Quasi-triangular Hopf algebras and Yangians
Hand-in exercise 5 (for 12-12)

J. Lamers

v4 (fixed several typos in hindsight),
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1. More string diagrams: quasi-triangularity

Let $(H, m, \eta, \Delta, \varepsilon)$ be a Hopf algebra. Recall the string diagrams from hand-in exercise 4. The convenient shorthand shown in Figure 1 below is justified by (co)associativity.

i) Convert the statement and the proof of Corollary 1.9 of [Maj] into string diagrams.

The remainder of this exercise refers to the first two pages of § 5 of [Maj].

Recall that a *universal R-matrix* for H is an invertible element $R \in H \otimes H$ satisfying some axioms. Being elements of the tensor product $H \otimes H$, R and its inverse can be depicted as shown in Figure 2. The axioms for R are then given in Figure 3. If such a universal R-matrix exists the Hopf algebra is called *quasi-triangular*.

ii) Show that cocommutative Hopf algebras are always quasi-triangular by considering $R = \mathbf{1} \otimes \mathbf{1}$.

iii) Express that R and R^{-1} are inverse to each other in terms of string diagrams.

iv) Turn the statement and proof of part 1 of Lemma 5.2 into string diagrams.

For us, the importance of quasi-triangular Hopf algebras lies in the fact that they give rise to a solution of the Yang-Baxter equation, cf. part 3 of Lemma 5.2.

v) Convert the statement and the proof the latter into string diagrams.

vi) Do the same for Lemma 5.3.

2. More Hopf algebras (and a Frobenius algebra)

So far we have encountered some examples of Hopf algebras related to quantum integrable spin chains. To get some more familiarity with Hopf algebras, here are some different examples. We start with a finite group G, not necessarily abelian.

Step 0: vector space. First we construct a vector space $\mathbf{C}[G]$, consisting of formal linear combinations of elements in G: by definition, $v \in \mathbf{C}[G]$ is of the form

$$v = \sum_{g \in G} c_g \, g \qquad \text{with} \;\; c_g \in \mathbf{C} \, .$$

Thus, the group elements form a basis $\{g\}_{g \in G}$ for $\mathbf{C}[G]$,[#1] and addition and complex multiplication are defined via the scalar structure of the coefficients: if $v = \sum_g c_g \, g$, $v' = \sum_g c'_g \, g$ and $\lambda \in \mathbf{C}$ then $v + \lambda \, v' := \sum_g (c_g + \lambda \, c'_g) \, g$. (This construction does not use that G is a group, and can be performed for any finite set to get the *free vector space* on that set.)

[1] If you find this double role of the g confusing you may write $\{e_g\}$ for the corresponding basis.

Step 1: algebra structures. We also have the group operation at our disposal. The vector space $\mathbf{C}[G]$ is turned into an algebra by defining multiplication on the basis by $m_1(g,h) \coloneqq g\,h$ and extending bilinearly to the rest of $\mathbf{C}[G]$. The resulting algebra is known as the *group algebra* of G over $\mathbf{C}$. Actually, there is a second multiplication that can be used to produce an algebra from the vector space $\mathbf{C}[G]$, given by $m_2(g,h) \coloneqq \delta_{g,h}\, g$, again extended bilinearly.

i) Give explicit expressions for $m_1, m_2 \colon \mathbf{C}[G] \otimes \mathbf{C}[G] \longrightarrow \mathbf{C}[G]$ by writing down the formulas for the coefficients c_g of the products $m_1(v,v')$ and $m_2(v,v')$ of $v, v' \in \mathbf{C}[G]$.

ii) Give explicit expressions for the corresponding unit maps $\eta_1, \eta_2 \colon \mathbf{C} \longrightarrow \mathbf{C}[G]$.

Step 2: coalgebra structures. Define $\Delta_1, \Delta_2 \colon \mathbf{C}[G] \longrightarrow \mathbf{C}[G] \otimes \mathbf{C}[G]$ by $\Delta_1(g) \coloneqq g \otimes g$ and

$$\Delta_2(g) \coloneqq \sum_{g', g'' \in G \text{ s.t. } g'\, g'' = g} g' \otimes g'' \ ,$$

both extended bilinearly.

iii) Check coassociativity for both comultiplications.

iv) Give explicit expressions for the corresponding counit maps $\varepsilon_1, \varepsilon_2 \colon \mathbf{C}[G] \longrightarrow \mathbf{C}$.

v) Check the three bialgebra axioms for $(\mathbf{C}[G], m_1, \eta_1, \Delta_1, \varepsilon_1)$ and $(\mathbf{C}[G], m_2, \eta_2, \Delta_2, \varepsilon_2)$. Which are (co)commutative?

vi) Check that $(\mathbf{C}[G], m_2, \eta_2, \Delta_1, \varepsilon_1)$ is nearly, but not quite, a bialgebra.

vii) Prove that $(\mathbf{C}[G], m_1, \eta_1, \Delta_2, \varepsilon_2)$ does not even satisfy the most important axiom for bialgebras (i.e. the axiom involving both m and Δ). You can do this by taking your favourite group and giving a counter example.

Intermezzo: Frobenius algebras. A *Frobenius algebra* is both an algebra and a coalgebra, with compatibility condition shown in Figure 4 below in terms of string diagrams (cf. hand-in exercise 4).

viii) Translate the Frobenius condition from Figure 4 into commutative diagrams.

ix) Prove that $(\mathbf{C}[G], m_1, \eta_1, \Delta_2, \varepsilon_2)$ is a Frobenius algebra.

Step 3: adding an antipode. Finally, inversion yields an antipode $S \colon \mathbf{C}[G] \longrightarrow \mathbf{C}[G]$ via $S(g) = g^{-1}$.

x) Give the explicit expression of $S(v)$ for arbitrary $v \in \mathbf{C}[G]$.

xi) By checking the Hopf algebra axiom, prove that the two bialgebras from (v) are Hopf algebras.

Step 4: quasi-triangularity. In which case do these Hopf algebras admit a universal R-matrix? Keeping in mind part (ii) of the first exercise, only one of the two Hopf algebras from (xi) is interesting. Given an element $R \in \mathbf{C}[G] \otimes \mathbf{C}[G]$, let's expand it in terms of the basis $\{g \otimes h\}_{g,h \in G}$ for $\mathbf{C}[G]^{\otimes 2}$ as

$$R = \sum_{g,\,h} R_{g,\,h}\, g \otimes h \ .$$

xii) Prove that the first axiom in Figure 3 is satisfied if G is abelian.

xiii) Prove that R is invertible precisely when $R_{g,\,h} \neq 0$ for all $g, h \in G$.

xiv) The remaining axioms from Figure 3 further impose conditions on the coefficients $R_{g,\,h}$ of R. Using the explicit expressions for the (co)multiplication for the interesting Hopf algebra, compare coefficients of basis elements $f \otimes g \otimes h \in \mathbf{C}[G]^{\otimes 3}$ to find these conditions on the $R_{g,\,h}$.

(It is not hard to show that the reverse of these three statements are true as well.)

3. Yangians

In this exercise we recap the material of last week's lecture. Write $\{E_{ij} \mid 1 \leq i,j \leq N\}$ for the standard basis of $\mathfrak{gl}_N$, with entries $(E_{ij})_{kl} = \delta_{ik}\,\delta_{jl}$. Defining the $N \times N$ matrix E with (i,j)th entry equal to E_{ij}, we have the following relation for the (tensor)powers of E in the universal enveloping algebra $U(\mathfrak{gl}_N)$:

$$[(E^{r+1})_{ij}, (E^s)_{kl}] - [(E^r)_{ij}, (E^{s+1})_{kl}] = (E^r)_{kj}(E^s)_{il} - (E^s)_{kj}(E^r)_{il}\ , \qquad r,s \geq 0\ .$$

Here we interpret $E^0 = \mathbf{1}$ as the $N \times N$ identity matrix.

i) Check this relation for $r, s \leq 1$.

Inspired by this, the *Yangian* $Y(\mathfrak{gl}_N)$ is abstractly defined as the (unital, associative) algebra generated by $\{t_{ij}^{(r)}\}_{r\geq 0}$ subject to the defining relations

$$[t_{ij}^{(r+1)}, t_{kl}^{(s)}] - [t_{ij}^{(r)}, t_{kl}^{(s+1)}] = t_{kj}^{(r)} t_{il}^{(s)} - t_{kj}^{(s)} t_{il}^{(r)} \qquad r,s \geq 0 \tag{1}$$

and with $t_{ij}^{(0)} \coloneqq \delta_{ij}$. This axiomatizes the above properties of $U(\mathfrak{gl}_N)$.

The Yang-Baxter equation. The Yangian is related to Yang-Baxter algebras in the following way (for the special case $N = 2$ we will find the XXX Yang-Baxter algebra). Package the $t_{ij}^{(r)}$ into generating functions $t_{ij}(u) \in Y(\mathfrak{gl}_N)[[u^{-1}]]$ defined by

$$t_{ij}(u) \coloneqq \sum_{r\geq 0} t_{ij}^{(r)}\, u^{-r}\ .$$

In terms of these, (1) is equivalent to the relation

$$(u-v)\,[t_{ij}(u), t_{kl}(v)] = t_{kj}(u)\,t_{il}(v) - t_{kj}(v)\,t_{il}(u) \tag{2}$$

in $Y(\mathfrak{gl}_N)[[u^{-1}, v^{-1}]]$.

ii) Prove that indeed (1) $\Longleftrightarrow$ (2).

The *monodromy matrix* $T(u) \in Y(\mathfrak{gl}_N)[[u^{-1}]] \otimes \mathrm{End}(\mathbf{C}^N)$ is defined as

$$T \coloneqq \sum_{i,j=1}^{N} t_{ij}(u) \otimes E_{ij}.$$

Further introduce the 'rational' (= XXX) R-matrix $R(u) \coloneqq \mathbf{1}\otimes\mathbf{1} - u^{-1}P \in \mathrm{End}(\mathbf{C}^N)\otimes\mathrm{End}(\mathbf{C}^N)[[u^{-1}]]$, where $P = \sum_{ij} E_{ij}\otimes E_{ji}$ is the permutation operator braiding the two copies of $\mathrm{End}(\mathbf{C}^N)$. In class it was shown that (2) is equivalent to the RTT or Yang-Baxter equation

$$R(u-v)\,T_1(u)\,T_2(v) = T_2(v)\,T_1(u)\,R(u-v) \tag{3}$$

on $Y(\mathfrak{gl}_N)[[u^{-1}, v^{-1}]] \otimes \mathrm{End}(\mathbf{C}^N) \otimes \mathrm{End}(\mathbf{C}^N)$.

Hopf algebra structure. The Yangian $Y(\mathfrak{gl}_N)$ is a Yang-Baxter (bi)algebra under the following operations: $\Delta(t_{ij}(u)) \coloneqq \sum_k t_{ik}(u)\otimes t_{kj}(u)$ and $\varepsilon(T(u)) \coloneqq \mathbf{1} \in \mathrm{End}(\mathbf{C}^N)$. Moreover, $S(T(u)) \coloneqq T^{-1}(u)$ defines an antipode for $Y(\mathfrak{gl}_N)$. Together, these operations endow the Yangian with the structure of a quasi-triangular Hopf algebra.

Here we have compactly written down relations for $Y(\mathfrak{gl}_N)[[u^{-1}]]$, that is, we have given formal power series in u^{-1} whose coefficients lie in $Y(\mathfrak{gl}_N)$. From these expressions, the operations on the actual generators $t_{ij}^{(r)}$ of the Yangian can be found by expanding both sides in u^{-1} and matching coefficients.

iii) Do this to find $\Delta(t_{ij}^{(r)})$, $\varepsilon(t_{ij}^{(r)})$ and $S(t_{ij}^{(r)})$ for $N = 2$ and $r \leq 1$.

Fig. 1

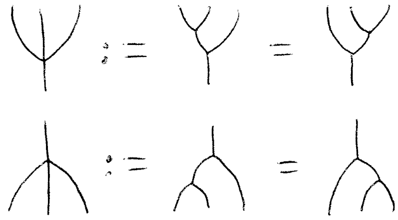

Fig. 2

Fig. 3

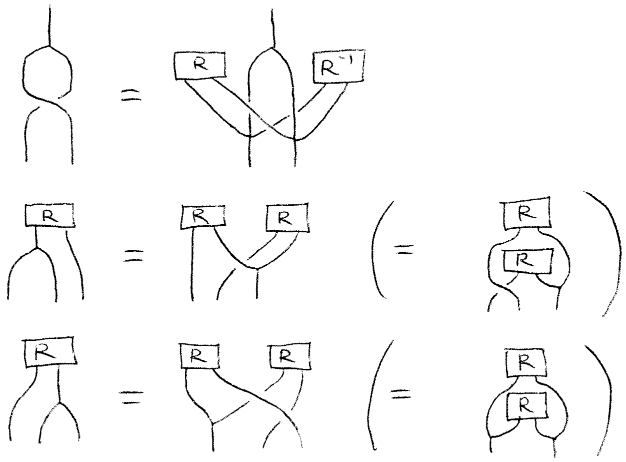

Fig. 4