{j\,j+1} + [h_{j\,j+1}, h_{j+1\,j+2}]\right) \quad \text{as we wanted to show.}$$