Comparison of notation and conventions in the literature on quantum integrable models

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This document provides a comparison of the different notations and conventions in the literature used for the Seminar on Quantum Integrable Models taught at Utrecht University in fall 2013; the sources referred to below can be found at the course webpage. Notice that the notation indicates where the reference is in the scale from mathematical to physical literature, and which material it roughly covers. I have tried to reproduce the notation of those references as closely as possible (unless that costs too much space). Please let me know if you spot any mistakes.

Contents

1 Hilbert space, Hamiltonian

To save some space in the table below I have replaced a part of the expressions for the Hamiltonian by '$\cdots$'. Further, 'vacuum' really means *pseudo*vacuum, i.e. the lowest (or sometimes highest) weight vector for the total spin operator S^z. In general the (Fock) space of states can be constructed from this vector in terms of spin waves (magnons). In the ferromagnetic regime of the XXX model, the pseudovacuum correspond to a physical vacuum state with minimal energy eigenvalue.

	Hilbert space				total spin (in z-direction)			Hamiltonian	
	local	basis	# sites	global	operator	# magnons	eigenspaces	operator	vacuum
[Bax]	–	–	N	–	–	n	–	$\mathscr{H} = -\frac{1}{2}\sum_{j=1}^{N}[J_x\sigma_j^x\sigma_{j+1}^x + \cdots]$	–
[GRS]	$V^{\frac{1}{2}}$	$\lvert\Uparrow\rangle, \lvert\Downarrow\rangle$	L	$\mathscr{H}^{(L)} = \bigotimes^{L} V^{\frac{1}{2}}$	$S^z_{\text{total}} = \Delta^{L-1}(S^z) = \sum_{i=1}^{L} S_i^z$	M	$\mathscr{H}_M^{(L)}$	$H_{XXX} = J\sum_{i=1}^{L}\left(\sigma_i^x\sigma_{i+1}^x + \cdots\right)$	$\lvert\Omega\rangle$
[JM]	V_k	v_+, v_-	N	$W_N = \bigotimes_{k\in\mathbf{Z}_N} V_k$	$S_N = \Delta^{(N-1)}(S^z) = \frac{1}{2}\sum_{k\in\mathbf{Z}_N}\sigma_k^z$	$s + N/2$	$W_N^{(s)}$	$\mathrm{H}_{XXZ} = -\frac{1}{2}\sum_{k}(\sigma_k^x\sigma_{k+1}^x + \cdots)$	Ω_-
[KBI]	–	$\lvert\uparrow\rangle_m, \lvert\downarrow\rangle_m$	M	–	$S_z = \frac{1}{2}\sum_{m=1}^{M}\sigma_z^{(m)}$	N	–	$H_0 = -\sum_{m=1}^{M}\left\{\sigma_x^{(m)}\sigma_x^{(m+1)} + \cdots - 1\right\}$	$\lvert 0'\rangle$
[Bat]	–	–	L	–	–	n	–	$H = \frac{J}{2}\sum_{i=1}^{L}(\sigma_i\cdot\sigma_{i+1} + 1)$	–
[DEFK]	$\mathbb{C}^2$	$\lvert\uparrow\rangle, \lvert\downarrow\rangle$	N	$W = (\mathbb{C}^2)^{\otimes N}$	$\mathrm{S}^z = \Delta^{(N)}(S^z) = \sum_{i=1}^{N} S_i^z$	M	–	$H = \sum_{i}(\mathrm{J}_x S_i^x S_{i+1}^x + \cdots)$	$\lvert\Omega\rangle$
[Fad]	$h_n = \mathrm{C}^2$	e_+, e_-	N	$\mathcal{H}_N = \prod_{n=1}^{N}\otimes h_n$	$S^z = \sum_n S_n^z$	l	–	$H = \sum_{\alpha,n}\left(S_n^\alpha S_{n+1}^\alpha - \frac{1}{4}\right)$	Ω
[Fra]	–	$\lvert\uparrow\rangle, \lvert\downarrow\rangle$	N	–	$S^z \equiv \sum_{n=1}^{N} S_n^z$	r	–	$\mathcal{H} = -J\sum_{n=1}^{N}\mathbf{S}_n\cdot\mathbf{S}_{n+1}$	$\lvert 0\rangle$
[KM]	–	$\lvert\uparrow\rangle, \lvert\downarrow\rangle$	N	–	$S_T^z \equiv \sum_{n=1}^{N} S_n^z$	r	–	$\mathcal{H} = -J\sum_{n=1}^{N}\mathbf{S}_n\cdot\mathbf{S}_{n+1}$	$\lvert F\rangle$
[Tol]	–	–	N	–	–	n	–	–	–

The differences in the column with the total spin operator are easy to understand: the spin operators S^x, S^y, S^z are usually given in terms of the Pauli matrices $\sigma^x, \sigma^y, \sigma^z$ (which form a basis for $\mathfrak{su}_2$) as $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$; by expressing the energy in units of Planck's constant we may arrange $\hbar = 1$.

Also notice the various conventions used for the Hamiltonian. With a particular goal in mind, it may be convenient to

- introduce an overall minus sign when one is interested in the antiferromagnetic regime, where the spins tend to anti-align;
- set the ground state energy equal to zero by shifting the Hamiltonian by a constant (this again depends on the regime one is interested in);
- absorb J by a rescaling of the spin operators.

2 Coordinate Bethe Ansatz, Bethe-Ansatz equations

	coordinate Bethe Ansatz	Bethe-Ansatz equations: 'S-matrix form'	Bethe-Ansatz equations: logarithmic form
[Bax]	$g(x_1,\dots,x_n)=\sum_P A_{p_1,\dots,p_n} z_{p_1}^{x_1}\dots z_{p_n}^{x_n}$	$z_j^N=(-)^{n-1}\prod_{\substack{l=1\\ \neq j}}^{n} s_{l,j}/s_{j,l}$	$Nk_j=2\pi I_j-\sum_{l=1}^{n}\Theta(k_j,k_l)$
[GRS]	$f(x_1,\dots,x_M)=\sum_{p\in\mathscr{S}_M} A_p\,\mathrm{e}^{i(k_{p(1)}x_1+\dots+k_{p(M)}x_M)}$	$\mathrm{e}^{ik_iL}=\prod_{\substack{j=1\\ j\neq i}}^{M}\hat{S}_{ji}(k_j,k_i)$	$Lk_i=2\pi I_i+\sum_{j=1}^{M}\Theta(k_i,k_j)$
[JM]	–	–	–
[KBI]	(really long, see below)	$\exp\{-ip_0(\lambda_\alpha)M\}=\exp\left\{i\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{N}\theta(\lambda_\alpha-\lambda_\beta)\right\}(-1)^{N-1}$	$Mp_0(\lambda_\alpha)+\sum_{\beta=1}^{N}\theta(\lambda_\alpha-\lambda_\beta)=2\pi n_\alpha$
[Bat]	$a(x_1,\dots,x_n)=\sum_{P\in S_n} A_P\exp\left(\sum_{j=1}^{n} ik_{p_j}x_j\right)$	$\exp ik_jL=(-1)^{n-1}\prod_{\ell\neq j}\frac{s_{\ell j}}{s_{j\ell}}$	–
[DEFK]	–	–	–
[Fad]	–	$e^{ip(\lambda_k)N}=\prod_{k=1}^{l}S(\lambda_k-\lambda_l)$	$Np(\lambda_j)=2\pi Q_j+\sum_{k=1}^{l}\varphi(\lambda_j-\lambda_k)$
[Fra]	$f(n_1,\dots,n_r)=\sum_{\mathcal{P}\in S_r}\exp\left(\mathrm{i}\sum_{j=1}^{r}k_{\mathcal{P}j}n_j+\frac{\mathrm{i}}{2}\sum_{l<j}\theta_{\mathcal{P}l\mathcal{P}j}\right)$	–	$Nk_j=2\pi\tilde{I}_j+\sum_{l\neq j}\theta_{jl}$
[KM]	$a(n_1,\dots,n_r)=\sum_{\mathcal{P}\in S_r}\exp\left(i\sum_{j=1}^{r}k_{\mathcal{P}j}n_j+\frac{i}{2}\sum_{i<j}\theta_{\mathcal{P}i\mathcal{P}j}\right)$	–	$Nk_i=2\pi\lambda_i+\sum_{j\neq i}\theta_{ij}$
[Tol]	$g(x_1,\dots,x_n)=\sum_{\sigma\in S_n} A_\sigma z_{\sigma(1)}^{x_1}\dots z_{\sigma(n)}^{x_n}$	$z_j^n=(-1)^{n-1}\prod_{l\neq i}^{n}\frac{s_{\ell,i}}{s_{i,\ell}}$	$NK_j=2\pi I_j-\sum_{\ell=1}^{n}\Theta(K_j,K_\ell)$

The coordinate Bethe Ansatz in [KBI] is written in a form that anticipates the (physical interpretation of the) result; see Chapter II and in particular equations (1.12) – (1.15). For completeness we quote their expression for the Bethe Ansatz:

$$\chi_N=\left\{\prod_{N\geq b>a\geq 1}\epsilon(m_b-m_a)\right\}\sum_{\mathcal{Q}}(-1)^{[\mathcal{Q}]}\exp\left\{-i\sum_{a=1}^{N}m_ap_0(\lambda_{\mathcal{Q}a})\right\}\exp\left\{\frac{-i}{2}\sum_{N\geq b>a\geq 1}\theta(\lambda_{\mathcal{Q}b}-\lambda_{\mathcal{Q}a})\epsilon(m_b-m_a)\right\}.$$

3 Rapidities, transfer matrix, more Bethe-Ansatz equations

	rapidities (= spectral parameters) symbol	rapidities (= spectral parameters) defined by	transfer matrix Boltzmann weight	transfer matrix tranfer	transfer matrix monodromy	Bethe-Ansatz equations in terms of rapidities (rational for XXX, hyperbolic for xxz/six-vertex)
[Bax]	α	$z_j = \exp(ik_j) = \dfrac{\sinh\frac{1}{2}(\lambda + i\alpha_j)}{\sinh\frac{1}{2}(\lambda - i\alpha_j)}$	$w(\mu,\alpha\|\beta,\mu')$	V	–	$\left[\dfrac{\sinh\frac{1}{2}(\lambda + i\alpha_j)}{\sinh\frac{1}{2}(\lambda - i\alpha_j)}\right]^N = -\prod_{l=1}^{n} \dfrac{\sinh[\frac{1}{2}i(\alpha_l - \alpha_j) - \lambda']}{\sinh[\frac{1}{2}i(\alpha_l - \alpha_j) + \lambda']}$
[GRS]	u or λ	$u_j = \dfrac{\gamma}{2}(\lambda_j - i)\,,\ \mathrm{e}^{ik_j} = \dfrac{\mathrm{e}^{\gamma\lambda_j} - \mathrm{e}^{i\gamma}}{\mathrm{e}^{\gamma\lambda_j + i\gamma} - 1}$	$W\begin{pmatrix} \beta & \nu \\ \mu & \alpha \end{pmatrix} \equiv \mathscr{R}^{\nu\beta}_{\mu\alpha}$	t	T	$\left(\dfrac{\sinh\frac{\gamma}{2}(\lambda_j + i)}{\sinh\frac{\gamma}{2}(\lambda_j - i)}\right)^L = \prod_{\substack{j=1 \\ j\neq i}}^{M} \dfrac{\sinh\frac{\gamma}{2}(\lambda_j - \lambda_k + 2i)}{\sinh\frac{\gamma}{2}(\lambda_j - \lambda_k - 2i)}$
[JM]	ζ	–	$R^{\varepsilon_1',\varepsilon_2'}_{\varepsilon_1,\varepsilon_2}$	T	$\mathcal{T}$	–
[KBI]	λ	$p_0(\lambda) = i\ln\left[\dfrac{\cosh(\lambda - i\eta)}{\cosh(\lambda + i\eta)}\right]$	–	τ	T	$\left(\dfrac{\lambda_j + ic/2}{\lambda_j - ic/2}\right)^M = \prod_{\substack{k=1 \\ k\neq j}}^{N} \dfrac{\lambda_j - \lambda_k + ic}{\lambda_j - \lambda_k - ic}$
[Bat]	–	–	–	–	–	–
[DEFK]	λ	–	–	t	T_a	$\left(\dfrac{\sinh(\lambda_i + i\mu s)}{\sinh(\lambda_i - i\mu s)}\right)^N = \prod_{i\neq j}^{M} \dfrac{\sinh(\lambda_i - \lambda_j + i\mu)}{\sinh(\lambda_i - \lambda_j - i\mu)}$
[Fad]	λ	–	–	F	$T_{N,a}$	$\left(\dfrac{\lambda_j + i/2}{\lambda_j - i/2}\right)^N = \prod_{k\neq j}^{l} \dfrac{\lambda_j - \lambda_k + i}{\lambda_j - \lambda_k - i}$
[Fra]	λ	$k_j = \dfrac{1}{\mathrm{i}}\ln\dfrac{\lambda_j + \mathrm{i}}{\lambda_j - \mathrm{i}}$	$\mathcal{L}^{\alpha'\gamma'}_{\alpha\gamma}$	$\mathbf{T}$	$\mathcal{T}$	$\left(\dfrac{\lambda_j + \mathrm{i}}{\lambda_j - \mathrm{i}}\right)^N = \prod_{l\neq j} \dfrac{\lambda_j - \lambda_l + 2\mathrm{i}}{\lambda_j - \lambda_l - 2\mathrm{i}}$
[KM]	–	–	–	–	–	–
[Tol]	–	–	$\omega(\mu\alpha\|\mu'\beta) = R^{\mu'\beta}_{\mu\alpha}$	V	–	–

Notice that [Bax] considers a lattice with M rows and N columns, and the transfer matrix V is obtained as a *horizontal* trace, i.e. by tracing over the vertices on a *row* of the lattice. Phrased differently: thinking of the transfer matrix as a (discrete) time evolution operator, the (periodic) time increases as we move up. The partition function is given by $Z = \mathrm{Trace}\, V^M$. On the other hand, the set-up of [JM] is the opposite: they take N rows and M columns, and compute the transfer matrix T as a *vertical* trace; in this picture time increases to the right. As a result we have that $Z = \mathrm{tr}\,(T^M)$ also in this case.

4 Conclusion: what do we do?

The existence of various notations and conventions of signs can be annoying, but it is a part of life — so we may as well get used to it. Nevertheless, to avoid unnecessary confusion, let us at least stick to the following throughout our seminar: we use L for the number of **l**attice sites, i.e. the **l**ength of the spin chain, and M for the number of **m**agnons.