**Orbispaces**

by

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## Abstract

In this thesis, I introduce a new definition for orbispaces based a notion of stratified fibration and prove it's equivalence with other existing definitions. I study the notion of orbispace structures on a given stratified space. I then set up two parallel theories of stratified fibrations, one for topological spaces, and one for simplicial sets.

Modulo a technical comparison between the two theories, I construct a classifying space for orbispace structures. Using a conjectural obstruction theory, I then prove that every compact orbispace is equivalent to the quotient of a compact space by the action of a compact Lie group.


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# Contents

# Chapter 1

# Introduction

This thesis naturally splits in two parts.

In the first part, we introduce a new, easily understandable definition for orbispaces (the topological analogue of orbifolds). From our point of view, an orbispace is a pair of spaces $E \to X$, whose fibers are classifying spaces of finite groups. The local model, being the Borel construction $(Y \times EG)/G$ mapping to the topological quotient $Y/G$ (also known as the coarse moduli space).

Other equivalent notions, such as stacks, topological groupoids or complexes of groups are well known. So in some sense, we are only add a new item to this list. In the first chapter of the thesis, we explain all these approaches in more detail, and provide constructions to go from one to the other. Later, in chapter 3, we prove the equivalence of our new definition at the level of 2-categories.

The map from $(Y \times EG)/G$ to $Y/G$ is not a fibration since the homotopy type of the fibers varies according to the size of the stabilizer group. We introduce a new notion of stratified fibration, which is adapted to this situation. We then show that an orbispace $E \to X$ is a stratified fibration in our sense.

Finally, we explain how one might use our definition of orbispaces in various circumstances. More specifically, we treat of the question of bundle theory, group actions, sheaf theory, and a little bit of elementary algebraic topology.

In the second part of this thesis, we describe a theory of stratified simplicial sets. Though we do not prove this here, we expect that the theories of stratified spaces and stratified simplicial sets correspond and thus we freely translate results between the two worlds.

The goal of this second half is, among other things, to outline the proof of the global quotient conjecture. It states that every compact orbispace is a quotient of a compact space by a compact Lie group. On our way to proving this conjecture, we introduce a few new notions of independent interest:

We define a stratified simplicial set to be a simlicial set equipped with a map to the nerve of that poset. We then introduce a new conjectural model structure on that category. We give an explicit construction of a stratified classifying space $\underline{\mathsf{Orb}}$ for orbispace structures on a stratified space. The simplicial set $\underline{\mathsf{Orb}}$ is stratified by the poset of isomorphism classes of finite groups. It comes with a universal orbispace structure $E_{\underline{\mathsf{Orb}}} \to \underline{\mathsf{Orb}}$, and has the property that homotopy classes of stratified maps into $\underline{\mathsf{Orb}}$ are in bijective correspondence with isomorphism classes of orbispace structures.

Given a topological group $K$ and a family of subgroups $\mathcal{F}$, we show that $B_{\mathcal{F}}K$, the well known classifying space for $\mathcal{F}$, is a stratified classifying space for the structure of "being of the form $Y/K$", where $Y$ is a $K$-space with stabilizers in $\mathcal{F}$. Given a compact orbispace $E \to X$, we then use obstruction theory to show that the representing map $X \to \underline{\mathsf{Orb}}$ always lifts to a map $X \to B_{\mathcal{F}}K$, for some appropriate $K$ and $\mathcal{F}$. In other words, every orbispace is a global quotient. The group $K$ can be taken to be a large unitary group.

Towards the end, we explain an interesting connection between vector bundles on orbispaces and global quotients by Lie groups. In particular, we show the the excision propoerty for $K$-theory is equivalent to the global quotient problem.

We also study global quotient by more general topological groups. We introduce the notion of a group which is contractible with respect to a family of subgroups. Then we show that if $K$ is contractible with respect to its finite subgroups, and if every finite groups occurs as a subgroup of $K$, then the categories of $K$-spaces and of orbipsaces are homotopy equivalent.

# Chapter 2

# Survey of existing definitions

In the next chapter, we will introduce a new definition for orbifolds and orbispaces. Before doing so, we survey the most important existing definitions and their interrelations.

The word "orbispace" can take different meanings, depending on which underlying category of "spaces" one works in, and which groups one allows. Morally, an orbispace is something that looks locally like the quotient of a space by a group. If we take "space" to mean manifold and "group" to mean finite group, this results in the usual notion of an orbifold [13][25]. If we interpret "space" to mean algebraic variety, we recover the notion of Deligne-Mumford stack [6][22]. If on the other hand we work with simplicial complexes, then we recover the notion of complex of groups of Bridson and Heafliger [4].

**Remark 2.1** We should warn the reader that sometimes the definition of orbifolds is stated in a way that requires them to be "effective" (for example [32][34]). Namely, they have to be be modeled by the quotient of a manifold by the action of a finite group acting *faithfully*. We shall stay away from this requirement which we find unnatural for our setting.

**Remark 2.2** In this chapter, we use topological spaces as our category of "spaces". But we only consider those spaces arising as geometric realizations of simplicial complexes. So for us, a space will always come along with a triangulation.

**Remark 2.3** Unlike geometers, when topologists glue two spaces together (for example when attaching a cell) they do not identify *open* subsets of these spaces. To account for this, we will modify the usual notion of cover. For us, a cover will always mean a cover by *closed* subspaces[1]. For example the intervals $[0, 1]$ and $[1, 2]$ form a cover of $[0, 2]$.

We now survey the most important definitions of orbispaces existing in the literature, and their connection to each other. Since the definitions listed below have their origin in some different category of "spaces", they been modified to fit the world of topology. We have tried to state them in a way that is as close as possible from

[1]More precisely, our Grothendieck topology is generated by proper surjective maps

the original setting, and hope that the reader will be able to adapt them back to the context (s)he is working in.

**Definition 2.4 (topological groupoids)** *A topological groupoid consists of two spaces $\mathcal{G}_0$ and $\mathcal{G}_1$ called objects and morphisms along with maps $s, t : \mathcal{G}_1 \to \mathcal{G}_0$ called source and target, a map $u : \mathcal{G}_0 \to \mathcal{G}_1$ called unit, and a map $\mu : \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \to \mathcal{G}_1$ called multiplication (all these maps are compatible with the triangulations). These maps satisfy the usual axioms for groupoids: unit, associativity, existence of inverse. We use the symbol $\mathcal{G}_1 \rightrightarrows \mathcal{G}_0$ to denote the groupoid $(\mathcal{G}_0, \mathcal{G}_1, s, t, u, \mu)$. Given a point $x \in \mathcal{G}_0$ the group $s^{-1}(x) \cap t^{-1}(x)$ is denoted $\mathrm{Aut}(x)$ and is called the automorphism group of the object $x$.*

*An orbispace is a topological groupoid, where all objects $x \in \mathcal{G}_0$ have finite automorphism groups and where the image of $u$ is a connected component of $\mathcal{G}_1$.*

**Definition 2.5 (stacks)** *A stack is a functor $F$ (in the sense of bicategories [2]) from the category of spaces to the category of groupoids. For any cover $\{V_i\}$ of a space $T$, the map*

$$F(T) \to \varprojlim \Big[ \coprod F(V_{ijk}) \overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}} \coprod F(V_{ij}) \leftleftarrows \coprod F(V_i) \Big] \tag{2.1}$$

*is an equivalence of groupoids. Here $V_{ij}$ and $V_{ijk}$ denote the double and triple intersections of $V_i$ and the limit is taken in the bicategorical sense.*

*To a space $X$ we associate the stack $Y(X)$ given by $Y(X)(T) := \mathrm{Hom}(T, X)$, where the set $\mathrm{Hom}(T, X)$ is viewed as a discrete groupoid. To a group $G$ we associate a stack $BG$ given by $BG(T) = \{G\text{-principal bundles on } T\}$.*

*A map of stacks $f : F \to F'$ is representable if for any point $p \in F(pt)$, the induced map $\mathrm{Aut}(p) \to \mathrm{Aut}(f(p))$ is injective. Let us call orbisimplex a stack of the form $Y(\Delta^n) \times BG$, where $\Delta^n$ denotes the $n$-simplex and $G$ is a finite group.*

*An orbispace is a stack $F = \varinjlim F^{(n)}$, where each $F^{(n)}$ is obtained from the previous one by taking a pushout*

$$\begin{tikzcd} \coprod \partial\tau_i \arrow[r] \arrow[d, "\coprod \alpha_i"'] & \coprod \tau_i \arrow[d] \\ F^{(n-1)} \arrow[r] & F^{(n)}. \end{tikzcd} \tag{2.2}$$

*Here the $\tau_i$ are orbisimplices $Y(\Delta^n) \times BG_i$, with boundaries $\partial\tau_i = Y(\partial\Delta^n) \times BG_i$. The attaching maps $\alpha_i : \partial\tau_i \to F^{(n-1)}$ are required to be representable.*

**Definition 2.6 (complexes of groups)** *A complex of groups is a space $X$ along with the following data. To each simplex $\sigma_i$ of $X$, we associate a group $G_i$. To each incidence $\sigma_i \supset \sigma_j$ we associate a group homomorphism $\phi_{ij} : G_i \to G_j$. To each double incidence $\sigma_i \supset \sigma_j \supset \sigma_k$ we associate a group element $g_{ijk} \in G_k$ satisfying*

$$\phi_{ik} = Ad(g_{ijk})\, \phi_{jk}\, \phi_{ij}. \tag{2.3}$$

*Finally, for each triple incidence $\sigma_i \supset \sigma_j \supset \sigma_k \supset \sigma_\ell$ the above elements must satisfy* *the cocycle condition*

$$g_{ij\ell}\ g_{jk\ell} = g_{ik\ell}\ \phi_{k\ell}(g_{ijk}). \tag{2.4}$$

*An orbispace is a complex of groups where the $G_i$ are finite and the $\phi_{ij}$ are injective.*

**Definition 2.7 (charts)** *An orbispace atlas on a space $X$ is a cover $\{U_i\}$, closed under finite intersections, along with the following data. For each $U_i$, we are given a "branched covering" $\pi_i : \widetilde{U}_i \to U_i$, with an action $G_i \circlearrowright \widetilde{U}_i$ by a finite group $G_i$. The maps $\pi_i$ are invariant under the action of $G_i$ and induce homeomorphisms between $G_i \backslash \widetilde{U}_i$ and $U_i$. For each inclusion $U_i \subset U_j$, we are given injective group homomorphisms $\phi_{ij} : G_i \to G_j$ and $\phi_{ij}$-equivariant maps $\alpha_{ij} : \widetilde{U}_i \to \widetilde{U}_j$ satisfying $\pi_i = \pi_j \circ \alpha_{ij}$. For any point $x \in \widetilde{U}_i$, the homomorphisms $\phi_{ij}$ induces an isomorphism between the $G_i$-stabilizer of $x$ and the $G_j$-stabilizer of its image $\alpha_{ij}(x) \in \widetilde{U}_j$. For each double inclusion $U_i \subset U_j \subset U_k$, we are given group elements $g_{ijk} \in G_k$ satisfying*

$$\alpha_{ik} = g_{ijk}\alpha_{jk}\alpha_{ij} \tag{2.5}$$

*and the cocycle conditions (2.3) and (2.4).*

*An orbispace is a space $X$ along with an orbispace atlas $(\{U_i\}, \{\widetilde{U}_i\}, \pi_i, \alpha_{ij}, \phi_{ij}, g_{ijk})$.*

## 2.1 Comparison between definitions

We now explain how to go from any one of the above definitions to another. This will be done in the following order:

$$\begin{array}{ccc} \textbf{(groupoids)} & \longrightarrow & \textbf{(stacks)} \\ \uparrow & & \downarrow \\ \textbf{(charts)} & \longleftarrow & \textbf{(complexes of groups)}. \end{array}$$

### 2.1.1 From groupoids to stacks

To go from a topological groupoid $\mathcal{G} = (\mathcal{G}_1 \rightrightarrows \mathcal{G}_0)$ to a stack $F = F(\mathcal{G})$, first consider the prestack $\widetilde{F}$ given by $\widetilde{F}(T) = \big(\mathrm{Hom}(T, \mathcal{G}_1) \rightrightarrows \mathrm{Hom}(T, \mathcal{G}_0)\big)$, and then stackify $\widetilde{F}$ to get $F$. More precisely, this is done by letting

$$F(T) = \varinjlim_{\{V_i\}} \left[ \varprojlim \left[ \coprod \widetilde{F}(V_{ijk}) \Lleftarrow \coprod \widetilde{F}(V_{ij}) \leftleftarrows \coprod \widetilde{F}(V_i) \right] \right],$$

where the colimit is taken over all covers $\{V_i\}$ of $T$ ordered by refinement, and the limit is as described in Definition 2.5.

**Proposition 2.8** *The stack $F$ is of the desired form.*

*Proof.* Let $\sigma \subset \mathcal{G}_0$ be an $n$-simplex. The assumption on the image of $u$ implies that $s^{-1}(\sigma)$ is a disjoint union of $n$-simplices. To see that, we show that paths $\gamma : [0,1] \to \sigma$

satisfy the unique lifting property. Consider two lifts $\tilde{\gamma}_1, \tilde{\gamma}_2 : [0,1] \to s^{-1}(\sigma)$ with same initial point $\tilde{\gamma}_1(0) = \tilde{\gamma}_2(0)$. The path $\delta(t) := \tilde{\gamma}_1(t)\tilde{\gamma}_2(t)^{-1}$ satisfies $\delta(0) \in \mathrm{Im}(u)$. Since $\mathrm{Im}(u)$ is a connected component on $\mathcal{G}_1$, all of $\delta$ lands in $\mathrm{Im}(u)$ and therefore $\tilde{\gamma}_1 = \tilde{\gamma}_2$. The same argument shows that $t$ maps $n$-simplices to $n$-simplices.

At this point, let us barycentrically subdivide $\mathcal{G}_0$ and $\mathcal{G}_1$. Now all simplices have an order on their vertices and the structure maps preserve that order.

An $n$-simplex $\sigma \subset \mathcal{G}_0$ determines a full subgroupoid $\underset{\sim}{\sigma} \subset \mathcal{G}$ given by $\underset{\sim}{\sigma}_0 = t(s^{-1}(\sigma))$ and $\underset{\sim}{\sigma}_1 = s^{-1}(\underset{\sim}{\sigma}_0)$. Since $s$ and $t$ map $n$-simplices to $n$-simplices and since all structure maps preserve a given order on the vertices of these simplices, $\underset{\sim}{\sigma}$ is just the direct product of a discrete groupoid $\underset{\sim}{G}$ with the simplex $\Delta^n$. Clearly $\underset{\sim}{G}$ is connected, so it is equivalent to the groupoid $G \rightrightarrows pt$, where $G = \mathrm{Aut}(x)$ for some point $x \in \sigma$. Let $\partial\underset{\sim}{\sigma}$ be the groupoid made from of all the boundaries of all the $n$-simplices of $\underset{\sim}{\sigma}_0$ and $\underset{\sim}{\sigma}_1$.

The $n$-skeleton $\mathcal{G}^{(n)} = (\mathcal{G}_0^{(n)} \rightrightarrows \mathcal{G}_1^{(n)})$ is obtained from the $(n-1)$-skeleton by a pushout diagram :

$$\begin{tikzcd} \coprod \partial\underset{\sim}{\sigma} \arrow[r] \arrow[d, "\coprod \alpha"'] & \coprod \underset{\sim}{\sigma} \arrow[d] \\ \mathcal{G}^{(n-1)} \arrow[r] & \mathcal{G}^{(n)}. \end{tikzcd} \tag{2.6}$$

Letting $F^{(n)}$ be the stack represented by $\mathcal{G}^{(n)}$, we get from (2.6) the desired pushout of stacks (2.2). Indeed, $\Delta^n$ represents $Y(\Delta^n)$, $\underset{\sim}{G} \simeq (G \rightrightarrows pt)$ represents $BG$, and therefore $\underset{\sim}{\sigma}$ represents $Y(\Delta^n) \times BG$ as desired.

Now we explain why the attaching maps $\alpha : \partial\underset{\sim}{\sigma} \to \mathcal{G}^{(n-1)}$ are representable. In the language of groupoids, we need to show that they induce monomorphisms on the automorphism groups of objects. This is actually quite trivial trivial since both $\partial\underset{\sim}{\sigma}$ and $\mathcal{G}^{(n-1)}$ are subgroupoids of $\mathcal{G}$ and $\alpha$ is just the inclusion. $\square$

### 2.1.2 From stacks to complexes of groups

Given a stack $F$, let $X := \pi_0 F(pt)$ be the underlying space of our complex of groups. For each closed orbisimplex $\sigma_i \subset F$, denote by $n_i$ its dimension and by $G_i$ its isotropy group. That is, we have $\sigma_i \simeq Y(\Delta^{n_i}) \times BG_i$. These are the groups that decorate the simplices of $X$. For each $\sigma_i$ we have

$$\sigma_i(\Delta^{n_i}) = \mathrm{Hom}(\Delta^{n_i}, \Delta^{n_i}) \times \{G_i\text{-principal bundles on } \Delta^{n_i}\}.$$

Let $p_i \in \sigma_i(\Delta^{n_i})$ be the object corresponding to $\mathrm{Id}_{\Delta^{n_i}} \times$ (trivial bundle). Note that $\mathrm{Aut}(p_i) \simeq G_i$. For each face $\sigma_j$ of $\sigma_i$ in $F$, we let $\sigma_i^j$ be the corresponding "abstract face" of $\sigma_i$, namely $\sigma_i^j \simeq Y(\Delta^{n_j}) \times BG_i$. It maps to $\sigma_j$ via the attaching map of $\sigma_i$. We let $p_i^j \in \sigma_j(\Delta^{n_j})$ be the image of $p_i$ under the composite $\sigma_i(\Delta^{n_i}) \to \sigma_i(\Delta^{n_j}) \supset$

$\sigma_i^j(\Delta^{n_j}) \to \sigma_j(\Delta^{n_j})$, where the first arrow is $\sigma_i(\Delta^{n_j} \hookrightarrow \Delta^{n_i})$ and the last one is the attaching map of $\sigma_i$ restricted to $\sigma_i^j$. We note that $\sigma_i(\Delta^{n_j} \hookrightarrow \Delta^{n_i})(p_i)$ lands in the essential image of the full subgroupoid $\sigma_i^j(\Delta^{n_j})$ of $\sigma_i(\Delta^{n_j})$, so $p_i^j$ is well defined up to unique isomorphism.

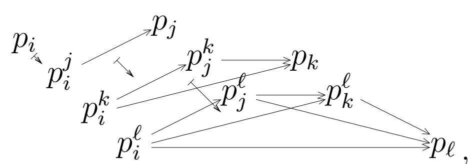
We let $[\sigma_i] := \pi_0\ \sigma_i(pt)$ denote the simplex of $X$ corresponding to $\sigma_i$. For each incidence $[\sigma_i] \supset [\sigma_j]$, we note that $p_i^j$ is isomorphic to $p_j$ in $\sigma_j(\Delta^{n_j})$. We pick such an isomorphism $\varphi_{ij} : p_i^j \to p_j$ and let

$$\phi_{ij} : G_i = \mathrm{Aut}(p_i) \longrightarrow \mathrm{Aut}(p_i^j) \xrightarrow{\varphi_{ij}(\ \ )\varphi_{ij}^{-1}} \mathrm{Aut}(p_j) = G_j.$$

These are the group homomorphisms that are part of the data of our complex of groups. They are injective because the attaching maps are representable.

A double incidence $[\sigma_i] \supset [\sigma_j] \supset [\sigma_k]$ leads to three morphisms $\varphi_{ij} : p_i^j \to p_j$, $\varphi_{jk} : p_j^k \to p_k$, and $\varphi_{ik} : p_i^k \to p_k$. Let $\varphi_{ij}^k : p_i^k \to p_j^k$ denote the restriction of $\varphi_{ij}$ to $\Delta^{n_k}$. We then define $g_{ijk} := \varphi_{ik}\left(\varphi_{ij}^k\right)^{-1}\varphi_{jk}^{-1} \in G_k = \mathrm{Aut}(p_k)$. These are the group elements that come in the definition of our complex of groups. It is straightforward to verify the condition $\phi_{ik} = Ad(g_{ijk})\,\phi_{jk}\,\phi_{ij}$ from the definition of $\phi_{ij}$ and $g_{ijk}$.

Now we need to show the cocycle identity for the $g_{ijk}$. A triple incidence $[\sigma_i] \supset [\sigma_j] \supset [\sigma_k] \supset [\sigma_\ell]$ leads to a diagram




where the arrows "$\to$" are the morphisms $\varphi$ and the arrows "$\mapsto$" denote restriction functors between groupoids. Each triangle in the above tetrahedron corresponds to a $g_{ijk}$, and the cocycle relation is then clear:

$$\begin{aligned} g_{ij\ell}\ g_{jk\ell} &= \left[\varphi_{i\ell}(\varphi_{ij}^\ell)^{-1}\varphi_{j\ell}^{-1}\right]\left[\varphi_{j\ell}(\varphi_{jk}^\ell)^{-1}\varphi_{k\ell}^{-1}\right] \\ &= \left[(\varphi_{i\ell}\varphi_{ik}^\ell)^{-1}\varphi_{k\ell}^{-1}\right]\varphi_{k\ell}\left[\varphi_{ik}^\ell(\varphi_{ij}^\ell)^{-1}(\varphi_{jk}^\ell)^{-1}\right]\varphi_{k\ell}^{-1} = g_{ik\ell}\ \phi_{k\ell}(g_{ijk}). \end{aligned}$$

Note that we have mapped $g_{ijk} = \varphi_{ik}\left(\varphi_{ij}^k\right)^{-1}\varphi_{jk}^{-1} \in \mathrm{Aut}(p_k)$ to the corresponding element $\varphi_{ik}^\ell(\varphi_{ij}^\ell)^{-1}(\varphi_{jk}^\ell)^{-1}$ of $\mathrm{Aut}(p_k^\ell)$.

### 2.1.3 From complexes of groups to charts

Let $X$ be a complex of groups. Given a simplex $\sigma$ in $X$, we let $U_\sigma$ be the union of all simplices $\tau$ of the barycentric subdivision of $X$ such that $\sigma \cap \tau$ is the barycenter of $\sigma$. The intersection $U_\sigma \cap U_{\sigma'}$ is either empty or equal to $U_{\sigma''}$, where $\sigma''$ is the simplex whose vertices is the union of the vertices of $\sigma$ and of $\sigma'$. So the $U_\sigma$ for a cover of $X$ which is closed under finite intersections.

For a simplex $\sigma_i$, let us use $U_i$ instead of $U_{\sigma_i}$. Also, we let $G_i^j := \phi_{ij}(G_i)$. For each $U_k$, we define the corresponding $\widetilde{U}_k$ to be the quotient

$$\widetilde{U}_k := \Big( \bigcup_{\sigma_j \supset \sigma_k} (\sigma_j \cap U_k) \times G_k/G_j^k \Big) \Big/ \sim ,$$

where the equivalence relation is generated by $(x, mG_i^k) \sim (x, m\, g_{ijk} G_j^k)$. The spaces $\widetilde{U}_j$ admit a left action of $G_j$ given by $h \cdot (x, mG_i^j) = (x, hmG_i^j)$ and the map $\pi_j : (x, mG_i^j) \mapsto x$ induces a homeomorphism $G_j \backslash \widetilde{U}_j \simeq U_j$. The $\phi_{ij}$ and $g_{ijk}$ required for Definition 2.7 are taken identical to those in Definition 2.6.

The $\alpha$'s are given by $\alpha_{jk}(x, mG_i^j) := (x, \phi_{jk}(m) g_{ijk}^{-1} G_i^k)$. To see that they are well defined we take two representatives $(x, mG_i^k)$ and $(x, m\, g_{ijk} G_j^k)$ of the same point of $\widetilde{U}_k$ and check using (2.4) that their values agree:

$$\begin{aligned} &\alpha_{k\ell}(x, mG_i^k) = \big(x, \phi_{k\ell}(m) g_{ik\ell}^{-1} G_i^\ell\big) \sim \big(x, \phi_{k\ell}(m) g_{ik\ell}^{-1}\, g_{ij\ell} G_j^\ell\big) \quad \text{and} \\ &\alpha_{k\ell}(x, m\, g_{ijk} G_j^k) = \big(x, \phi_{k\ell}(m\, g_{ijk}) g_{jk\ell}^{-1} G_j^\ell\big). \end{aligned}$$

One then checks the $\phi_{jk}$-equivariance:

$$\begin{aligned} \alpha_{jk}\big(h \cdot (x, mG_i^j)\big) &= \alpha_{jk}\big(x, hmG_i^j\big) \\ &= (x, \phi_{jk}(hm)\, g_{ijk}^{-1}\, G_i^k) \\ &= \big(x, \phi_{jk}(h)\, \phi_{jk}(m)\, g_{ijk}^{-1}\, G_i^k\big) \\ &= \phi_{jk}(h) \cdot \big(\alpha_{jk}(x, mG_i^j)\big) \end{aligned}$$

and the compatibility with the $g_{ijk}$:

$$\begin{aligned} g_{jk\ell}\, \alpha_{k\ell}\, \alpha_{jk}(x, mG_i^j) &= \big(x, g_{jk\ell}\, \phi_{k\ell}(\phi_{jk}(m) g_{ijk}^{-1}) g_{ik\ell}^{-1} G_i^\ell\big) \\ &= \big(x, g_{jk\ell}\, \phi_{k\ell}\, \phi_{jk}(m) \phi_{k\ell}(g_{ijk}^{-1}) g_{ik\ell}^{-1} G_i^\ell\big) \\ &= \big(x, \phi_{j\ell}(m) g_{jk\ell} (g_{ik\ell}\, \phi_{k\ell}(g_{ijk}))^{-1} G_i^\ell\big) \\ &= \big(x, \phi_{j\ell}(m) g_{ij\ell}^{-1} G_i^\ell\big) \\ &= \alpha_{j\ell}(x, mG_i^j). \end{aligned}$$

Finally, we need to check that $\phi_{ij}$ induces an isomorphism between the stabilizer of a point and that of its image under $\alpha_{ij}$. Recall that $G_i^j$ denotes $\phi_{ij}(G_i)$.

**Lemma 2.9** *If $x$ is in the interior of $\sigma_k$, then the stabilizer of $(x, G_k^\ell) \in \widetilde{U}_\ell$ under the action of $G_\ell$ is exactly $G_k^\ell$.*

*Proof.* Let $\approx$ denote the relation given by $(x, mG_j^\ell) \approx (x, m' G_{j'}^\ell)$ if the element $(m\, g_{jk\ell})^{-1}\, m'\, g_{j'k\ell}$ belongs to $G_k^\ell$, where $k$ is chosen such that $x \in \sigma_k \setminus \partial\sigma_k$. We claim that $\approx$ is the equivalence relation generated by $\sim$. This implies the statement about the stabilizer of $(x, G_k^\ell)$ since $\approx$ doesn't identify distinct points of $\{x\} \times G_\ell / G_k^\ell$.

We first show that $\approx$ extends $\sim$. For two points $(x, mG_i^\ell) \sim (x, m\, g_{ij\ell} G_j^\ell)$. We check by (2.4) that $(m\, g_{ik\ell})^{-1}\, m\, g_{ij\ell}\, g_{jk\ell} \in G_k^\ell$ and therefore $(x, mG_i^\ell) \approx (x, m\, g_{ij\ell} G_j^\ell)$. Clearly $\approx$ is reflexive and symmetric, so we check transitivity. Consider the situation $(x, mG_j^\ell) \approx (x, m' G_{j'}^\ell) \approx (x, m'' G_{j''}^\ell)$. Since both group elements $(m\, g_{jk\ell})^{-1}\, m'\, g_{j'k\ell}$ and $(m'\, g_{j'k\ell})^{-1}\, m''\, g_{j''k\ell}$ lie in $G_k^\ell$, the same holds for their product, and we deduce that $(x, mG_j^\ell) \approx (x, m'' G_{j''}^\ell)$. $\square$

By the above Lemma, the $G_j$-stabilizer of a point $(x, mG_i^j) \in \widetilde{U}_j$ is exactly $m\, G_i^j\, m^{-1} \subset G_j$, provided that $x$ sits in the interior of $\sigma_i$. The stabilizer of its image $\alpha_{jk}(x, mG_i^j) = (x, \phi_{jk}(m) g_{ijk}^{-1} G_i^k)$ is then equal to $\phi_{jk}(m) g_{ijk}^{-1} G_i^k\, g_{ijk}\, \phi_{jk}(m)^{-1}$. We now check using (2.3) that $\phi_{jk}$ induces isomorphisms between these two groups:

$$\begin{aligned}
\phi_{jk}\big(mG_i^j m^{-1}\big) &= \phi_{jk}\big(m\, \phi_{ij}(G_i) m^{-1}\big) \\
&= \phi_{jk}(m) \phi_{jk}(\phi_{ij}(G_i)) \phi_{jk}(m)^{-1} \\
&= \phi_{jk}(m) g_{ijk}^{-1}\, \phi_{ik}(G_i) g_{ijk}\, \phi_{jk}(m)^{-1} = \phi_{jk}(m) g_{ijk}^{-1} G_i^k\, g_{ijk}\, \phi_{jk}(m)^{-1}.
\end{aligned}$$

### 2.1.4 From charts to groupoids

Let $X$ be an orbispace given by charts $U_i \simeq G_i \backslash \widetilde{U}_i$. We construct a topological groupoid $\mathcal{G}$ as follows. Its object space is given by $\mathcal{G}_0 = \coprod \widetilde{U}_i$. Its arrows are generated by elements $[h] : x \to h \cdot x$ and $[\alpha_{ij}] : x \to \alpha_{ij}(x)$ subject to the relations

$$[h][h'] = [hh'], \qquad [\alpha_{ij}][h] = [\phi_{ij}(h)][\alpha_{ij}], \quad \text{and} \quad [\alpha_{ik}] = [g_{ijk}][\alpha_{jk}][\alpha_{ij}]. \tag{2.7}$$

These generate a topological groupoid $\mathcal{G}_1 \rightrightarrows \mathcal{G}_0$ whose orbit space $\mathcal{G}_0/\mathcal{G}_1$ is homeomorphic to $X$. We have implicitly used that $\alpha_{ij} h = \phi_{ij}(h) \alpha_{ij}$ and $\alpha_{ik} = g_{ijk} \alpha_{jk} \alpha_{ij}$ since otherwise, (2.7) would not be equations between morphisms sharing the same source and target.

We need to show that the automorphism groups of objects are finite and that the image of the unit map $u$ is clopen of $\mathcal{G}_1$. Both these facts will be a direct consequence of the following proposition:

**Proposition 2.10** *For $x, y \in \widetilde{U}_i \subset \mathcal{G}_0$, the only morphisms $x \to y$ are of the form $[h] : x \to h \cdot x = y$. Moreover, two morphisms $[h], [h'] : x \to y$ are equal in $\mathcal{G}_1$ if and only if $h = h'$ in $G_i$. In other words*

$$s^{-1}(\widetilde{U}_i) \cap t^{-1}(\widetilde{U}_i) = G_i \times \widetilde{U}_i. \tag{2.8}$$

First we show that the automorphism groups are finite. Let $x \in \widetilde{U}_i$. By Proposition 2.10, all the self arrows $x \to x$ are given by elements of $G_i$. Since $G_i$ is finite, we conclude that $\mathrm{Aut}(x)$ is finite. Now we explain why $Im(u)$ is clopen in $\mathcal{G}_1$. The image of $u$ is the disjoint union of $Im(u|_{\widetilde{U}_i})$. By (2.8), each one is clopen in $s^{-1}(\widetilde{U}_i) \cap t^{-1}(\widetilde{U}_i)$. Since the $s^{-1}(\widetilde{U}_i) \cap t^{-1}(\widetilde{U}_i)$ are the intersections of two clopen subsets of $\mathcal{G}_1$, $Im(u|_{\widetilde{U}_i})$ is clopen in $\mathcal{G}_1$. We conclude that $Im(u)$ is clopen.

*Proof of proposition 2.10.* Let $\pi : \mathcal{G} \to X$ denote the projection. If $\pi(x) \neq \pi(y)$ there are no morphisms between $x$ and $y$ and so there is nothing to show. If $\pi(x) = \pi(y)$

then $x$ and $y$ lie in the same $G_i$-orbit, and so there exists an arrow $[h_0] : x \to y$, where $h_0 \in G_i$. By composing with $[h_0^{-1}]$, we can restrict ourselves to the case $x = y$.

Consider the inclusion $\iota$ of the stabilizer group $G_x$ of $x$, into the groupoid $\mathcal{G}_x := \pi^{-1}(\pi(x)) \subset \mathcal{G}$. The first statement of our proposition claims that $\iota$ is full, and the second one the $\iota$ is faithful. So we are reduced to proving the following lemma:

**Lemma 2.11** *The inclusion functor $\iota : G_x \to \mathcal{G}_x$ is an equivalence of groupoids.*

*Proof.* We construct a retraction $r : \mathcal{G}_x \to G_x$ and a natural transformation $\nu$ from $\imath \circ r$ to the identity. Recall that $x \in \widetilde{U}_i$. For each object $y$ of $\mathcal{G}_x$ with $y \in \widetilde{U}_j$ we pick an object $z \in \widetilde{U}_k$, where $U_k = U_i \cap U_j$, and two group elements $a \in G_i$ and $b \in G_j$ such that $a\,x = \alpha_{ki}(z)$ and $y = b\,\alpha_{kj}(z)$. This data can be pictured as follows :

$$\begin{tikzcd} x \arrow[r, "{[a]}"] & \alpha_{ki}(z) & z \arrow[l, mapsto, "{[\alpha_{ki}]}"'] \arrow[r, mapsto, "{[\alpha_{kj}]}"] & \alpha_{kj}(z) \arrow[r, "{[b]}"] & y, \end{tikzcd} \tag{2.9}$$

where we use the arrows "$\to$" and "$\mapsto$" to distinguish between the two kinds of generators for $\mathcal{G}_1$.

We now define $r : \mathcal{G}_x \to G_x$. At the level of objects, $r$ sends everything to $x$. For the arrows of the form $[h] : y \to y' = h \cdot y$ we let $z, a, b$ and $z', a', b'$ be the choices corresponding to $y$ and $y'$ respectively. The arrow $r([h])$ is the only possible one that makes the following diagram commutative:

$$\begin{tikzcd} x \arrow[r, "{[a']}"] & \alpha_{ki}(z') & z' \arrow[l, mapsto, "{[\alpha_{ki}]}"'] \arrow[r, mapsto, "{[\alpha_{kj}]}"] & \alpha_{kj}(z') \arrow[r, "{[b']}"] & y' \\ x \arrow[u, "{r([h])}"] \arrow[r, "{[a]}"'] & \alpha_{ki}(z) \arrow[u, "{[\phi_{ki}(\tilde{h})]}"] & z \arrow[u, "{[\tilde{h}]}"] \arrow[l, mapsto, "{[\alpha_{ki}]}"] \arrow[r, mapsto, "{[\alpha_{kj}]}"'] & \alpha_{kj}(z) \arrow[u, "{[\phi_{kj}(\tilde{h})]}"] \arrow[r, "{[b]}"'] & y. \arrow[u, "{[h]}"'] \end{tikzcd} \tag{2.10}$$

More precisely, since $\phi_{kj}$ induced isomorphisms on stabilizers, there is a unique element $\tilde{h} \in G_k$ such that $\phi_{kj}(\tilde{h}) = b'^{-1} h\, b$. We then let $r([h]) := [a'^{-1} \phi_{ki}(\tilde{h})\, a]$.

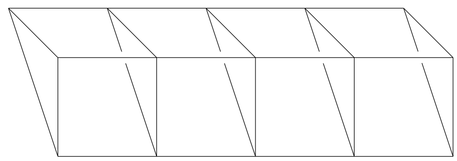
To define $r([\alpha_{jj'}])$, we proceed similarly. Let $z, a, b$ and $z', a', b'$ be the choices corresponding to $y$ and $y'$. The diagram that we want to make commutative now looks like this:

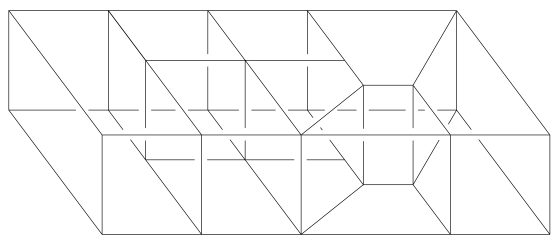
$$\begin{tikzcd} x \arrow[r, "{[a']}"] & \alpha_{k'i}(z') & z' \arrow[l, mapsto] \arrow[r, mapsto] & \alpha_{k'j'}(z') \arrow[rr] & & y' \\ & \alpha_{k'i}(\alpha_{kk'}(z)) \arrow[u, "{[\phi_{k'i}(\tilde{h})]}"] \arrow[dd, "{[g_{kk'i}]}"'] & \alpha_{kk'}(z) \arrow[u, "{[\tilde{h}]}"] \arrow[l, mapsto] \arrow[r, mapsto] & \alpha_{k'j'}(\alpha_{kk'}(z)) \arrow[u, "{[\phi_{k'j'}(\tilde{h})]}"] \arrow[d, "{[g_{kk'j'}]}"'] & & \\ & & & \alpha_{kj'}(z) & \alpha_{jj'}(\alpha_{kj}(z)) \arrow[l, "{[g_{kjj'}]}"] \arrow[uur, "{[\phi_{jj'}(b)]}"] & \\ x \arrow[uuu, "{r([\alpha_{jj'}])}"] \arrow[r, "{[a]}"'] & \alpha_{ki}(z) & z \arrow[l, mapsto] \arrow[uu, mapsto] \arrow[ur, mapsto] \arrow[rr, mapsto] & & \alpha_{kj}(z) \arrow[u, mapsto] \arrow[r, "{[b]}"'] & y. \arrow[uuu, mapsto, "{[\alpha_{jj'}]}"'] \end{tikzcd} \tag{2.11}$$

As before, there is a unique element $\tilde{h} \in G_{k'}$ satisfying $\phi_{k'j'}(\tilde{h}) = b'^{-1}\phi_{jj'}(b)\, g_{kjj'}^{-1} g_{kk'j'}$, and we let $r([\alpha_{jj'}]) = [a'^{-1}\phi_{k'i}(\tilde{h})\, g_{kk'i}^{-1} a]$.

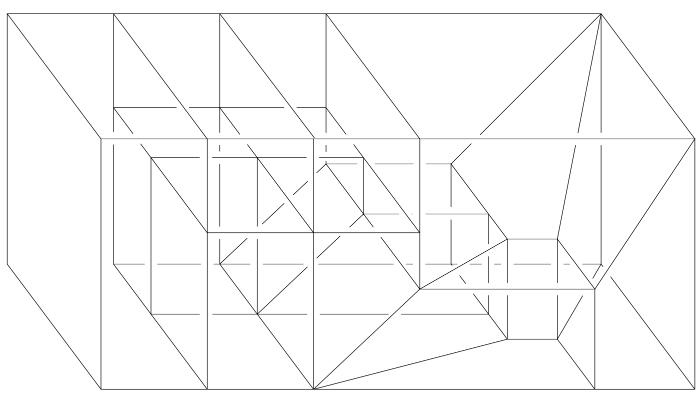
We now go back to (2.9) to define our natural transformation $\nu : \imath \circ r \to 1$. It is given by $\nu_y := [b][\alpha_{kj}][\alpha_{ki}]^{-1}[a] : x \to y$. Using (2.10) and (2.11), it is immediate to verify that $\nu$ is natural with respect to the morphisms $[h]$ and $[\alpha_{jj'}]$.

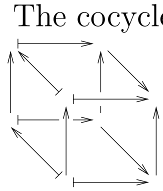
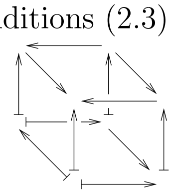
So far, we have not used the cocycle conditions (2.3) and (2.4). These are needed to verify that $r$ is well defined, namely that $r([h])$ and $r([\alpha_{jj'}])$ satisfy the three relations (2.7). This is done by a careful diagram chase, where the diagrams look like this:


,


,

and


.

In all these cases, we start with the data that the rightmost face commutes, and slowly work our way to the left until we show that the leftmost face commutes. For each 3-cell, we use one of the properties of $\phi_{ij}$ and $g_{ijk}$ in order to show that all the 2-faces commute. The cocycle conditions (2.3) and (2.4) are used when encountering cells of the form  and  respectively. $\square$

# Chapter 3

# Orbispaces from the point of view of the Borel construction

We will now describe our new definition of orbispaces and relate it to the existing definitions discussed in Chapter 1. Parts of this chapter rely on the technology of Chapter 4.

Recall that the Borel construction of a $G$-space $Y$ is the quotient $(Y \times EG)/G$ of the product of $Y$ with a contractible space $EG$ that has a free action of $G$. The Borel construction is also called the homotopy quotient of $Y$ by $G$. An explicit model for $EG$ is provided by the geometric realization of the simplicial space $\cdots G^3 \substack{\to\\\to\\\to} G^2 \rightrightarrows G$, where the face maps are the projections omitting a factor and the degeneracy maps repeat an entry. Using this particular model of $EG$, we get a model for the Borel construction as the geometric realization of the nerve of the groupoid $Y \times G \rightrightarrows Y$. Indeed, the product $Y \times EG$ is the realization of the nerve of the groupoid $Y \times G^2 \rightrightarrows Y \times G$. Since quotients commutes with geometric realizations therefore

$$(Y \times EG)/G = |Y \times G^2 \rightrightarrows Y \times G|/G = |Y \times G^2/G \rightrightarrows Y \times G/G| = |Y \times G \rightrightarrows Y|.$$

Note that $Y \times EG$ could have been replaced by any free $G$-space $\tilde{Y}$ admitting an equivariant map to $Y$ which is an acyclic fibration. Hence forward, it is this more general construction that we shall call the Borel construction.

Given a topological group $K$ and an action $Y \circlearrowleft K$ whose stabilizers are finite, we let $[Y/K]$ be the orbispace quotient (see example 3.2). The Borel construction $\tilde{Y}/G$ remembers a large amount of the orbispace homotopy type of $[Y/G]$. For example, the ordinary cohomology of $[Y/G]$ is nothing else than the cohomology of $\tilde{Y}/G$. However, the $K$-theory of $K^*[Y/G] = K^*_G(Y)$ is typically not isomorphic to $K^*(\tilde{Y}/G)$. It is instead the completion of $K^*[Y/G]$ at the ideal consisting of virtual vector bundles of dimension zero. However, we can keep track of the whole orbispace homotopy type if we remember the topological quotient $Y/G$. This leads to the following definition:

**Definition 3.1** *An orbispace is an object in the following 2-category:*

*An object is a map of spaces $p : E \to X$ which is locally isomorphic to the Borel construction of a finite group $G$ acting on a space. In other words $X$ has a cover*

*$\{U_i\}$ by closed*[1] *subspaces, where each restriction $p^{-1}(U_i) \to U_i$ is homeomorphic to the Borel construction $\tilde{Y}_i/G_i \to Y_i/G_i$ of some action $Y_i \circlearrowright G_i$. The space $E$ is called the total space and $X$ the topological quotient.*

*A morphism of orbispaces $(E, X) \to (E', X')$ is a commutative diagram*

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "p"] & E' \arrow[d, "p'"] \\ X \arrow[r, "g"] & X'. \end{tikzcd}$$

*If $g = g'$, there may exist 2-morphisms $(E, X) \Downarrow (E', X')$ between maps $(f, g)$ and $(f', g')$. A 2-morphism is a homotopy $h : E \times [0, 1] \to E'$ such that $p' \circ h_t = g \circ p$ for all $t \in [0, 1]$*

$$\begin{tikzcd} E \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow h"] \arrow[d] & E' \arrow[d] \\ X \arrow[r, "g"] & X' . \end{tikzcd} \tag{3.1}$$

*If $g \neq g'$, there are no 2-morphisms. Two 2-morphisms are considered to be the same if they are homotopic to each other relatively to the endpoints.*

**Example 3.2** Let $K$ be a topological group and $Y$ a $K$-space with finite isotropy groups. Then $[Y/K] := \big((Y \times EK)/K \to Y/K\big)$ is an orbispace.

Note that a map $(f, g) : (E, X) \to (E', X')$ is entirely determined by $f : E \to E'$. So we will use $f : (E, X) \to (E', X')$ as a shorthand notation.

**Remark 3.3** An equivalent way or recording the information is to have $E$ foliated by the fibers of $p$. The leaf space of this foliation is $X$. We will often switch between these two equivalent ways of giving the data.

Given an orbispace $p : E \to X$, the fibers $p^{-1}(x)$ are all $K(\pi, 1)$'s, where the $\pi$ are finite groups depending on the point $x \in X$. This induces a stratification of $X$ by the isomorphism type of $\pi = \pi_1(p^{-1}(x))$.

**Definition 3.4** *Let $X$ be a space equipped with a stratification by the poset of isomorphism classes of finite groups (see Definition 4.4).*

*An orbispaces structure on $X$ is an orbispace $p : E \to X$ such that for every $x \in X$, the group $\pi_1(p^{-1}(x))$ is isomorphic to the group indexing the stratum of $x$.*

Having the correct homotopy type of fibers is not quite enough to be an orbispace. For example, the map

$$\big(K(G, 1) \times [0, 1]\big)\Big/(y, 1) \sim (y', 1) \longrightarrow [0, 1] \tag{3.2}$$

[1] If we used open covers instead, we would get an equivalent definition (see Remark 3.6).

is not an orbispace (unless $G$ is trivial). More generally, if $M_f$ is the mapping cylinder of a fibration $f : K(G,1) \twoheadrightarrow K(H,1)$, then $M_f \to [0,1]$ is an orbispace if and only if $\pi_1(f) : G \to H$ is injective.

The following theorem gives the exact conditions when $E \to X$ is an orbispace structure on $X$.

**Theorem 3.5** *A map $p : E \to X$ is an orbispace if and only if the following conditions are satisfied:*

- *The map $p$ is a stratified fibration in the sense of Definition 4.9.*
- *The fibers of $p$ are $K(\pi,1)$'s and their fundamental groups are all finite.*
- *Let $F_x$ and $F_y$ be the fibers over points $x$, $y \in X$ and let $\gamma : [0,1] \to X$ be a directed[2] path from $x$ to $y$. Then the corresponding map[3] $\nabla_\gamma : F_x \to F_y$ is injective on $\pi_1$.*

*Proof.* We first show that orbispaces satisfy the above conditions. All the conditions are local (for the first one, this is the content of Lemma 4.19), so we may assume that $(E,X) = (\tilde{Y}/G, Y/G)$, where $\tilde{Y}$, $Y$ and $G$ are as in Definition 3.1. We first check that $p : \tilde{Y}/G \to Y/G$ is a stratified fibration. Let $q : \tilde{Y} \to Y$ be the projection. Let $\Lambda^n \hookrightarrow \Delta^n$ be a generating directed cofibration and consider the lifting diagram

$$\begin{tikzcd} \Lambda^n \arrow[r] \arrow[d, hook] & \tilde{Y}/G \arrow[d, "q"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & Y/G. \end{tikzcd} \tag{3.3}$$

Since $\Lambda^n$ is simply connected and $\tilde{Y}$ is a free $G$-space, we can lift $\Lambda^n \to \tilde{Y}/G$ to a map $\Lambda^n \to \tilde{Y}$. There is a lift $\Delta^n \to Y$ by Lemma 4.20. There is a lift $\Delta^n \to \tilde{Y}$ because $q$ is a fibration. Finally we compose with the projection $\tilde{Y} \to \tilde{Y}/G$ to get our desired lift $\Delta^n \to \tilde{Y}/G$. This diagram chase is best visualized as follows:

$$\begin{tikzcd} \Lambda^n \arrow[r] \arrow[dd, hook] \arrow[rr, dotted, bend left, "①"] & \tilde{Y}/G \arrow[dd, "p"] & \tilde{Y} \arrow[l] \arrow[d, "q"] \\ & & Y \arrow[d] \\ \Delta^n \arrow[r] \arrow[uur, dotted, "④"] \arrow[uurr, dotted, "③"] \arrow[urr, dotted, "②"] & Y/G \arrow[r, equal] & Y/G. \end{tikzcd}$$

This finishes the proof that $p$ is a stratified fibration. Its fibers have the required homotopy type since

$$p^{-1}([y]) = p^{-1}\big(yG/G\big) = q^{-1}(yG)/G = q^{-1}(y)/\mathrm{Stab}_G(y)$$

[2]See Definition 4.22.

[3]See Lemma 4.23.

and $q^{-1}(y)$ is contractible.

We now show that $\nabla_\gamma$ is injective on $\pi_1$. By definition, $\nabla_\gamma$ is the composite $F_x \times \{1\} \hookrightarrow F_x \times [0,1] \xrightarrow{\ell} \tilde{Y}/G$ where $\ell$ is a lift

$$\begin{tikzcd} F_x \times \{0\} \arrow[rr, "\iota"] \arrow[d, hook] & & \tilde{Y}/G \arrow[d] \\ F_x \times [0,1] \arrow[r] \arrow[urr, dotted, "\ell"] & {[0,1]} \arrow[r, "\gamma"] & Y/G. \end{tikzcd}$$

Since $\nabla_\gamma$ is homotopic to $\iota$, it's enough to show that $\iota_* : \pi_1(F_x) \to \pi_1(\tilde{Y}/G)$ is injective. Let $\hat{x} \in Y$ be a representative of $x$. The fiber $q^{-1}(\hat{x})$ is a universal cover for $F_x$. The map $\iota$ lifts to an inclusion $q^{-1}(\hat{x}) \hookrightarrow \tilde{Y}$, so $\iota_*$ in injective.

Now, let's assume that $p : E \to X$ satisfies the three conditions in the statement of the theorem. We want to show that it's an orbispace. Given a point $x \in X$, we need to find a neighborhood of $U$ such that $p^{-1}(U) \to U$ is homeomorphic to a Borel construction $\tilde{Y}/G \to Y/G$.

Let $U$ be a star-shaped closed neighborhood of $x$. By picking $U$ small enough, we can make sure that for all points $y$ in $U$, the straight path from $y$ to $x$ is directed. Let $G$ be the fundamental group of $p^{-1}(U)$, let $\tilde{Y}$ be its universal cover, and let $Y$ be the leaf space of $\tilde{Y}$ with respect to the foliation inherited from $U$. Since $p^{-1}(U)$ deformation retracts to $F_x$, we have $\pi_1(F_x) = \pi_1(p^{-1}(U)) = G$. The map $\pi_1(F_y) \to \pi_1(p^{-1}(U))$ is a monomorphism, so the preimages of $F_y$ in $\tilde{Y}$ have contractible connected components. This proves that the fibers of the projection $q : \tilde{Y} \to Y$ are all contractible. Clearly $\tilde{Y} \circlearrowleft G$ is free and $(p^{-1}(U), U) \simeq (\tilde{Y}/G, Y/G)$.

We now show that $q : \tilde{Y} \to Y$ is a fibration. By Lemma 4.30, it's enough to show that $q$ is a stratified fibration. Let $\Lambda^n \hookrightarrow \Delta^n$ a directed cofibration and consider the lifting problem

$$\begin{tikzcd} \Lambda^n \arrow[r] \arrow[d, hook] & \tilde{Y} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & Y. \end{tikzcd}$$

Since $p$ is a stratified fibration,there is a lift $\Delta^n \to p^{-1}(U)$. We can then lift it to $\tilde{Y}$ because $\Delta^n$ is simply connected. This finishes the verification that $p : E \to X$ in an orbispace. $\square$

**Remark 3.6** For each point $x \in X$, we have constructed a closed neighborhood $U_x$ such that $p^{-1}(U_x) \to U_x$ is homeomorphic to a Borel construction. The $U_x$ form a closed cover of $X$ as required for definition 3.1, but if we replaced $U_x$ by their interiors, we would get an open cover. This shows that the notion of orbispace doesn't depend on the choice of open versus closed covers.

We will soon establish the equivalence of Definition 3.1 with the other definitions presented in Chapter 2. But before, we would like explain how to do various constructions one might be interested in.

## 3.1 Bundles and pullbacks

A bundle on an orbispace $(E, X)$ is a bundle $P \to E$ along with a leaf-wise flat connection. Namely, for each path $\gamma$ in a leaf of $E$, we are given an isomorphism $\nabla_\gamma : P_{\gamma(0)} \to P_{\gamma(1)}$. This isomorphism only depends on the homotopy class of $\gamma$ and is compatible with composition of paths. The space of $P$ is itself the total space of an orbispace. The topological quotient is given by $P/\sim$, where $x \sim y$ if there is a path $\gamma$ with $\nabla_\gamma(x) = y$.

The pull-back of a bundle $P \to E$ along a map $f : (E, X) \to (E', X')$ is the usual pullback $f^*(P)$. The leaf-wise flat connection is given by

$$\nabla_\gamma : \big(f^*(P)\big)_x = P_{f(x)} \xrightarrow{\nabla_{f\circ\gamma}} P_{f(y)} = \big(f^*(P)\big)_y.$$

As an example, we explain how to build the tangent bundle of an orbifold.

**Example 3.7** Let $p : E \to X$ be an orbifold. It is locally isomorphic to the Borel construction of a smooth action on a manifold. Recall that $E$ comes foliated by the fibers of $p$. Given a point $x \in E$ we consider a small neighborhood $U$ of $x$ and the induced foliation on that neighborhood. If $U$ is chosen conveniently, its leaf space $U/\sim$ is a manifold, and thus we may let $T_x$ be the tangent space $T_{[x]}(U/\sim)$.

We first show that $U$ may be chosen so that its leaf space is a manifold. By definition, $E$ is locally of the form $\tilde{M}/G$ for some $\tilde{M}$ mapping to a manifold $M$. Pick a preimage $\hat{x} \in \tilde{M}$ of $x \in \tilde{M}/G$. The action of $G$ on $\tilde{M}$ is proper and free, so we may pick a neighborhood $V$ of $\hat{x}$ that doesn't intersect any of its translates by $G$. Let us also pick $V$ so that its intersection with any leaf of $q : \tilde{M} \to M$ is connected. Clearly, the leaf space of $V$ is then $q(V)$, which is a manifold. Since $V$ does not intersect any of its translates, it is homeomorphic to its image $U$ in $M'/G$. This gives us the desired neighborhood of $x$. If $U' \subset U$, is a smaller neighborhood of $x$, then $U'/\sim$ will be an open submanifold of $U/\sim$ and thus the construction doesn't depend on the choice of neighborhood.

We now construct the leaf-wise connection $\nabla$. Clearly, it is enough to do it locally. So for each point $x \in E$ we need a neighborhood $W$ in $p^{-1}(x)$, and for each point $x' \in W$ an isomorphism $T_x \simeq T_{x'}$. Given $x \in E$, we may pick $W$ to be the intersection of $U$ with the leaf of $x$, where $U$ is as above. If $x'$ is another point of $W$, then both $T_x$ and $T_{x'}$ are given by $T_{[x]}(U/\sim)$, so they come with a preferred isomorphism between them: the identity.

Pulling back general maps of orbispaces is done in a slightly different way. The construction is similar to homotopy pullbacks, the only difference being that instead of arbitrary paths, one only uses those that stay in a fixed leaf. Consider the following diagram of orbispaces:

$$\begin{tikzcd} E' \arrow[r, "f"] \arrow[d, "p'"] & E \arrow[d, "p"] & E'' \arrow[l, "f'"'] \arrow[d, "p''"] \\ X' \arrow[r, "g"] & X & X'' \arrow[l, "g'"'] \end{tikzcd} \tag{3.4}$$

The pullback $(E''', X''')$ is then given by

$$E''' := \big\{(x, y, \gamma) \in E' \times E'' \times E^{[0,1]} \,\big|\, f(x) = \gamma(0), f'(y) = \gamma(1),\ p \circ \gamma \text{ is constant}\big\}/\sim \tag{3.5}$$

where $(x, y, \gamma) \sim (x, y, \gamma')$ if $\gamma$ and $\gamma'$ are leaf-wise homotopic relatively to their endpoints. Two points $(x_0, y_0, \gamma_0)$ and $(x_1, y_1, \gamma_1)$ are in the same leaf if there exists leaf-wise homotopy $(x_t, y_t, \gamma_t)_{t\in[0,1]}$ from one to the other. In (3.5), we could have omitted the operation of modding out by $\sim$. Indeed, the equivalence classes of $\sim$ are all contractible, so we would just get a different model for the pullback.

If either $f$ or $f'$ is a leaf-wise fibration, we can also use the usual pullback $E' \times_E E''$. The leaves are then the connected components of the pullbacks of leaves.

## 3.2 Group actions

Since orbispaces form a 2-category, there are two possibly different notions of actions. Let $G$ be a topological group with multiplication $\mu : G^2 \to G$. A strict action of $G$ on an orbispace $(E, X)$ consists of actions on $E$ and on $X$ commuting with the projection. A weak action of $G$ on $(E, X)$ is a map $\nu : G \times (E, X) \to (E, X)$ and an associator 2-morphism $\alpha : \nu \circ (\mu \times 1) \to \nu \circ (1 \times \nu)$. The associator is required to make the following diagram commute

$$\begin{tikzcd}
\nu \circ (\mu^{(2)} \times 1) \arrow[r, "\alpha\circ(1\times\mu\times1)"] \arrow[d, "\alpha\circ(\mu\times1\times1)"'] & \nu \circ (1 \times \nu) \circ (1 \times \mu \times 1) \arrow[d, "\nu\circ(1\times\alpha)"] \\
\nu \circ (\mu \times \nu) \arrow[r, "\alpha\circ(1\times1\times\nu)"'] & \nu \circ (1 \times \nu) \circ (1 \times 1 \times \nu),
\end{tikzcd} \tag{3.6}$$

where $\mu^{(2)} : G^3 \to G$ denotes the multiplication. Note that a weak action on $(E, X)$ induces a usual (strict) action on $X$.

A map $f : (E, X) \to (E', X')$ between two orbispaces equipped with weak $G$-actions is said to be $G$-equivariant if we also have an intertwiner $\beta : f \circ \nu \to \nu' \circ (1 \times f)$ that makes the following diagram commute :

$$\begin{tikzcd}
f \circ \nu \circ (\mu \times 1) \arrow[r, "f\circ\alpha"] \arrow[dd, "\beta\circ(\mu\times1)"'] & f \circ \nu \circ (1 \times \nu) \arrow[d, "\beta\circ(1\times\nu)"] \\
& \nu' \circ (1 \times f) \circ (1 \times \nu) \arrow[d, "\nu'\circ(1\times\beta)"] \\
\nu' \circ (\mu \times f) \arrow[r, "\alpha'\circ(1\times1\times f)"'] & \nu' \circ (1 \times \nu') \circ (1 \times 1 \times f).
\end{tikzcd} \tag{3.7}$$

**Remark 3.8** The notion of weak group action is the natural specialization of the notion of $A_\infty$-action to the world of 2-categories (as opposed to $\infty$-categories). We refer the reader to [23] and [3] for more details on the theory of $A_\infty$-spaces and their actions. We could also consider weak group objects and their actions. These come with their own associator $\mu \circ (\mu \times 1) \to \mu \circ (1 \times \mu)$, and so the diagram (3.6) would then be replaced by a pentagon diagram.

We now show that the notions of strict and weak actions are equivalent. This is a special case of the rectification procedure for strictifying algebras over operads [3, Thm. 4.49] [5, Sec. 18.3]. The corresponding result for stacks has appeared in [31].

**Theorem 3.9** *Given a weak action $(\nu, \alpha)$ of a topological group $G$ on an orbispace $(E, X)$, there exists an equivalent orbispace $(\widetilde{E}, X)$ carrying a strict action $\tilde{\nu}$ of $G$. Moreover, the equivalence $f : (\widetilde{E}, X) \to (E, X)$ is $G$-equivariant in the weak sense.*

*Proof.* Let $WG$ be the topological monoid given as follows (see [3],[33]). Its elements consist of a collection of points $0 = x_0 \leq x_1 \leq \ldots \leq x_k$ on the positive real line satisfying $x_i - x_{i-1} \leq 1$. Moreover, each point $x_i$ is decorated by a group element $g_i \in G$. The points of $WG$ are denoted $(x_0 \ldots x_k; g_0 \ldots g_k)$. If $x_i = x_{i+1}$, we also identify $(\ldots x_i, x_i \ldots; \ldots g_i, g_{i+1} \ldots)$ with $(\ldots x_i \ldots; \ldots g_i g_{i+1} \ldots)$. The multiplication is given by

$$(x_0 \ldots x_k; g_0 \ldots g_k) \cdot (x'_0 \ldots x'_\ell; g'_0 \ldots g'_\ell) = (x_0 \ldots x_k, y_0 \ldots y_\ell; g_0 \ldots g_k, g'_0 \ldots g'_\ell),$$

where $y_i = x_k + 1 + x'_i$. We have a natural homomorphism $p_G : WG \to G$ given by $(x_0 \ldots x_k; g_0 \ldots g_k) \mapsto g_0 \cdots g_k$ and a section $G \to WG$, $g \mapsto (0; g)$. This section is not a homomorphism. The map $WG \times [0,1] \to WG : ((x_0 \ldots x_k; g_0 \ldots g_k), t) \mapsto (tx_0 \ldots tx_k; g_0 \ldots g_k)$ is a deformation retraction on the image of $G$, hence $p_G$ is a homotopy equivalence[4]. Since the fibers of $p_G$ are contractible, $(WG, G)$ is actually a (strict) monoid in the category of orbispaces.

Note that $WG$ is a colimit $WG = \varinjlim WG^{(n)}$, where each $WG^{(n)}$ is obtained from $WG^{(n-1)}$ by gluing an "n-cell" $G^{n+1} \times [0,1]^n$ and freely generating a monoid from it. Combinatorially, $WG^{(n)}$ is the subspace of $WG$ where at most $n$ consecutive $x_i$'s have distance $< 1$, and the "n-cell" that generates it consists of element $(x_0 \ldots x_k; g_0 \ldots g_k)$ with $k \leq n$.

Now, let us relate all this to our weak action of $G$ on $(E, X)$. We use the skeleta $WG^{(n)}$ to inductively define a strict action of $WG$ on $E$. First, since $WG^{(0)}$ is free on $G$ our map $\nu : G \times E \to E$ defines an action of $WG^{(0)}$. The associator map $\alpha : G^2 \times [0,1] \times E \to E$ is exactly what we need to extend this to an action of $WG^{(1)}$. Now, the identity (3.6) claims the existence of a fiber-wise homotopy between two maps $G^3 \times [0,1] \times E \to E$. These two maps can be assembled into a single map $G^3 \times \partial[0,1]^2 \times E \to E$, and then (3.6) claims that it extends to a map defined on the whole of $G^3 \times [0,1]^2 \times E$. Again, this is exactly what we need to extend our action to $WG^{(2)}$. Note that so far all our maps commute with the projections on $G \times X$ and on

[4]In fact, $WG$ is the canonical cofibrant replacement of $G$ in the model category of topological monoids (assuming $G$ is cofibrant as a space), and can be constructed as a monadic bar construction $WG = B(*, \text{Free monoid}, G)$.

$X$, so we are actually on our way to defining a (strict) action of $(WG, G)$ on $(E, X)$ in the category of orbispaces.

To finish our construction, recall that all higher cells are of the form $G^{n+1} \times [0,1]^n$. Suppose that we have built the action of $WG^{(n-1)}$. That action provides a map $G^{n+1} \times \partial[0,1]^n \times E \to E$ and we wish to extend it over $G^{n+1} \times [0,1]^n \times E \to E$. Our extension problem looks like this:

$$\begin{tikzcd} G^{n+1} \times \partial[0,1]^n \times E \arrow[r] \arrow[d, hook] & E \arrow[d] \\ G^{n+1} \times [0,1]^n \times E \arrow[r] \arrow[ur, dotted] & X \end{tikzcd}$$

At this point, we note that each fiber over $G \times X$ of the inclusion $G^{n+1} \times \partial[0,1]^n \times E \hookrightarrow G^{n+1} \times [0,1]^n \times E$ is 2-connected and that the fibers of $E \to X$ are 1-truncated. So by obstruction theory (i.e. Theorem 4.29), our desired map $G^{n+1} \times [0,1]^n \times E \to E$ exists. This finishes our inductive construction of the strict action of $(WG, G)$ on $(E, X)$.

Now we define our orbispace $\widetilde{E}$ by

$$\widetilde{E} := (G \times EWG \times E)/WG,$$

where $WG$ acts on $G \times EWG \times Y$ by $\tilde{g} \cdot (g, x, y) = (g\tilde{g}^{-1}, \tilde{g}x, \tilde{g}y)$. The strict action $G \circlearrowright \widetilde{E}$ is given by $h \cdot (g, x, y) = (hg, x, y)$. The space $\widetilde{E}$ comes with a $G$-equivariant map $f$ to $E$ given by $f(g, x, y) = gy$. Since $WG \simeq G$, the map $f : (\widetilde{E}, X) \to (E, X)$ is a fiber-wise homotopy equivalence, and therefore an equivalence of orbispaces.

To show that $f : (\widetilde{E}, X) \to (E, X)$ is $G$-equivariant in the weak sense, we introduce an auxiliary orbispace $(E', X)$. It is given by $E' := (WG \times EWG \times E)/WG \simeq EWG \times E$ and admits strictly $WG$-equivariant equivalences

$$(\widetilde{E}, X) \xleftarrow{\sim} (E', X) \xrightarrow{\sim} (E, X).$$

Since strict $WG$-actions are the same thing as weak $G$-actions, one sees that these maps are weakly $G$-equivariant. Now we just need to check that the inverse of a weakly equivariant map is also weakly equivariant, and that the composite of two weakly equivariant maps is weakly equivariant. These routine verifications are left to the reader. $\square$

## 3.3 Sheaf cohomology

A sheaf $\mathcal{F}$ on an orbispace $(E, X)$ is a sheaf on $X$ along with the following additional data. For each open $U \subset E$ and leaf-wise homotopy from $h : U \times [0,1] \to E$ from $h_0 = \mathrm{Id}_U$ to some map $h_1 : U \to V$ (not necessarily an inclusion) we are given a map $\mathcal{F}(h) : \mathcal{F}(V) \to \mathcal{F}(U)$. Here, by leaf-wise homotopy, we mean that for any point $y \in U$, the path $\{y\} \times [0,1]$ stays in a fixed leaf of $E$. The maps $\mathcal{F}(h)$ are compatible with restriction and composition of homotopies. Replacing $\mathcal{F}$ by its étale space $|\mathcal{F}| \to E$, this additional data is equivalent to a leaf-wise (flat) connection on $|\mathcal{F}|$, as considered in section 3.1. This notion of sheaf is equivalent to what is known as an étale sheaf (see [24] for a treatment of sheaves from the point of view of étale groupoids).

Let $\mathsf{Sh}(E,X)$ denote the category of sheaves on $(E,X)$. Given a map $f : (E,X) \to (E',X')$, we have the classical operations on sheaves $f^*$ and $f_*$. The pullback functor $f^* : \mathsf{Sh}(E',X') \to \mathsf{Sh}(E,X)$ is most easily defined by pulling back the étale space of the sheaf $|f^*\mathcal{F}| := f^*|\mathcal{F}|$.

The pushforward functor $f_* : \mathsf{Sh}(E,X) \to \mathsf{Sh}(E',X')$ agrees with the usual pushforward of sheaves if $f$ is a leaf-wise fibration. Otherwise, it is defined by $f_*(\mathcal{F})(U) := \Gamma(P; p^*(\mathcal{F}))$, where $P$ is the pullback (3.5) of $E \to E' \hookleftarrow U$ and $p : P \to X$ is the projection. We have the usual adjunction $f^* \dashv f_*$.

Given a sheaf of abelian groups $\mathcal{A}$ on $(E,X)$, the sheaf cohomology $H^n((E,X);\mathcal{A})$ is the derived functor of global sections. The following result relates the sheaf cohomology of $(E,X)$ to that of its total space $E$ (see [26] and [7, chapt 6] for analogous results).

**Theorem 3.10** *If $(E,X)$ is an orbispace, $p : (E,E) \to (E,X)$ the canonical map, and $\mathcal{A}$ a sheaf of abelian groups on $(E,X)$, then $p$ induces a natural isomorphism of sheaf cohomology groups $H^*((E,X);\mathcal{A}) \simeq H^*((E,E);p^*\mathcal{A})$.*

Note that the sheaf cohomology of the orbispace $(E,E) = (\mathrm{Id} : E \to E)$ is the usual sheaf cohomology $H^*(E;\mathcal{A})$.

*Proof.* The map $p : (E,E) \to (E,X)$ may be made into a leaf-wise fibration by replacing $(E,E)$ by $(Z,E)$, where

$$Z := \left\{\gamma \in E^{[0,1]} \,\middle|\, p \circ \gamma \text{ is constant}\right\}/\text{homotopy rel. endpoints.}$$

Since all the fibers of $Z$ are contractible, $(Z,E)$ is indeed equivalent to $(E,E)$. Call $q : (Z,E) \to (E,X)$ the evaluation map $q(\gamma) := \gamma(1)$ and call $\pi : (E,X) \to pt$ the unique map to the point. We want to show that $H^n((E,E); f^*\mathcal{A}) = H^n((Z,E); q^*\mathcal{A}) := R^n(\pi q)_*(q^*\mathcal{A})$ is isomorphic to $H^n((E,X);\mathcal{A}) := R^n\pi_*\mathcal{A}$.

We examine the Grothendieck spectral sequence

$$R^n\pi_* R^m q_*(q^*\mathcal{A}) \Rightarrow R^{n+m}(\pi q)_*(q^*\mathcal{A}). \tag{3.8}$$

Let us consider a point $x \in E$ and look at the stalk of $R^m q_*(q^*\mathcal{A})$ at $x$. Call $i : \{x\} \to (E,X)$ the inclusion, $F \subset Z$ the fiber of $q$ over $x$, with map $j : (F,F) \to (Z,E)$ and projection $\bar{q} : (F,F) \to \{x\}$. These all fit into the following commutative diagram:

$$\begin{tikzcd}
(E,E) \arrow[dr, "p"] & (Z,E) \arrow[l, "\sim"'] \arrow[d, "q"] & (F,F) \arrow[l, "j"'] \arrow[d, "\bar{q}"] \\
pt & (E,X) \arrow[l, "\pi"'] & \{x\} \arrow[l, "i"']
\end{tikzcd}$$

Note that a sheaf on $(F,F)$ is really the same thing as a sheaf on $F$. And since $F$ is a contractible the stalk at $x$ of $R^m q_*(q^*\mathcal{A})$ can be computed:

$$\imath^* R^m q_*(q^*\mathcal{A}) = R^m \bar{q}_* \jmath^*(q^*\mathcal{A}) = H^m(F, \jmath^* q^*\mathcal{A}) = H^m(F, \bar{q}^* \imath^* \mathcal{A}) = \begin{cases} \imath^*\mathcal{A} & \text{if} \quad m = 0 \\ 0 & \text{otherwise,} \end{cases}$$

where the first equality holds because localization is exact, and the last one holds because $F$ is contractible and $\bar{q}^*\imath^*\mathcal{A}$ is a constant sheaf. So the Grothendieck spectral sequence (3.8) degenerates and thus

$$\begin{aligned} H^n((E,X),\mathcal{A}) =& R^n\pi_*\mathcal{A} = (R^n\pi_*)q_*q^*\mathcal{A} \\ =& R^n(\pi q)_* q^*\mathcal{A} = H^n((Z,E); q^*\mathcal{A}) = H^n((E,E), p^*\mathcal{A}). \end{aligned}$$

□

This proof can be interpreted in the following way. The map $(E,E) \to (E,X)$ is a fibration with contractible fibers, so the Leray spectral sequence $H^*(base; H^*(fiber)) \Rightarrow H^*(total\ space)$ degenerates and produces the required isomorphism.

## 3.4 Algebraic invariants

The homology, cohomology, homotopy groups of $(E,X)$ are defined to be those of $E$. The universal cover of $(E,X)$ is $(\widetilde{E}, \widetilde{E}/\sim)$, where $\widetilde{E}$ is the universal cover of $E$ and $\widetilde{E}/\sim$ is its leaf space with respect to the foliation induced from $E$. A cover of $(E,X)$ is again just a cover of $E$, and the usual correspondence between subgroups of $\pi_1(E,X)$ and covers of $(E,X)$ carries over from spaces. A good treatment of the above subjects, both from the point of view of étale groupoids and from the points of view of the Borel construction, is available in Moerdijk's paper [25]. See [14] for an account of rational cohomology of orbispaces, and [4, pp 604-608] for the relationship between $\pi_1$ and covering spaces.

If $X$ is compact, the $K$-theory of $(E,X)$ is the Grothendieck group of orbi-vector bundles, as described in the section 3.1. If $(F,Y) \subset (E,X)$ is a pair (i.e. if $F = E|_Y$), a relative $K$-class is given by $\mathbb{Z}/2$-graded vector bundles $V = V_0 \ominus V_1$ on $(E,X)$ and an isomorphism $f : V_0|_{(F,Y)} \to V_1|_{(F,Y)}$. The fact that this defines a cohomology theory is surprisingly tricky (see Proposition 6.11 for the proof of the excision axiom), and relies on the fact that all compact orbispaces are global quotients (Theorem 6.6). A similar attempt to define $K$-theory for Lie orbispaces would fail the excision axiom.

## 3.5 Equivalence of our definitions

This section shows the equivalence of Definition 3.1 with Definition 2.4 at the level of bicategories. To do this, we first need to complete Definition 2.4 and explain what the morphisms and the two morphisms are. This is quite tricky and can be done in two different ways.

The first approach is to take continuous functors $\mathcal{G} \to \mathcal{G}'$ as our morphisms and continuous natural transformations as our 2-morphisms. One then needs to formally invert (in a weak sense) a class of morphisms called weak equivalences (see [28], [29]). The second approach is to define a morphism $\mathcal{G} \to \mathcal{G}'$ via its "graph" $\mathcal{G}_0 \leftarrow \Gamma \to \mathcal{G}'_0$. The space $\Gamma$ admits commuting actions of $\mathcal{G}_1$ and $\mathcal{G}'_1$ subject to certain conditions. A 2-morphism $\Gamma \to \Gamma'$ is then a map commuting with the actions of $\mathcal{G}_1$ and $\mathcal{G}'_1$ (see [16], [24], [27]). We will use the first approach.

### 3.5.1 The bicategory of topological groupoids

Let $\mathsf{Orb}$ be the 2-category given in Definition 3.1, and let $\mathsf{Gpd}$ denote the 2-category of topological groupoids satisfying the conditions of Definition 2.4, continuous functors and continuous natural transformations. We will show that $\mathsf{Orb}$ is equivalent to the bicategory of fractions $\mathsf{Gpd}[W^{-1}]$ introduced by Pronk [28], [29].

**Definition 3.11** *A map of spaces $X \to X'$ is topologically surjective if there exists a closed cover[5] $\{U_i\}$ of $X'$ for which each $U_i$ admits a section $U_i \to X$.*

**Definition 3.12** *A weak equivalence is a continuous functor $f : \mathcal{G} \to \mathcal{H}$ for which the map*

$$\mathcal{G}_0 \times_{\mathcal{H}_0} \mathcal{H}_1 \to \mathcal{H}_0 \tag{3.9}$$

*is topologically surjective (the functor is essentially surjective), and the diagram*

$$\begin{tikzcd} \mathcal{G}_1 \arrow[r, "f"] \arrow[d, "{(s,t)}"'] & \mathcal{H}_1 \arrow[d, "{(s,t)}"] \\ \mathcal{G}_0 \times \mathcal{G}_0 \arrow[r, "f\times f"] & \mathcal{H}_0 \times \mathcal{H}_0. \end{tikzcd} \tag{3.10}$$

*is a pullback (the functor is fully faithful). A weak equivalence is denoted by the symbol $\xrightarrow{\sim}$. We let $W \subset \mathsf{Gpd}_1$ be the class of all weak equivalences.*

One way of constructing weak equivalences is by pulling back a groupoid $\mathcal{G}$ along a topologically surjective map $f : V \twoheadrightarrow \mathcal{G}_0$.

**Lemma 3.13** *Let $\mathcal{G}$ be a groupoid and $f : V \to \mathcal{G}_0$ a topologically surjective map. Let $f^*\mathcal{G}_1$ be the space $V \times_{\mathcal{G}_0} \mathcal{G}_1 \times_{\mathcal{G}_0} V$. Then $f^*\mathcal{G} := (f^*\mathcal{G}_1 \rightrightarrows V)$ is a groupoid, and the projection functor $f^*\mathcal{G} \to \mathcal{G}$ is a weak equivalence.*

*Proof.* The multiplication $f^*\mathcal{G}_1 \times_V f^*\mathcal{G}_1 \to f^*\mathcal{G}_1$ is given by

$$\begin{aligned} V \times_{\mathcal{G}_0} \mathcal{G}_1 \times_{\mathcal{G}_0} V \times_V V \times_{\mathcal{G}_0} \mathcal{G}_1 \times_{\mathcal{G}_0} V &= V \times_{\mathcal{G}_0} \mathcal{G}_1 \times_{\mathcal{G}_0} V \times_{\mathcal{G}_0} \mathcal{G}_1 \times_{\mathcal{G}_0} V \\ &\to V \times_{\mathcal{G}_0} \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \times_{\mathcal{G}_0} V \to V \times_{\mathcal{G}_0} \mathcal{G}_1 \times_{\mathcal{G}_0} V, \end{aligned}$$

where the second map is the projection and the last map is the multiplication in $\mathcal{G}$. The groupoid axioms are easy to check. The map

$$\big(V \times_{\mathcal{G}_0} \mathcal{G}_1 \times_{\mathcal{G}_0} V\big) \times_{\mathcal{G}_0} \mathcal{G}_1 \to \mathcal{G}_0$$

[5] We work with closed covers, but the arguments are identical to those using open covers.

is topologically surjective because all the maps in all the pullbacks are. The diagram

$$\begin{tikzcd}
V \times_{\mathcal{G}_0} \mathcal{G}_1 \times_{\mathcal{G}_0} V \arrow[r] \arrow[d, "{(s,t)}"'] & \mathcal{G}_1 \arrow[d, "{(s,t)}"] \\
V \times V \arrow[r] & \mathcal{G}_0 \times \mathcal{G}_0
\end{tikzcd}$$

a pullback diagram. We have checked the two conditions in Definition 3.12, so the functor $f^*\mathcal{G} \to \mathcal{G}$ is indeed a weak equivalence. $\square$

We also have the following well known result.

**Lemma 3.14** *Let $f : \mathcal{G} \to \mathcal{H}$ be a weak equivalence. Then the corresponding map $\bar{f} : \mathcal{G}_0/\mathcal{G}_1 \to \mathcal{H}_0/\mathcal{H}_1$ is a homeomorphism.*

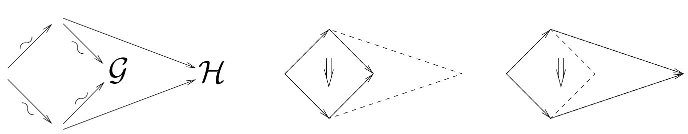
*Proof.* The two maps $\mathcal{H}_0 \to \mathcal{H}_0/\mathcal{H}_1$ and $\mathcal{G}_0 \times_{\mathcal{H}_0} \mathcal{H}_1 \to \mathcal{H}_0$ are topologically surjective. So we can triangulate $\mathcal{H}_0/\mathcal{H}_1$ such that every simplex $\sigma : \Delta^n \to \mathcal{H}_0/\mathcal{H}_1$ lifts to a map $\tilde{\sigma} : \Delta^n \to \mathcal{G}_0 \times_{\mathcal{H}_0} \mathcal{H}_1$. We define $\bar{f}^{-1}(\sigma(t))$ to be the image of $\tilde{\sigma}(t)$ in $\mathcal{G}_0/\mathcal{G}_1$. Assuming that $\bar{f}^{-1}$ is well defined, it is clear from the construction that $\bar{f}^{-1}(\bar{f}(x)) = x$ and $\bar{f}(\bar{f}^{-1}(y)) = y$.

We now show that $\bar{f}^{-1}$ is well defined. Suppose that we have two sections $\Delta^n \to \mathcal{G}_0 \times_{\mathcal{H}_0} \mathcal{H}_1$. The two maps from $\Delta^n \to \mathcal{H}_0$ agree in $\mathcal{H}_0/\mathcal{H}_1$ hence differ by a map to $\mathcal{H}_1$ (after refining the triangulation). Assemble the two maps to $\mathcal{G}_0 \times_{\mathcal{H}_0} \mathcal{H}_1$ and the map to $\mathcal{H}_1$ to a map $\Delta^n \to \mathcal{G}_0 \times_{\mathcal{H}_0} \mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{G}_0$. Compose with the multiplication map to get a map $\Delta^n \to \mathcal{G}_0 \times_{\mathcal{H}_0} \mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{G}_0$. By (3.10), $\mathcal{G}_0 \times_{\mathcal{H}_0} \mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{G}_0 = G_1$. We have a map $\Delta^n \to \mathcal{G}_1$ whose source and target are the two maps $\Delta^n \to \mathcal{G}_0$. Their projections in $\mathcal{G}_0/\mathcal{G}_1$ are therefore equal. This diagram chase is best visualized by the following picture:

$$\begin{tikzcd}
\mathcal{H}_0/\mathcal{H}_1 & \mathcal{H}_0 \arrow[l] & & \mathcal{H}_1 \arrow[ll] \\
\Delta^n \arrow[u, "1"] \arrow[ur, dotted, shift left, "2"] \arrow[ur, dotted, shift right] \arrow[urrr, dotted, "3"] \arrow[r, dotted, shift left, "4"] \arrow[r, dotted, shift right] \arrow[rr, dotted, bend left, "5"] \arrow[dr, dotted, shift left, "6"] \arrow[dr, dotted, shift right] \arrow[drr, dotted, "7"] \arrow[d, dotted, "8"] & \mathcal{G}_0 \times_{\mathcal{H}_0} \mathcal{H}_1 \arrow[u] \arrow[d] & \mathcal{G}_0 \times_{\mathcal{H}_0} \mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{G}_0 \arrow[l, Rightarrow] \arrow[ur] \arrow[d, "1\times\mu^{(2)}\times 1"] & \\
\mathcal{G}_0/\mathcal{G}_1 & \mathcal{G}_0 \arrow[l] & \mathcal{G}_1 = \mathcal{G}_0 \times_{\mathcal{H}_0} \mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{G}_0 \arrow[l] &
\end{tikzcd}$$

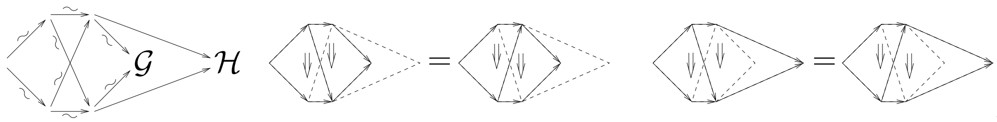
$\square$

We now explain how the localized bicategory $\mathsf{Gpd}[W^{-1}]$ is constructed. We follow Pronk [28]. The objects of $\mathsf{Gpd}[W^{-1}]$ are those of $\mathsf{Gpd}$. The morphisms of $\mathsf{Gpd}[W^{-1}]$ are diagrams of the form $\mathcal{G} \xleftarrow{\sim} \mathcal{K} \to \mathcal{H}$. Finally, the 2-morphisms of $\mathsf{Gpd}[W^{-1}]$ are equivalence classes of 2-diagrams



,

where two two such diagrams are equivalent if they fit into a bigger 2-diagram:




The various compositions in $\mathsf{Gpd}[W^{-1}]$ involve cumbersome diagrams which are detailed in [28].

### 3.5.2 The functor $\mathsf{Gpd}[W^{-1}] \to \mathsf{Orb}$

The bicategory $\mathsf{Gpd}[W^{-1}]$ has the following universal property shown in [28, section 3.3]. For any bicategory $\mathcal{D}$, the bicategory of functors from $\mathsf{Gpd}$ to $\mathcal{D}$ that send elements of $W$ to equivalences is equivalent to the bicategory of functors from $\mathsf{Gpd}[W^{-1}]$ to $\mathcal{D}$. We now recall a result of Pronk [28], [29].

**Theorem 3.15** *Let $\mathcal{C}$, $\mathcal{D}$ be bicategories and $W \subset \mathcal{C}_1$ a class of weak equivalences. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor which sends elements of $W$ to equivalences. Then the corresponding functor $\tilde{F} : \mathcal{C}[W^{-1}] \to \mathcal{D}$ is an equivalence if and only if the following conditions hold:*

- *$F$ is essentially surjective on objects.*
- *For every 1-morphism $f$ in $\mathcal{D}$, there exists a $w \in W$ such that $Fg \overset{\sim}{\Rightarrow} f \circ Fw$ for some $g$ in $\mathcal{C}_1$.*
- *$F$ is fully faithful on 2-morphisms.* □

So, in order to show that $\mathsf{Orb}$ and $\mathsf{Gpd}[W^{-1}]$ are equivalent, we need to construct a (weak) functor $F : \mathsf{Gpd} \to \mathsf{Orb}$ and show that it satisfies the conditions of Theorem 3.15. A first approximation of $F(\mathcal{G})$ is given by $(|N\mathcal{G}|, \mathcal{G}_0/\mathcal{G}_1)$. This would be very convenient since $F$ would then be a strict functor. But $|N\mathcal{G}| \to \mathcal{G}_0/\mathcal{G}_1$ is unfortunately not a stratified fibration, so we use Quillen's small object argument to replace it by a stratified fibration $|N\mathcal{G}|' \to \mathcal{G}_0/\mathcal{G}_1$. The category of regular CW-complexes doesn't have enough colimits so there are further technical details to make this argument work precisely. We then let

$$F(\mathcal{G}) := (|N\mathcal{G}|', \mathcal{G}_0/\mathcal{G}_1). \tag{3.11}$$

To show that $F(\mathcal{G}) \in \mathsf{Orb}$, we need to show that it satisfies the three conditions of Theorem 3.5. The map $p : |N\mathcal{G}|' \to \mathcal{G}_0/\mathcal{G}_1$ is a stratified fibration by construction and the fibers of $p$ are homotopy equivalent to those of $\tilde{p} : |N\mathcal{G}| \to \mathcal{G}_0/\mathcal{G}_1$. So it's enough to check the second and third conditions on $\tilde{p}$. Let $y\mathcal{G}_0/\mathcal{G}_1$ be a point represented by $\tilde{y} \in \mathcal{G}_0$. The fiber $\tilde{p}^{-1}(y)$ is the realization of the full subgroupoid $\mathcal{G}_y \subset \mathcal{G}$ on the $\mathcal{G}_1$-orbit of $\tilde{y}$. Since $\mathrm{Aut}_{\mathcal{G}}(\tilde{y})$ is finite, we have

$$p^{-1}(y) \simeq \tilde{p}^{-1}(y) = |N\mathcal{G}_y| \simeq K\big(\mathrm{Aut}_{\mathcal{G}}(\tilde{y}), 1\big). \tag{3.12}$$

Now we show that the maps $\nabla_\gamma : p^{-1}(x) \to p^{-1}(y)$ are injective on $\pi_1$. Again, it's enough to check it on the corresponding map $\tilde{p}^{-1}(x) \to \tilde{p}^{-1}(y)$. We show this locally. Since $\mathcal{G}_0$ and $\mathcal{G}_1$ admit triangulations making all structure maps simplicial, there exists a neighborhood $y \in \bar{U} \subset \mathcal{G}_0/\mathcal{G}_1$ and a deformation retraction to $\bar{U} \searrow \{y\}$, covered by $U_0 \searrow (\mathcal{G}_y)_0$ and $U_1 \searrow (\mathcal{G}_y)_1$, where $U_0$ and $U_1$ are the preimages of $\bar{U}$ in $\mathcal{G}_0$ and $\mathcal{G}_1$ respectively. The groupoid $U := (U_1 \rightrightarrows U_0)$ is a full subgroupoid of $\mathcal{G}$, and we can assemble the above maps to a deformation retraction of groupoids $r : U \searrow \mathcal{G}_y$. Given a point $x \in \mathcal{G}_0/\mathcal{G}_1$, the map $\tilde{p}^{-1}(x) \to \tilde{p}^{-1}(y)$ is the realization of $r : \mathcal{G}_x \to \mathcal{G}_y$. So we need to show that $r$ induces monomorphisms on the automorphism groups of objects. If this was not the case, we would have a non-identity $g \in (\mathcal{G}_x)_1$ whose image $r(g)$ is an identity. Since $r$ is a deformation retraction, we would also get a path from $g$ to $r(g)$. But this contradicts the fact that $Im(u)$ is a connected component of $\mathcal{G}_1$. We conclude that $\tilde{p}^{-1}(x) \to \tilde{p}^{-1}(y)$ is injective on $\pi_1$. This finishes the proof that $F(\mathcal{G}) \in \mathsf{Orb}$.

We now continue the definition of our functor $F$. A continuous functor $f : \mathcal{G} \to \mathcal{H}$ produces maps $(|Nf|, \bar{f}) : (|N\mathcal{G}|, \mathcal{G}_0/\mathcal{G}_1) \to (|N\mathcal{H}|, \mathcal{H}_0/\mathcal{H}_1)$. We compose it with the inclusion $|N\mathcal{H}| \hookrightarrow |N\mathcal{H}|'$. The inclusion $|N\mathcal{G}| \hookrightarrow |N\mathcal{G}|'$ is a directed cofibration with respect to the stratification inherited from $\mathcal{H}_0/\mathcal{H}_1$, so we can extend $|Nf|$ to

$$F(f) : (|N\mathcal{G}|', \mathcal{G}_0/\mathcal{G}_1) \to (|N\mathcal{H}|', \mathcal{H}_0/\mathcal{H}_1). \tag{3.13}$$

If $f$ is an identity, pick $F(f)$ to be an identity.

A continuous natural transformation $h : f \Rightarrow g : \mathcal{G} \overset{\Downarrow}{\rightrightarrows} \mathcal{H}$ gives a simplicial homotopy $Nf \Rightarrow Ng$ and hence a homotopy $|Nf| \Rightarrow |Ng|$. The existence of $h$ also implies that $\bar{f} = \bar{g}$, so we get a diagram

$$\begin{tikzcd} |N\mathcal{G}| \arrow[r, bend left, "|Nf|"] \arrow[r, bend right, "|Ng|"'] \arrow[r, phantom, "\Downarrow_{|Nh|}"] \arrow[d] & |N\mathcal{H}| \arrow[d] \\ \mathcal{G}_0/\mathcal{G}_1 \arrow[r, "\bar{f}=\bar{g}"] & \mathcal{H}_0/\mathcal{H}_1 . \end{tikzcd} \tag{3.14}$$

We then compose (3.14) with the inclusion $|N\mathcal{H}| \hookrightarrow |N\mathcal{H}|'$. Assembling (3.11) and (3.14), we get a map

$$F(f) \cup |Nh| \cup F(g) : \; |N\mathcal{G}|' \cup \big(|N\mathcal{G}| \times [0,1]\big) \cup |N\mathcal{G}|' \to |N\mathcal{H}|'. \tag{3.15}$$

The inclusion $|N\mathcal{G}|' \cup (|N\mathcal{G}| \times [0,1]) \cup |N\mathcal{G}|' \hookrightarrow |N\mathcal{G}|' \times [0,1]$ is a directed cofibration with respect to the stratification inherited from $\mathcal{H}_0/\mathcal{H}_1$ so we can extend (3.15) to the whole $|N\mathcal{G}|' \times [0,1]$. This is our 2-morphism $F(h) : F(f) \Rightarrow F(g)$.

To finish the construction of $F$, we still need 2-morphisms $\varphi_{f,g} : F(g) \circ F(f) \Rightarrow F(g \circ f)$ (recall that all 2-morphisms are invertible). If $f : \mathcal{G} \to \mathcal{H}$ and $g : \mathcal{H} \to \mathcal{K}$

are two composable 1-arrows, so we get a diagram

$$\begin{tikzcd} |N\mathcal{G}| \arrow[r, "|Nf|"] \arrow[d, hook] & |N\mathcal{H}| \arrow[r, "|Ng|"] \arrow[d, hook] & |N\mathcal{K}| \arrow[d, hook] \\ |N\mathcal{G}|' \arrow[r, "F(f)"] & |N\mathcal{H}|' \arrow[r, "F(g)"] & |N\mathcal{K}|' . \end{tikzcd} \tag{3.16}$$

Both maps $F(g) \circ F(f)$ and $F(g \circ f)$ extend $|N(g \circ f)|$ from $|N\mathcal{G}|$ to $|N\mathcal{G}|'$. By the same argument as before, the map

$$\big(F(g)\circ F(f)\big) \cup \big((|N(g\circ f)|)\circ \mathrm{pr}_1\big) \cup F(g\circ f) : |N\mathcal{G}|' \cup (|N\mathcal{G}| \times [0,1]) \cup |N\mathcal{G}|' \to |N\mathcal{K}|'$$

extends to $|N\mathcal{G}|' \times [0,1]$. This is our 2-morphism $\varphi_{f,g} : F(g) \circ F(f) \Rightarrow F(g \circ f)$.

To show that $F$ is a functor, we still need to show that it preserves the various composition. The identities are sent to identities, so we don't have to worry about them.

Let $h_1 : f \Rightarrow g : \mathcal{G} \overset{\Downarrow}{\rightrightarrows} \mathcal{H}$ and $h_2 : g \Rightarrow k : \mathcal{G} \overset{\Downarrow}{\rightrightarrows} \mathcal{H}$ be vertically composable 2-morphisms, and let $\bullet$ denote vertical composition. The 2-morphisms $F(h_2) \bullet F(h_1)$ and $F(h_2 \bullet h_1)$ are represented by maps $|N\mathcal{G}|' \times [0,1] \to |N\mathcal{H}|'$. They agree on $(|N\mathcal{G}| \times [0,1]) \cup (|N\mathcal{G}|' \times \{0,1\})$, so we get a map

$$(|N\mathcal{G}| \times [0,1]^2) \cup (|N\mathcal{G}|' \times \partial[0,1]^2) \to |N\mathcal{H}|'. \tag{3.17}$$

The map (3.17) extends to $|N\mathcal{G}|' \times [0,1]^2$, which shows that $F(h_2) \bullet F(h_1)$ and $F(h_2 \bullet h_1)$ are homotopic relatively to the end points. This shows that $F(h_2) \bullet F(h_1) = F(h_2 \bullet h_1)$ on $\mathsf{Orb}$.

Now suppose that $h_1 : f_1 \Rightarrow g_1 : \mathcal{G} \overset{\Downarrow}{\rightrightarrows} \mathcal{H}$ and $h_2 : f_2 \Rightarrow g_2 : \mathcal{H} \overset{\Downarrow}{\rightrightarrows} \mathcal{K}$ are horizontally composable. We need to show that

$$F(h_2 \circ h_1) \bullet \varphi_{f_1,f_2} = \varphi_{g_1,g_2} \bullet \big(F(h_2) \circ F(h_1)\big). \tag{3.18}$$

The two maps in (3.18) agree on the subspace $(|N\mathcal{G}| \times [0,1]) \cup (|N\mathcal{G}|' \times \{0,1\}) \subset |N\mathcal{G}|' \times [0,1]$. So, by the same argument as (3.17) they're equal as 2-morphisms of $\mathsf{Orb}$. This finishes the construction of $F : \mathsf{Gpd} \to \mathsf{Orb}$.

**Theorem 3.16** *Let* $\mathsf{Orb}$ *be the 2-category given in Definition 3.1, and* $\mathsf{Gpd}$ *be the 2-category of topological groupoids satisfying the conditions of Definition 2.4. Let* $W \subset \mathsf{Gpd}_1$ *be the weak equivalences, as given in Definition 3.12.*

*Then the functor* $F : \mathsf{Gpd} \to \mathsf{Orb}$ *constructed above extends to an equivalence* $\tilde{F} : \mathsf{Gpd}[W^{-1}] \to \mathsf{Orb}$.

*Proof.* Let $f : \mathcal{G} \xrightarrow{\sim} \mathcal{H}$ be a weak equivalence. We first explain why

$$\begin{tikzcd} |N\mathcal{G}|' \arrow[r, "F(f)"] \arrow[d, "p_{\mathcal{G}}"] & |N\mathcal{H}|' \arrow[d, "p_{\mathcal{H}}"] \\ \mathcal{G}_0/\mathcal{G}_1 \arrow[r, "\bar{f}"] & \mathcal{H}_0/\mathcal{H}_1 \end{tikzcd} \tag{3.19}$$

is an equivalence in $\mathsf{Orb}$. By Lemma 3.14, $\bar{f}$ is a homeomorphism. Given a point $x \in \mathcal{G}_0/\mathcal{G}_1$ with preimage $\hat{x} \in \mathcal{G}_0$, its fiber $p_{\mathcal{G}}^{-1}(x)$ is a $K(\mathrm{Aut}_{\mathcal{G}}(\hat{x}), 1)$ by (3.12). The map $\mathrm{Aut}_{\mathcal{G}}(\hat{x}) \to \mathrm{Aut}_{\mathcal{H}}(f(\hat{x}))$ is an isomorphism by (3.10), so $F(f)$ is a homotopy equivalence on each fiber. By Theorem 4.21, the map $F(f)$ is has a homotopy inverse relatively to the projections to $\mathcal{H}_0/\mathcal{H}_1$. This is exactly what we wanted to show.

We have checked that $F : \mathsf{Gpd} \to \mathsf{Orb}$ sends weak equivalences to equivalences. So by the universal property of $\mathsf{Gpd}[W^{-1}]$, the functor $F$ extends to a functor $\tilde{F} : \mathsf{Gpd}[W^{-1}] \to \mathsf{Orb}$. To show that $\tilde{F}$ is an equivalence, we verify the conditions of Theorem 3.15.

First, we show that $F$ is essentially surjective. Let $p : E \to X$ be an orbispace and let $\{U_i\}$ be a closed cover such that $(p^{-1}(U_i), U_i) \simeq (\tilde{Y}_i/G_i, Y_i/G_i)$. Let $q_i : \tilde{Y}_i \to Y_i$ be the projections and let $s_i : Y_i \to \tilde{Y}_i$ be sections. Let $\mathcal{G}_0$ be the disjoint union of the $Y_i$'s and $s : \mathcal{G}_0 \to E$ be induced by the $s_i$. We then let $\mathcal{G}$ be the topological groupoid with objects $\mathcal{G}_0$ and arrows given by

$$\mathrm{Hom}_{\mathcal{G}}(x, y) = \big\{\gamma \in E^{[0,1]} \,\big|\, s(x) = \gamma(0),\, s(y) = \gamma(1),\, p \circ \gamma \text{ is constant}\,\big\}\big/\sim, \tag{3.20}$$

where $\gamma \sim \gamma'$ if they are homotopic relatively to their endpoints, and within their $p$-fiber. In other words, $\mathcal{G}$ is the groupoid pulled back from fib-$\Pi_1(E)$ along the map $s : \mathcal{G}_0 \to E$, where fib-$\Pi_1(E)$ is the fiber-wise fundamental groupoid of $E$.

We now show that $\mathcal{G}$ satisfies the two conditions in Definition 2.4. The automorphism groups $\mathrm{Aut}_{\mathcal{G}}(x)$ are finite since they agree with the fundamental groups of the fibers of $p$. We now show that the subspace of identities is a union of connected components of $\mathcal{G}_1$. The space $\mathcal{G}_1$ is the disjoint union of the subspaces $\mathcal{G}_1(i,j) := \mu^{-1}(Y_i \times Y_j)$. Since $\mathcal{G}_1(i,j)$ intersects $Im(u)$ only when $i = j$, we can restrict our attention to the full subgroupoids $\mathcal{G}_1(i,i) \rightrightarrows Y_i$. These are isomorphic to the action groupoids $Y_i \times G_i \rightrightarrows Y_i$. The identities are $Y_i \times \{e\}$ which is indeed a connected component of $Y_i \times G_i$. we have shown that $\mathcal{G} \in \mathsf{Gpd}$.

We now build an equivalence $F(\mathcal{G}) \to (E, X)$. Since $\mathcal{G}$ is a subgroupoid of fib-$\Pi_1(E)$ we get a diagram

$$\begin{tikzcd} |N\mathcal{G}|' \arrow[drr] & |N\mathcal{G}| \arrow[l, hook'] \arrow[r] \arrow[dr] & \big|N\big(\text{fib-}\Pi_1(E)\big)\big| \arrow[d] & E \arrow[l] \arrow[dl] \\ & & \mathcal{G}_0/\mathcal{G}_1 = X, & \end{tikzcd} \tag{3.21}$$

where all the top horizontal maps are fiber-wise homotopy equivalences. We first apply Theorem 4.21 to get a homotopy inverse $|N(\text{fib-}\Pi_1(E))| \to E$, and then extend it to $|N\mathcal{G}|'$ using Theorem 4.29. The resulting map $(|N\mathcal{G}|', \mathcal{G}_0/\mathcal{G}_1) \to (E, X)$ is then an equivalence by Theorem 4.21. This finishes the proof that $F$ is essentially surjective.

We now verify the second condition of Theorem 3.15. Let $\mathcal{G}, \mathcal{H}$ be topological groupoids, and let $f : F(\mathcal{G}) \to F(\mathcal{H})$ be an orbispace morphism. We need to find a groupoid $\tilde{\mathcal{G}} \xrightarrow{\sim} \mathcal{G}$ and a continuous functor $\tilde{f} : \tilde{\mathcal{G}} \to \mathcal{H}$ making the diagram

$$\begin{tikzcd} F(\tilde{\mathcal{G}}) \arrow[r, "w", "\sim"'] \arrow[rr, bend left, "F(\tilde{f})"] & F(\mathcal{G}) \arrow[r, "f"] & F(\mathcal{H}) \end{tikzcd} \tag{3.22}$$

commute up to a 2-morphism.

Let $\iota : \mathcal{H}_0 \hookrightarrow |N\mathcal{H}|'$ be the inclusion, and $q : \mathcal{H}_0 \to \mathcal{H}_0/\mathcal{H}_1$ and $p : |N\mathcal{H}|' \to \mathcal{H}_0/\mathcal{H}_1$ be the projections. We claim that $|N\mathcal{H}|'$ has a closed cover $\{V_i\}$ that fiber-wise retracts into the image of $\iota$. Let $\{U_i\}$ be a cover of $\mathcal{H}_0$ which is fine enough so that $p^{-1}(q(U_i)) \to q(U_i)$ are homeomorphic to Borel constructions $\tilde{Y}_i/G_i \to Y_i/G_i$. By picking the cover fine enough, we can also make sure that all the $U_i$ and $\tilde{Y}_i$ are simply connected. Let $v_i : \tilde{Y}_i \to \tilde{Y}_i/G_i = p^{-1}(q(U_i))$ be the projection. Since $\tilde{Y}_i$ is the universal cover of $p^{-1}(q(U_i))$, the map $\iota : U_i \to p^{-1}(q(U_i))$ lifts to a map $\tilde{\iota} : U_i \to \tilde{Y}_i$. Let $u_i$ denote the composite of $\tilde{\iota}$ with the projection to $Y_i$. Let $V_i$ be the pullback $u_i^*\tilde{Y}_i$ with map $\tilde{u}_i : V_i \to \tilde{Y}_i$ and projection $r_i : V_i \to U_i$. Since $r_i$ has contractible fibers, we can pick a section $s_i : U_i \to V_i$.

$$\begin{tikzcd} V_i = u_i^*\tilde{Y}_i \arrow[d, "r_i"] \arrow[rr, bend left, "\tilde{u}_i"] & p^{-1}q(U_i) \arrow[d] & \tilde{Y}_i \arrow[l, two heads, "v_i"'] \arrow[d] \\ U_i \arrow[u, bend left, "s_i"] \arrow[ur, "\iota"] \arrow[urr, "\tilde{\iota}"'] \arrow[rr, bend left, "u_i"] \arrow[r, two heads] & qU_i & Y_i \arrow[l, two heads] \end{tikzcd}$$

The map $s_i \circ r_i$ is fiber-wise homotopic to the identity on $V_i$, so $v_i \circ \tilde{u}_i$ is fiber-wise homotopic to

$$v_i \circ \tilde{u}_i \circ s_i \circ r_i = v_i \circ \tilde{\iota} \circ r_i = \iota \circ r_i.$$

Let $V$ denote the disjoint union of the $V_i$'s and let $r : V \to \mathcal{H}_0$ and $\alpha : V \to |N\mathcal{H}|'$ be the maps induced from the $r_i$ and $v_i \circ \tilde{u}_i$ respectively. Since each $v_i \circ \tilde{u}_i$ is topologically surjective, so is $\alpha$. We have constructed a cover $V \to |N\mathcal{H}|'$ that fiber-wise retracts into the image of $\iota$:

$$\begin{tikzcd} V \arrow[rr, two heads, "\alpha"] \arrow[dr, "r"'] & \arrow[d, phantom, "\Downarrow h"] & {|N\mathcal{H}|'} \\ & \mathcal{H}_0 \arrow[ur, hook, "\iota"'] & \end{tikzcd} \tag{3.23}$$

Let $\alpha^*\mathcal{G}_0$ be the pullback of $\mathcal{G}_0 \hookrightarrow |N\mathcal{G}|' \xrightarrow{f} |N\mathcal{H}|' \twoheadleftarrow V$, and let $\tilde{\mathcal{G}}$ be groupoid pulled back from $\mathcal{G}$ along the map $\alpha^*\mathcal{G}_0 \to \mathcal{G}_0$. By Lemma 3.13, the projection $\tilde{\mathcal{G}} \to \mathcal{G}$ is a weak equivalence.

We now build a functor $\tilde{f} : \tilde{\mathcal{G}} \to \mathcal{H}$. On the objects, it is given by $r \circ (\alpha^* f) : \alpha^*\mathcal{G}_0 \to V \to \mathcal{H}_0$. An arrow in $\tilde{\mathcal{G}}$ consists of an arrow $g \in \mathcal{G}_1$, and two points

$v_0, v_1 \in V$ such that $f(s(g)) = \alpha(v_0)$ and $f(t(g)) = \alpha(v_1)$. Let $f(g)$ denote the image of the path $\{g\} \times \Delta^1 \subset |N\mathcal{G}|'$ under the map $f$. The composition $h(v_0)^{-1} \cdot f(g) \cdot h(v_1)$ is then a path from $\iota(r(v_0))$ to $\iota(r(v_1))$, where $h$ is the homotopy in (3.23). This path lies entirely in a fiber of $|N\mathcal{H}|'$. Since $\mathcal{H}$ is a full subgroupoid of fib-$\Pi_1|N\mathcal{H}|'$, the path $h(v_0)^{-1} \cdot f(g) \cdot h(v_1)$ gives us a morphism in from $r(v_0)$ to $r(v_1)$. This is what we define $\tilde{f}(g, v_0, v_1)$ to be.

To check that $\tilde{f} : \tilde{\mathcal{G}} \to \mathcal{H}$ is a functor, we pick two composable arrows $(g_1, v_0, v_1)$ and $(g_2, v_1, v_2)$ in $\tilde{\mathcal{G}}$. The two paths

$$\begin{aligned} &\tilde{f}(g_1, v_0, v_1)\tilde{f}(g_2, v_1, v_2) = h(v_0)^{-1} \cdot f(g_1) \cdot h(v_1) \cdot h(v_1)^{-1} \cdot f(g_2) \cdot h(v_2) \quad \text{and} \\ &\tilde{f}\big((g_1, v_0, v_1)(g_2, v_1, v_2)\big) = \tilde{f}(g_1 g_2, v_0, v_2) = h(v_0)^{-1} \cdot f(g_1 g_2) \cdot h(v_2) \end{aligned} \tag{3.24}$$

are clearly homotopic, and so they represent the same element in $\mathcal{H}_1$. This finishes the construction of $\tilde{f}$.

We have constructed all the maps in (3.22). Now we need to provide the 2-morphism $H : f \circ w \Rightarrow F(\tilde{f})$. On $\tilde{\mathcal{G}}_0 \subset |N\tilde{\mathcal{G}}|'$, it is given by

$$H : f \circ w = \alpha \circ (\alpha^* f) \overset{h}{\Rightarrow} \iota \circ r \circ (\alpha^* f) = \iota \circ \tilde{f} = F(\tilde{f}). \tag{3.25}$$

To extend $H$ to the one skeleton $|N\tilde{\mathcal{G}}|^{(1)}$, we consider the lifting problem

$$\begin{tikzcd} \big(\tilde{\mathcal{G}}_0 \times [0,1]\big) \cup \big(|N\tilde{\mathcal{G}}|^{(1)} \times \{0,1\}\big) \arrow[r, "h \cup (f\circ w) \cup F(\tilde{f})"] \arrow[d, hook] & |N\mathcal{H}|' \arrow[d, "p"] \\ |N\tilde{\mathcal{G}}|^{(1)} \times [0,1] \arrow[r] \arrow[ur, dotted] & \mathcal{H}_0/\mathcal{H}_1 . \end{tikzcd} \tag{3.26}$$

Since the fibers of $p$ are $K(\pi, 1)$'s the only obstructions come from the 2-cells. The boundary of the 2-cells are given by paths $f \circ w(g, v_0, v_1) = f(g)$, $\tilde{f}(g, v_0, v_1)$, $h(v_0)$ and $h(v_1)$, where $(g, v_0, v_1) \in \tilde{\mathcal{G}}_1$ is some arrow. But the two paths $\tilde{f}(g, v_0, v_1)$ and $h(v_0)^{-1} \cdot f(g) \cdot h(v_1)$ are homotopic by construction. So there are no obstructions. Similarly, there are no obstructions to extending $H$ to the whole $|N\mathcal{H}|'$. This finishes the construction of the 2-arrow $H : f \circ w \Rightarrow F(\tilde{f})$.

Now we check the last condition of Theorem 3.15, namely that $F$ is fully faithful on 2-morphisms. Let $\mathcal{G}$, $\mathcal{H}$ be topological groupoids and $f, g : \mathcal{G} \to \mathcal{H}$ be continuous functors. We want to construct an inverse $F^{-1}$ to the natural map

$$F : \text{2-Hom}_{\mathsf{Gpd}}(f, g) \to \text{2-Hom}_{\mathsf{Orb}}\big(F(f), F(g)\big). \tag{3.27}$$

Given a 2-morphism $h : F(f) \Rightarrow F(g)$ and an object $x \in \mathcal{G}_0$, we let $F^{-1}(h) : f(x) \to g(x)$ be the arrow corresponding to the path $h(x)$ in $|N\mathcal{H}|'$. This is an inverse to (3.27), and thus proves that $F$ is fully faithful on 2-morphisms.

We have checked the three conditions in Theorem 3.15, so $\tilde{F} : \mathsf{Gpd}[W^{-1}] \to \mathsf{Orb}$ is an equivalence of bicategories. $\square$

# Chapter 4

# Stratified fibrations

## 4.1 Triangulations

One of our main working tools, will be triangulations. So we will work in the subcategory of spaces which are regular CW-complexes (see [10] for basic background). These are the spaces arising as geometric realization of simplicial sets. It is also convenient to restrict the class of morphisms. So we make the following working definition:

**Definition 4.1** *The objects of the category* spaces *are topological spaces arising as geometric realizations of simplicial sets. The morphisms are the continuous maps which are semi-algebraic of each simplex of the source i.e. the graph of the map is defined (locally) by finitely many algebraic equalities and inequalities.*

We now define what we mean by a triangulation of a space. Since our definition varies slightly from the usual one, we use a different terminology.

**Definition 4.2** *Let $X$ be an object of* spaces. *An oriented triangulation of $X$ is a simplicial set $Y$ and an isomorphism between $X$ and $|Y|$.*

We have the following convenient lemma about cofibrations.

**Lemma 4.3** *Let $i : A \hookrightarrow B$ be a cofibration. Then there exist spaces $C$, $D$ and maps $f : C \to A$, $g : C \to D$ such that $B \simeq A \cup_f (C \times [0,1]) \cup_g E$. More precisely, we get a commutative diagram*

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, equal] & A \cup_f (C \times [0,1]) \cup_g D \arrow[d, "\simeq"] \\ A \arrow[r, hook, "i"] & B \end{tikzcd} \tag{4.1}$$

*where the top arrow is the obvious inclusion. Moreover, if $i$ is a weak equivalence, then $g$ is a weak equivalence.*

*Proof.* Triangulate $B$ is a way compatible with $A$ and let $D$ be the union of all simplices that do not intersect $A$. The map $t : A \sqcup D \to [0,1]$ sending $A$ to 0 and $D$ to 1 extends by linearity to a map on the whole of $B$. Let $C := t^{-1}(1/2)$.

An $n$-simplex of $B$ has $k+1$ vertices in $A$ and $\ell+1$ vertices in $A$ for some $k, \ell$ satisfying $k+\ell+1=n$. The intersection of that simplex with $C$ is then isomorphic to $\Delta^k \times \Delta^\ell$. We then assemble the maps $\Delta^k \times \Delta^\ell \twoheadrightarrow \Delta^k \hookrightarrow \Delta^n$ and $\Delta^k \times \Delta^\ell \twoheadrightarrow \Delta^\ell \hookrightarrow \Delta^n$ to maps $f : C \to A$ and $g : C \to D$. Since $\Delta^n = \Delta^k * \Delta^\ell$, we get a map $\Delta^k \times \Delta^\ell \times [0,1] \twoheadrightarrow \Delta^n$ which is a homeomorphism over the interior of $[0,1]$. Assembling these maps over all the simplices of $B$, produces a map $C \times [0,1] \to B$. This provides the isomorphism $A \cup_f (C \times [0,1]) \cup_g D \xrightarrow{\sim} B$.

If $i$ is a weak equivalence, then so is the bottom arrow in the following diagram.

$$\begin{tikzcd}
C \times \{1\} \arrow[r, "g"] \arrow[d, hook] & D \arrow[d, hook] \\
A \cup_f (C \times [0,1]) \arrow[r, "\sim"] & A \cup_f (C \times [0,1]) \cup_g D
\end{tikzcd} \tag{4.2}$$

Since (4.2) is a homotopy pushout diagram, $g$ is also a weak equivalence. □

## 4.2 Stratifications

**Definition 4.4** *Let $\mathbb{J}$ be a poset and $X$ a topological space. A stratification of $X$ by $\mathbb{J}$ is an upper semi-continuous function $s : X \to \mathbb{J}$ which takes finitely many values on each compact subspace. It defines a partition of $X$ into strata $X_j := s^{-1}(j)$. We also introduce the notations*

$$X_{\leq j} := s^{-1}\{i \in \mathbb{J} | i \leq j\} \qquad X_{<j} := s^{-1}\{i \in \mathbb{J} | i < j\}$$
$$X_{\geq j} := s^{-1}\{i \in \mathbb{J} | i \geq j\} \qquad X_{>j} := s^{-1}\{i \in \mathbb{J} | i > j\}.$$

*The subsets $X_{\leq j}$ and $X_{<j}$ are open while $X_{\geq j}$ and $X_{>j}$ are closed.*

*Let $(X, s)$ and $(X', s')$ be two $\mathbb{J}$-stratified spaces. A continuous map $f : X \to X'$ is stratified if it satisfies $s' \circ f = s$, or equivalently if it sends $X_j$ to $X'_j$.*

**Definition 4.5** *Let $(X, s)$ be a stratified space. An oriented triangulation $\mathcal{T}$ is compatible with the stratification if given any simplex $\sigma : \Delta^n \to X$ of $\mathcal{T}$, the composite $s \circ \sigma$ is constant on the subsets*

$$\Big\{(t_0, \ldots, t_n) \in \Delta^n \,\Big|\, t_i = 0 \text{ for } i < j,\ t_j \neq 0\Big\} \subset \Delta^n.$$

On compact spaces, the existence of compatible oriented triangulations is an easy corollary of the existence of (usual) triangulation.

**Lemma 4.6** *Given a compact stratified space $X$, there exists a compatible oriented triangulation.*

*Proof.* By [17], there exists a (usual) triangulation $\mathcal{T}$ of $X$ such that each stratum is a union of open simplices of $\mathcal{T}$. The barycentric subdivision of $\mathcal{T}$ is then a compatible oriented triangulation. □

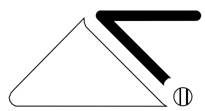
Let $\mathbb{J}^{op}$ be the opposite poset to $\mathbb{J}$, with elements denoted $j^{op}$. To each $\mathbb{J}$-stratification $s : X \to \mathbb{J}$ there is an associated $\mathbb{J}^{op}$-stratification $s^{op} : X \to \mathbb{J}^{op}$, defined up to some appropriate notion of weak equivalence.

**Definition 4.7** *Let $X$ be an $\mathbb{J}$-stratified space and $\mathcal{T}$ a compatible oriented triangulation. The opposite stratification $s^{op} : X \to \mathbb{J}^{op}$ is given by*

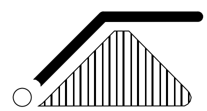
$$s^{op}(x) := \left[\max_{y \in \Delta^n} s(\sigma(y))\right]^{op} = \min_{y \in \Delta^n} s(\sigma(y))^{op}, \tag{4.3}$$

*where $\sigma : \Delta^n \to X$ is the smallest non-degenerate simplex of $\mathcal{T}$ in the image of which $x$ lies. The space $X$, equipped with this new stratification will be denoted $X^{op}$. We also define $X^{op}_j := X^{op}_{j^{op}}$, $X^{op}_{\geq j} := X^{op}_{\leq j^{op}}$ and similarly for $X^{op}_{>j}$, $X^{op}_{\leq j}$ and $X^{op}_{<j}$.*

We illustrate here an example of a stratified space $X$ and its opposite $X^{op}$:



$X$



$X^{op}$

The poset for this example is $\mathbb{J} = ($ *"white"* $<$ *"black"* $<$ *"stripes"* $)$.

Note that in general, the strata $X^{op}_j$ are homotopy equivalent to $X_j$, but the closure relations are reversed. We also have

$$X^{op}_{\leq j} \simeq X_{\leq j}, \qquad X^{op}_{<j} \simeq X_{<j}, \qquad X^{op}_{\geq j} \simeq X_{\geq j}, \quad \text{and} \quad X^{op}_{>j} \simeq X_{>j}. \tag{4.4}$$

## 4.3 Directed cofibrations

Before introducing stratified fibrations, we recall the classical notion of (Serre) fibrations. The symbols $\twoheadrightarrow$ and $\hookrightarrow$ will be used to denote fibrations and cofibrations respectively.

Let $\Delta^n$ be the topological $n$-simplex and $\Lambda^n := Cone(\partial\Delta^{n-1})$ be the $n$-horn. A map $E \to X$ is a fibration if for any maps $\Delta^n \to X$ and $\Lambda^n \to E$ making the diagram

$$\begin{tikzcd} \Lambda^n \arrow[r] \arrow[d, hook] & E \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & X \end{tikzcd} \tag{4.5}$$

commute, there exists a lift $\Delta^n \to E$ extending the given map on $\Lambda^n$. The inclusions $\Lambda^n \hookrightarrow \Delta^n$ are called the generating acyclic cofibrations.

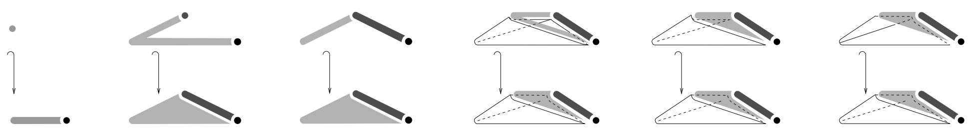
Just like Serre fibrations, we define stratified fibrations by a lifting property. The maps which play the role of acyclic cofibrations are called directed cofibrations. We begin by defining a set of generating directed cofibrations. We then define stratified fibrations to be the maps satisfying the right lifting property with respect to the generating directed cofibrations, and directed cofibrations as those satisfying the left lifting property with respect to stratified fibrations.

**Definition 4.8** *Let $\Delta^n$ be the $n$-simplex and $\Lambda^{n,i} \subset \Delta^n$ be the union of all the facets containing the $i$th vertex. Let $s : \Delta^n \to \mathbb{J}$ be a stratification which is constant on the subsets*

$$\{(t_0, \ldots, t_n) \in \Delta^n \mid t_i = 0 \text{ for } i < j,\, t_j \neq 0\} \tag{4.6}$$

*and $s' := s|_{\Lambda^{n,i}}$. Then $(\Lambda^{n,i}, s') \hookrightarrow (\Delta^n, s)$ is a generating directed cofibration if $i < n$.*

We illustrate the generating directed cofibrations for $n = 1, 2, 3$ :



(4.7)

Note that $(\Lambda^{n,n}, s') \hookrightarrow (\Delta^n, s)$ is sometimes a generating directed cofibration. It is one if and only if $s$ is constant on the edge linking the $n$th and $n-1$st vertex of $\Delta^n$. Indeed, the linear map flipping that edge is then a stratified homeomorphism between the pairs $(\Delta^n, \Lambda^{n,n})$ and $(\Delta^n, \Lambda^{n,n-1})$.

We now introduce a new notion of stratified fibration. It is much stronger than the stratified fibrations of Huges [20] and Friedman [9].

**Definition 4.9** *A stratified map $p : E \to X$ is a stratified fibration if it satisfies the right lifting property with respect to the generating directed cofibrations. In other words, for every generating directed cofibration $\Lambda^n \hookrightarrow \Delta^n$ and every commutative diagram (4.5) of stratified maps, there has to exist a lift $\Delta^n \to E$ making both triangles commute.*

*A stratified map $A \to B$ is a directed cofibration if it satisfies the left lifting property with respect to stratified fibrations. In other words, for every diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & E \arrow[d, two heads] \\ B \arrow[r] \arrow[ur, dotted] & X, \end{tikzcd} \tag{4.8}$$

*where $E \to X$ is a stratified fibration, there exists a lift $B \to E$.*

To be able to work with the above definitions, we need a better understanding of directed cofibrations. The following Lemma is classical from the theory of model categories [19, section 2.1] [11, section I.4].

**Lemma 4.10** *a. If $A \to B$ is a directed cofibration and $A \to C$ is an arbitrary stratified map, then $C \to B \cup_A C$ is a directed cofibration.*

*b. If $A_i \to A_{i+1}$ are directed cofibrations, then $A_0 \to \varinjlim A_i$ is a directed cofibration. A similar statement holds when $i$ ranges over some arbitrary ordinal, but then we should also insist that $A_\beta = \varinjlim_{\alpha<\beta} A_\alpha$ whenever $\beta$ is a limit ordinal.*

*c. Let $A \to B$ be a directed cofibration and let $A' \to B'$ be a retract of it. In other words, suppose that we have a commuting diagram*

$$\begin{tikzcd} A' \arrow[r] \arrow[d, hook] & A \arrow[r] \arrow[d, hook] & A' \arrow[d, hook] \\ B' \arrow[r] & B \arrow[r] & B', \end{tikzcd} \tag{4.9}$$

*where the horizontal composites are identities. Then $A' \to B'$ is a directed cofibration.*

*Proof.* *a.* Given a stratified fibration $E \to X$, and a commutative square $C \to E$, $B \cup_A C \to X$, we first find a lift $B \to E$

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & C \arrow[r] \arrow[d, hook] & E \arrow[d] \\ B \arrow[r] \arrow[urr, dotted] & B \cup_A C \arrow[r] & X. \end{tikzcd} \tag{4.10}$$

Then we assemble the maps $B \to E$ and $C \to E$ into a map $B \cup_A C \to E$.

*b.* Given a stratified fibration $E \to X$ and a commutative square $A_1 \to E$, $\varinjlim A_i \to X$, we inductively build partial lifts $A_i \to E$

$$\begin{tikzcd} A_i \arrow[r] \arrow[d, hook] & E \arrow[d] \\ A_{i+1} \arrow[r] \arrow[ur, dotted] & X. \end{tikzcd} \tag{4.11}$$

These maps assemble to the desired map $\varinjlim A_i \to E$.

*c.* Given a stratified fibration $E \to X$ and a commutative square $A' \to E$, $B' \to X$, we first find a lift $B \to E$

$$\begin{tikzcd} A \arrow[d, hook] & A' \arrow[l, shift right] \arrow[l, leftarrow, bend right, shift left] \arrow[r] \arrow[d, hook] & E \arrow[d] \\ B \arrow[urr, dotted] & B' \arrow[l, shift right] \arrow[l, leftarrow, bend right, shift left] \arrow[r] & X. \end{tikzcd} \tag{4.12}$$

Composing the map $B' \to B$ with the lift $B \to E$ gives a solution to our problem. $\square$

Lemma 4.10 also has a converse [19, Corollary 2.1.15].

**Lemma 4.11** *The class of maps that can be obtained, starting from the class of generating directed cofibration, and using the constructions of Lemma 4.10 is exactly the class of directed cofibrations.* $\square$

The following Lemma provides some basic examples of directed cofibrations.

**Lemma 4.12** *Let $(X, A)$ be a pair of spaces and let $s : X \times [0,1] \to \mathbb{J}$ be a stratification such that $s|_{\{x\}\times[0,1)}$ is constant for all $x \in X$. Then the inclusion*

$$\big(X \times \{0\}\big) \cup \big(A \times [0,1]\big) \hookrightarrow X \times [0,1] \tag{4.13}$$

*is a directed cofibration.*

*Proof.* Let $\mathcal{T}$ be an oriented triangulation of $X$ which is compatible with the subspace $A$ and with the stratification of $X \times [0,1]$. Let $S_n \subset X \times [0,1]$ be the subspaces given by

$$S_n := (X \times \{0\}) \cup \big((A \cup X^{(n)}) \times [0,1]\big), \tag{4.14}$$

where $X^{(n)}$ denotes the $n$-skeleton of $\mathcal{T}$. It is enough by Lemma 4.10.$b$ to show that each inclusion $S_{n-1} \hookrightarrow S_n$ is a directed cofibration. We can write it as a pushout

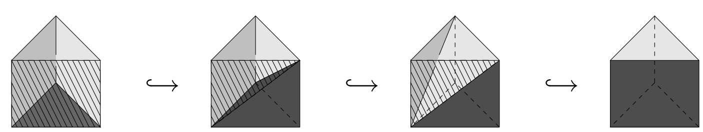
$$\begin{tikzcd} \coprod\Big(\big(\Delta^n \times \{0\}\big) \cup \big(\partial\Delta^n \times [0,1]\big)\Big) \arrow[r] \arrow[d, hook] & S_{n-1} \arrow[d, hook] \\ \coprod\big(\Delta^n \times [0,1]\big) \arrow[r] & S_n, \end{tikzcd} \tag{4.15}$$

where the coproducts run over the set of $n$-simplices of $X$. So by Lemma 4.10.$a$ we are reduced to the case $X = \Delta^k$ and $A = \partial\Delta^k$.

We wish to show that $(\Delta^n \times \{0\}) \cup (\partial\Delta^n \times [0,1]) \hookrightarrow \Delta^n \times [0,1]$ is a directed cofibration. Using the standard triangulation of $\Delta^n \times [0,1]$, we find sequence of spaces

$$\big(\Delta^n \times \{0\}\big) \cup \big(\partial\Delta^n \times [0,1]\big) = Y_0 \hookrightarrow Y_1 \hookrightarrow \ldots \hookrightarrow Y_{n+1} = \Delta^n \times [0,1]$$

illustrated below for $n = 2$

$Y_0 =$  $= Y_3.$

The space $Y_i$ contains the $i$ first $(n+1)$-simplices of $\Delta^n \times [0,1]$ and we have a sequence of pushout diagrams

$$\begin{tikzcd} \Lambda^{n+1,n-i} \arrow[r] \arrow[d, hook] & Y_i \arrow[d, hook] \\ \Delta^{n+1} \arrow[r] & Y_{i+1}. \end{tikzcd} \tag{4.16}$$

Each inclusion $\Lambda^{n+1,n-i} \hookrightarrow \Delta^{n+1}$ is a generating directed cofibration. Therefore by Lemma 4.10.$ab$, so is $Y_{i-1} \to Y_i$ and so is $(\Delta^n \times \{0\}) \cup (\partial\Delta^n \times [0,1]) \hookrightarrow \Delta^n \times [0,1]$. $\square$

**Corollary 4.13** *If $X \times [0,1]$ is stratified with two strata $X \times [0,1)$ and $X \times \{1\}$, then the inclusion $X \times \{0\} \hookrightarrow X \times [0,1]$ is a directed cofibration.*

**Lemma 4.14** *Let $\iota : A \hookrightarrow B$ be an inclusion which is also a homotopy equivalence. Give both $A$ and $B$ a constant stratification. Then $\iota$ is a directed cofibration.*

*Proof.* Let $r : B \to A$ be a deformation retraction, and $h : B \times [0,1] \to B$ be the homotopy between $\iota \circ r$ and $\mathrm{Id}_B$. Let $r' : B \times \{0\} \cup A \times [0,1] \to A$ be the map which is $r$ on $B$ and projection on $A \times [0,1]$, and let $C$ be the pushout of the diagram

$$\begin{tikzcd} B \times \{0\} \cup A \times [0,1] \arrow[r, "r'"] \arrow[d, hook, "\iota'"] & A \arrow[d, hook] \\ B \times [0,1] \arrow[r] & C. \end{tikzcd} \tag{4.17}$$

The inclusion $B \times \{0\} \cup A \times [0,1]$ is a directed cofibration by Lemma 4.12, and so is $\iota'$ by Lemma 4.10.$a$.

Now we note that, since $h$ is constant on $A$, it factor through a map $h' : C \to B$. Let $i_1 : B \times \{1\} \to C$ denote the inclusion. Then the diagram

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, hook] & A \arrow[r, equal] \arrow[d, hook] & A \arrow[d, hook] \\ B \arrow[r, "i_1"] & C \arrow[r, "h"] & B \end{tikzcd} \tag{4.18}$$

expresses $\iota$ as a retract of $\iota'$. So $\iota$ is a directed cofibration by Lemma 4.10.$c$. □

**Theorem 4.15** *Let $(B, s)$ be a $\mathbb{J}$-stratified space and $A \subset B$ a subspace. Suppose that the image of $s$ has no infinite descending chains (for example $B$ is compact). Then the following are equivalent:*

1. *$A_{\leq j} \to B_{\leq j}$ is a homotopy equivalence for all $j \in \mathbb{J}$.*
2. *The map*
$$\bigcup_{j \in S} A_{\leq j} \to \bigcup_{j \in S} B_{\leq j} \tag{4.19}$$
*is a homotopy equivalence for all subsets $S \subset \mathbb{J}$.*
3. *$A \hookrightarrow B$ is a directed cofibration.*

*Proof.* Since $\mathbb{J}$ doesn't have infinite descending chains, we can extend the partial order to a well order $\mathbb{J}'$. Let $\iota : \mathbb{J} \to \mathbb{J}'$ denote the identity.

The implication 2. $\Rightarrow$ 1. is trivial so we show 1. $\Rightarrow$ 2. Assume that 1. holds. We show that (4.19) is a homotopy equivalence by induction on the first element $\alpha \in \mathbb{J}'$ which is not in $\iota(S)$. If $\alpha$ is a limit ordinal, then

$$\bigcup_{j \in S} A_{\leq j} = \varinjlim_{\beta < \alpha} \bigcup_{\substack{j \in S, \\ \iota(j) \leq \beta}} A_{\leq j} \simeq \varinjlim_{\beta < \alpha} \bigcup_{\substack{j \in S, \\ \iota(j) \leq \beta}} B_{\leq j} = \bigcup_{j \in S} B_{\leq j},$$

where the middle equivalence holds by the inductions hypothesis. If $\alpha = \beta + 1$, we let $k := \iota^{-1}(\beta)$ and write

$$\bigcup_{j\in S} A_{\leq j} = \Big(\bigcup_{\substack{j\in S\\ j\neq k}} A_{\leq j}\Big) \cup_{\left(\bigcup_{j\in S,\, j<k} A_{\leq j}\right)} A_{\leq k} \simeq \Big(\bigcup_{\substack{j\in S\\ j\neq k}} B_{\leq j}\Big) \cup_{\left(\bigcup_{j\in S,\, j<k} B_{\leq j}\right)} B_{\leq k} = \bigcup_{j\in S} B_{\leq j}$$

where the middle equivalence holds by the inductions hypothesis and because $A_{\leq k} \simeq B_{\leq k}$.

We now show 2. $\Rightarrow$ 3. Let

$$A_\alpha := \bigcup_{j:\,\iota(j)<\alpha} A^{op}_{\leq j} \qquad \text{and} \qquad B_\alpha := \bigcup_{j:\,\iota(j)<\alpha} B^{op}_{\leq j}.$$

By (4.4) and (4.19), the cofibration $A_\alpha \hookrightarrow B_\alpha$ is a homotopy equivalence. Consider the spaces $C_\alpha := A \cup B_\alpha$. We have a sequence of inclusions

$$A = C_0 \hookrightarrow C_1 \hookrightarrow \ldots \to \varinjlim C_j = B, \tag{4.20}$$

so by Lemma 4.10.$b$, it's enough to show that each inclusion $C_\alpha \hookrightarrow C_{\alpha+1}$ is a directed cofibration. The horizontal arrows in

$$\begin{tikzcd} A_\alpha \arrow[r, hook, "\sim"] \arrow[d, hook] & B_\alpha \arrow[d, hook] \\ A_{\alpha+1} \arrow[r, hook, "\sim"] & B_{\alpha+1} \end{tikzcd} \tag{4.21}$$

are homotopy equivalences, so that diagram is homotopy cocartesian. The inclusion $A_{\alpha+1} \cup_{A_\alpha} B_\alpha \hookrightarrow B_{\alpha+1}$ is therefore a homotopy equivalence. Using Lemma 4.3 we write $B_{\alpha+1}$ as a homotopy pushout

$$B_{\alpha+1} = \big(A_{\alpha+1} \cup_{A_\alpha} B_\alpha\big) \cup_f \big(D \times [0,1]\big) \cup_g E, \tag{4.22}$$

where $g : D \to E$ is a homotopy equivalence. Since $C_\alpha = A \cup_{A_{\alpha+1}} \big(A_{\alpha+1} \cup_{A_\alpha} B_\alpha\big)$ and $C_{\alpha+1} = A \cup_{A_{\alpha+1}} B_{\alpha+1}$, we also get

$$C_{\alpha+1} = C_\alpha \cup_f \big(D \times [0,1]\big) \cup_g E. \tag{4.23}$$

The stratification is $s \circ f \circ \mathrm{pr}_1$ on $D \times (0,1)$ and $j$ on $E$. Now consider the following sequence of inclusions illustrated in (4.26):

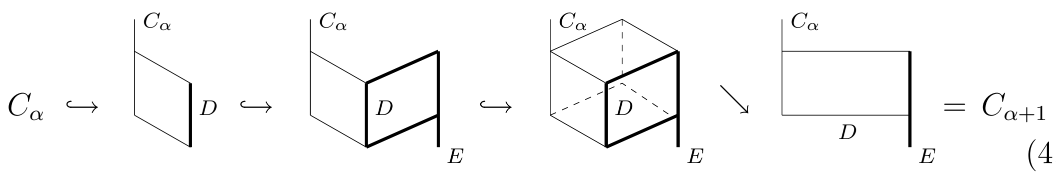
$$\begin{aligned} C_\alpha &\hookrightarrow C_\alpha \cup_f (D \times [0,1]) \\ &\hookrightarrow C_\alpha \cup_f \big(D \times ([0,1] \times \{0\} \cup \{1\} \times [0,1])\big) \cup_g E \\ &\hookrightarrow C_\alpha \cup_f (D \times [0,1]^2) \cup_g E. \end{aligned} \tag{4.24}$$

The stratification is $s \circ f \circ \mathrm{pr}_1$ on $D \times [0,1) \times [0,1]$ and is $j$ on $D \times \{1\} \times [0,1]$ and on $E$.

The three inclusions in (4.24) are directed cofibrations. The first one is by Lemma 4.12. The second one is because $D \times \{1\} \subset D \times [1,2] \cup_g E$ is a homotopy equivalence, and so we can apply Lemmas 4.14, and 4.10.$a$. To see that the third one is, we apply Lemma 4.12 to $D \times ([0,1] \times \{0\} \cup \{1\} \times [0,1]) \hookrightarrow D \times [0,1]^2$ and finish by Lemma 4.10.$a$. By Lemma 4.10.$b$ the composite

$$C_\alpha \hookrightarrow C_\alpha \cup_f \left(D \times [0,1]^2\right) \cup_g E \tag{4.25}$$

is also a directed cofibration.




(4.26)

We now observe that $C_\alpha \hookrightarrow C_{\alpha+1} = C_\alpha \cup_f (D \times [0,1]) \cup_g E$ is a retract of (4.25) by using the diagonal map $\Delta : [0,1] \to [0,1]^2$ and the projection $\mathrm{pr}_1 : [0,1]^2 \to [0,1]$. So by Lemma 4.10.$c$, we have that $C_\alpha \hookrightarrow C_{\alpha+1}$ is a directed cofibration. We now go back to (4.20) and apply Lemma 4.10.$b$. This finishes the proof that $A \hookrightarrow B$ is a directed cofibration.

We now show 3. $\Rightarrow$ 2. Suppose that $A \to B$ is a directed cofibration, we want to show that (4.19) is a homotopy equivalence. By Lemma 4.11, it is enough to check it for generating cofibrations, and to check that the constructions of Lemma 4.10 preserve that property. The latter is straightforward, so we concentrate on the former.

Let $\Lambda^{n,i} \hookrightarrow \Delta^n$ be a generating directed cofibration. The sets (4.19) are either empty or of the form

$$\left\{ t \in \Lambda^{n,i} \,\middle|\, \sum_{j=0}^{r} t_j \neq 0 \right\} \qquad \textit{and} \qquad \left\{ t \in \Delta^{n} \,\middle|\, \sum_{j=0}^{r} t_j \neq 0 \right\} \tag{4.27}$$

for some appropriate $r$ depending on $S$. Let $f_i : \Delta^n \to \Delta^n$ be the projection

$$f_i : (t_0, t_1, \ldots, t_n) \mapsto (t_0, \ldots, t_i + t_n, \ldots, t_{n-1}, 0). \tag{4.28}$$

The straight line homotopy between Id and $f_i$ is a deformation retraction from the spaces (4.27) onto the facet $t_n = 0$. This shows that they are both contractible, and in particular that they are homotopy equivalent. $\square$

Here are some more examples of directed cofibrations (which include the generating one):

**Example 4.16** Let $Z$ be a space and $a \leq b : Z \to \mathbb{R}_{\geq 0}$ be two upper semi-continuous functions. Let

$$A := \{(z,t) \in Z \times \mathbb{R}_{\geq 0} \,|\, t \leq a(z)\} \qquad \text{and} \qquad B := \{(z,t) \in Z \times \mathbb{R}_{\geq 0} \,|\, t \leq b(z)\}. \tag{4.29}$$

Let $s : B \to \mathbb{J}$ be a stratification such that $s|_{\{z\} \times [0,b(z)]}$ is increasing for all $z \in Z$.

Then the inclusion $A \hookrightarrow B$ is a directed cofibration.

*Proof.* The fibers of the projections $A_{\leq j} \to (Z \times \{0\})_{\leq j}$ and $B_{\leq j} \to (Z \times \{0\})_{\leq j}$ are intervals, so we have homotopy equivalences $A_{\leq j} \simeq (Z \times \{0\})_{\leq j} \simeq B_{\leq j}$. The inclusion $A \hookrightarrow B$ satisfies the first condition of Theorem 4.15, and is therefore a directed cofibration. $\square$

The following corollary of Theorem 4.15 will be used frequently in future proofs:

**Lemma 4.17** *Let $p : E \to X$ be a stratified fibration and consider the lifting problem*

$$\begin{tikzcd} S^k \arrow[r, "\alpha"] \arrow[d, hook] & E \arrow[d] \\ D^{k+1} \arrow[r, "\beta"] \arrow[ur, dotted, "\ell"] & X. \end{tikzcd} \tag{4.30}$$

*Suppose that we have a solution $\tilde{\ell}$ to a similar lifting problem*

$$\begin{tikzcd} S^k \arrow[r, "\alpha"] \arrow[d, hook] & E \arrow[d] \\ D^{k+1} \arrow[r, "\tilde{\beta}"] \arrow[ur, dotted, "\tilde{\ell}"] & X, \end{tikzcd} \tag{4.31}$$

*and that $\tilde{\beta}$ factors as $\tilde{\beta} = \beta \circ b$. Suppose moreover that $b|_{S^k}$ is the identity and that the fibers of $b : D^{k+1} \to D^{k+1}$ are all contractible. Then our original lifting problem (4.30) admits a solution.*

*Proof.* Let $C$ be the mapping cylinder of $b$ and $\iota : D^{k+1} \to C$ be the inclusion of the codomain. By theorem 4.15, the inclusion $D^k \cup (\partial D^k \times [0,1]) \hookrightarrow C$ is a directed cofibration. It follows that

$$\begin{tikzcd} & D^k \cup \big(\partial D^k \times [0,1]\big) \arrow[r] \arrow[d, hook] & D^k \arrow[r, "\tilde{\ell}"] & E \arrow[d] \\ D^{k+1} \arrow[r, "\iota"] & C \arrow[r] \arrow[urr, dotted, "h"] & D^{k+1} \arrow[r, "\beta"] & X. \end{tikzcd} \tag{4.32}$$

admits a solution $h : C \to E$. Now letting $\ell := h \circ \iota$ produces the solution to (4.30). $\square$

## 4.4 Stratified fibrations

The statements dual to Lemma 4.10 hold by the same formal arguments.

**Lemma 4.18** 1. *The pullback of a stratified fibration along a stratified map is a stratified fibration.*

2. *The composite of stratified fibrations is a stratified fibration.*

*3. A retract of a stratified fibration is a stratified fibration.*

The property of being a stratified fibration is a local property with respect to closed covers.

**Lemma 4.19** *Let $p : E \to X$ be a map and $\{V_\alpha\}$ a closed cover of $X$. If all the restrictions $p|_{V_\alpha} : E|_{V_\alpha} \to V_\alpha$ are stratified fibrations, then so is $p$.*

*Proof.* Let $a, b : Z \to \mathbb{R}_{\geq 0}$ and $\iota : A \hookrightarrow B$ be as in (4.29). The generating acyclic cofibrations are of that form, it's enough to show that

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook, "\iota"'] & E \arrow[d] \\ B \arrow[r, "f"] \arrow[ur, dotted] & X \end{tikzcd} \tag{4.33}$$

has a lift. Triangulate $B$ so that each simplex in that triangulation maps to a given $V_\alpha$ and so that the projection $\mathrm{pr} : B \to Z$ is simplicial.

We can build $B$ from $A$ by successively adding each simplex of that triangulation. More precisely, we can write $\iota$ as a sequence

$$A = A_0 \hookrightarrow A_1 \hookrightarrow \ldots \hookrightarrow A_r = B \tag{4.34}$$

where each $A_i$ is of the form $\{(z,t) \in Z \times \mathbb{R}_{\geq 0} \,|\, t \leq a_i(z)\}$ for some appropriate functions $a_i : Z \to \mathbb{R}_{\geq 0}$. Each layer $A_{i+1} \setminus A_i = \{(z,t) \,|\, a_i(z) < t \leq a_{i+1}(z)\}$ maps to some $V_\alpha$, and so does it closure $\overline{A_{i+1} \setminus A_i}$. The inclusion $\overline{A_{i+1} \setminus A_i} \cap A_i \hookrightarrow \overline{A_{i+1} \setminus A_i}$ is of the form (4.29), and is therefore a directed cofibration.

We now build lifts $A_i \to E$ inductively on $i$. Suppose that we have $A_i \to E$. In order to extend it to $A_{i+1}$, we find some $V_\alpha$ containing $f(\overline{A_{i+1} \setminus A_i})$ and write down the following commutative diagram

$$\begin{tikzcd} \overline{A_{i+1} \setminus A_i} \cap A_i \arrow[r] \arrow[d, hook] \arrow[drr] & E|_{V_\alpha} \arrow[d] \arrow[drr] & & \\ \overline{A_{i+1} \setminus A_i} \arrow[r] \arrow[ur, dotted] \arrow[drr] & V_\alpha \arrow[drr] & A_i \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\ & & A_{i+1} \arrow[r] \arrow[ur, dotted] & X. \end{tikzcd}$$

The lift $\overline{A_{i+1} \setminus A_i} \to E|_{V_\alpha}$ exists since $p|_{V_\alpha}$ is a stratified fibration. Assembling it with the existing map $A_i \to E$ produces the desired lift $A_{i+1} \to E$. □

Ramified covers provide the first interesting examples of stratified fibrations.

**Lemma 4.20** *A map of stratified spaces $E \to X$, which is a covering when restricted to each stratum $X_j$, is a stratified fibration.*

*Proof.* Let $\Lambda^{n,i} \hookrightarrow \Delta^n$ be a generating directed cofibration and consider the lifting problems

$$\begin{tikzcd} \Lambda^{n,i} \arrow[r, "f"] \arrow[d, hook] & E \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & X. \end{tikzcd} \tag{4.35}$$

Without loss of generality, we can assume that $\mathbb{J} = \{0, 1, \ldots, n\}$.

Since we're not in the case $i = n = 1$, both $(\Lambda^{n,i})_0$ and $(\Delta^n)_0$ are contractible. The map $E_0 \to X_0$ is a cover, so we have a (unique) lift

$$\begin{tikzcd} (\Lambda^{n,i})_0 \arrow[r, "f"] \arrow[d, hook] & E_0 \arrow[d] \\ (\Delta^n)_0 \arrow[r] \arrow[ur, dotted, "\ell"] & X_0. \end{tikzcd} \tag{4.36}$$

Taking the closure of the graph of $\ell$ produces a map $\bar{f} : \overline{(\Delta^n)_0} = \Delta^n \to E$.

We now show that $\bar{f}$ agrees with $f$ on $\Lambda^{n,i}$. Clearly, $\bar{f} = f$ on $\overline{(\Lambda^{n,i})_0}$. If $i = 0$, that's all of $\Lambda^{n,i}$. Otherwise $\overline{(\Lambda^{n,i})_0} = (\Lambda^{n,i})_0 \cup d^0(\Lambda^{n-1,i-1})$, where $d^0 : \Delta^{n-1} \to \Delta^n$ is the 0th coface map. Since we're not in the case $i = n = 2$, both $(d^0(\Lambda^{n-1,i-1}))_1$ and $(d^0(\Delta^{n-1}))_1$ are contractible. The map $E_1 \to X_1$ is a cover, so the diagram

$$\begin{tikzcd} \big(d^0(\Lambda^{n-1,i-1})\big)_1 \arrow[r, "f=\bar{f}"] \arrow[d, hook] & E_1 \arrow[d] \\ \big(d^0(\Delta^n)\big)_1 \arrow[r] \arrow[ur, dotted, "\exists!"] & X_1. \end{tikzcd} \tag{4.37}$$

has a unique lift. Both $f$ and $\bar{f}$ are solutions of (4.37), so they agree on $(d^0(\Delta^n))_1$. This shows that $f = \bar{f}$ on $\overline{(\Lambda^{n,i})_0} \cup (d^0(\Delta^n))_1$. Since $\overline{(\Lambda^{n,i})_0} \cup (d^0(\Delta^n))_1$ is dense in $\Lambda^{n,i}$, $\bar{f}$ agrees with $f$ on $\Lambda^{n,i}$. $\square$

**Theorem 4.21 (Whitehead's Theorem)** *Let*

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd} \tag{4.38}$$

*be a commuting diagram where $p_1$ is a stratified fibration. Assume that $f$ induces homotopy equivalences on all fibers. Then $f$ has a right homotopy inverse $g$ relatively to $X$. Namely $p_1 \circ g = p_2$ and $f \circ g$ is homotopic to $\mathrm{Id}_{E_2}$ by a homotopy $h : E_2 \times [0, 1] \to E_2$ satisfying $p_2 \circ g = p_2 \circ \mathrm{pr}_1$.*

*If $p_2$ is also a stratified fibration, then $E_1$ and $E_2$ are homotopy equivalent as spaces over $X$.*

*Proof.* Let $\mathcal{T}$ be an oriented triangulation of $E_2$ compatible with the stratification. We build the $g : E_2 \to E_1$ by induction on the skeleta $E_2^{(k)}$ of the above triangulation.

Assume that $g$ has been defined on $E_2^{(k-1)}$, along with the corresponding homotopy $h : f \circ g \Rightarrow \mathrm{Id}$. To extend it to $E_2^{(k)}$, we do it on each $k$-simplex individually. Let $i_0, i_1 : \Delta^k \to \Delta^k \times [0,1]$ be the two inclusions, and $r$ the retraction of $\Delta^k$ onto its last vertex $e_k$. Let $\sigma : \Delta^k \to E_2$ be a simplex, and let $F_1$, $F_2$ be the fibers of $p_1$, $p_2$ over the point $p_2(\sigma(e_k)) \in X$. We want to define $g \circ \sigma : \Delta^k \to E_1$ in a way compatible with the existing map $g \circ \sigma|_{\partial\Delta^k}$.

We pick a lift $m$ of

$$\begin{tikzcd} \partial\Delta^k \arrow[r, "\sigma"] \arrow[d, hook, "i_0"'] & E_2^{(k-1)} \arrow[r, "g"] & E_1 \arrow[d, "p_1"] \\ \partial\Delta^k \times [0,1] \arrow[r, "r"] \arrow[urr, dotted, "m"] & \Delta^k \arrow[r, "p_2\circ\sigma"] & X. \end{tikzcd} \tag{4.39}$$

The element $\alpha := f \circ m \circ i_1 : \partial\Delta^k \to E_2$ represents an element of $\pi_{k-1}(F_2)$. The three maps $\sigma$, $h \circ \sigma|_{\partial\Delta^k}$ and $f \circ m$ assemble to a disk $\eta : D^k \to E_2$ bounding $\alpha$. This shows that $\alpha$ is nullhomotopic in $E_2$.

The class $\alpha$ is also nullhomotopic in $F_2$. Let $j_0, j_1 : D^k \to D^k \times [0,1]$ be the inclusions, $r' : D^k \times [0,1] \to D^k$ be a retraction of $D^k$ onto its basepoint, and $n$ be a lift of

$$\begin{tikzcd} (D^k \times \{0\}) \cup (S^{k-1} \times [0,1]) \arrow[r, "\mathrm{pr}_1"] \arrow[d, hook, "j_0"] & D^k \arrow[r, "\eta"] & E_2 \arrow[d, "p_2"] \\ D^k \times [0,1] \arrow[r, "r'"] \arrow[urr, dotted, "n"] & D^k \arrow[r, "p_2\circ\eta"] & X. \end{tikzcd} \tag{4.40}$$

Then $n \circ j_1$ produces such a nullhomotopy.

Since $\alpha = f \circ m \circ i_1$ is zero in $\pi_{k-1}(F_2)$, we know by assumption that $m \circ i_1$ is zero in $\pi_{k-1}(F_1)$. Let $\beta : D^k \to\subset E_1$ be a disk bounding it, and make sure that $f \circ \beta$ is homotopic in $F_2$ to the map $n \circ j_1 : D^k \to E_2$ provided by (4.40).

The maps $m$ and $\beta$ assemble to a disk $\tau : D^k \to E_1$ bounding $g \circ \sigma|_{\partial\Delta^k}$. This disk can not be used to define $g \circ \sigma$ because its composite with $p_1$ does not agree with $p_2 \circ \sigma$. However, we are in a situation where we can apply Lemma 4.17. The map $\tau$ plays the role of $\ell$ in (4.30), and the map $b : D^k \to \sigma$ is assembled from $r$ and from the constant map at $e_k$. So we get our map $g \circ \sigma : \Delta^k \to E_1$ making the diagram (4.38) commute. This extends $g$ to each $k$-simplex $\sigma$, and thus to the hole of $E_2^{(k)}$.

We now extend the homotopy $h$. Again, we extend it to each $k$-simplex $\sigma$ individually. Consider the map $n : D^k \times [0,1] \to E_2$ of (4.40), the homotopy between $n \circ j_1 : D^k \to E_2$ and $f \circ \beta$, and the composite of $f$ with the map $C \to E_1$ constructed in the proof of Lemma 4.17. These three maps assemble to a homotopy $D^k \times [0,1] \to E_2$ between $\sigma$ and $g \circ f \circ \sigma$. This homotopy satisfies the assumptions of Lemma 4.17, so we get a new homotopy $h \circ (\sigma \times [0,1]) : D^k \times [0,1] \to E_2$, compatible with the projection to $X$. We have extended $h$ to each $k$-simplex $\sigma$, thus finishing the inductive construction of $g$ and $h$.

Now assume that $p_2$ is also a stratified fibration. By applying the first part of the theorem to the map $g$, we learn that it also has a right homotopy inverse. The map $f$ had a right homotopy inverse, which itself has a right homotopy inverse. So, by a well known argument, we deduce that $f$ has a two sided homotopy inverse. In other words, $f$ is a homotopy equivalence. □

Here's an alternate proof of Theorem 4.21 which uses Theorem 4.29 (and a bit of non-abelian cohomology).

*Proof.* Let $C$ be the mapping cone of $f$ and $r$ its obvious map to $X$. The obstructions to the existence of a lift

$$\begin{tikzcd} E_1 \arrow[r, equal] \arrow[d, hook] & E_1 \arrow[d, "p"] \\ C \arrow[r, "r"] \arrow[ur, dotted, "\ell"] & X \end{tikzcd} \tag{4.41}$$

lie in the relative cohomology group $H^*(C, E_1; \mathcal{F})$, where $\mathcal{F}$ denotes the cosheaf $\pi_*(\textit{Fiber of}\, p)$. This relative cohomology group can be computed by the Leray-Serre spectral sequence

$$H^*\big(X; H^*(\textit{Fiber of}\, r,\ \textit{Fiber of}\, p; \mathcal{F})\big) \Rightarrow H^*(C, E_1; \mathcal{F}). \tag{4.42}$$

Since $\mathcal{F}$ is constant on each fiber $r^{-1}(x)$, and since the inclusions $p^{-1}(x) \to r^{-1}(x)$ are homotopy equivalences, we have $H^*(r^{-1}(x), p^{-1}(x); \mathcal{F}) = 0$ for all $x \in X$. The spectral sequence (4.42) is identically zero, and so are the obstruction groups $H^*(C, E_1; \mathcal{F})$. Composing the solution $\ell$ of (4.41) with the inclusion $E_2 \to C$ produces a right homotopy inverse to $f$. □

## 4.5 The fundamental category

If $E \to X$ is a Serre fibration, then any path in the base $X$ induces a homotopy class of homotopy equivalences between the fibers over the endpoints. More precisely, we get a functor from the fundamental groupoid of $X$ with values in the homotopy category of spaces. Something similar happens for stratified fibrations.

Given a stratified space $(X, s)$, let us say that a path $\gamma : [0,1] \to X$ is directed if $s \circ \gamma : [0,1] \to \mathbb{J}$ is (weakly) increasing. The composition of directed paths is directed, so we get:

**Definition 4.22** *The fundamental category $\Pi_1(X)$ of a stratified space $X$ has an object for each point $x \in X$ and a morphism $x \to y$ for each directed path $\gamma$ from $x$ to $y$. Two paths from $x$ to $y$ are identified in $\Pi_1(X)$ if they are homotopic through directed paths. The composition of morphisms is given by concatenation of paths.*

Two objects are isomorphic in $\Pi_1(X)$ if and only if they are in the same connected component of a the same stratum $X_i$.

**Lemma 4.23** *Let $p : E \to X$ be a stratified fibration. Then the assignment $x \mapsto p^{-1}(x)$ extends to a functor $\nabla$ from $\Pi_1(X)$ to the homotopy category of spaces.*

*Proof.* Let $x$, $y$ be points in $X$ and let $\gamma$ be a path from $x$ to $y$. Let $F_x$ and $F_y$ denote the fibers over $x$ and $y$ respectively. To build $\nabla_\gamma : F_x \to F_y$, we consider the lifting problem

$$\begin{tikzcd} F_x \times \{0\} \arrow[rr] \arrow[d, hook] & & E \arrow[d] \\ F_x \times [0,1] \arrow[r, "\mathrm{pr}_2"] \arrow[urr, dotted, "\ell"] & {[0,1]} \arrow[r, "\gamma"] & X, \end{tikzcd} \tag{4.43}$$

where the top map is the inclusion. We define $\nabla_\gamma$, to be the composite $F_x \times \{1\} \hookrightarrow F_x \times [0,1] \to E$.

We now show that $\nabla_\gamma$ only depends on the homotopy class of $\gamma$. Let $\gamma' : [0,1] \to X$ be another path, and let $h : [0,1]^2 \to X$ be a homotopy between $\gamma$ and $\gamma'$. Precomposing the solution to

$$\begin{tikzcd} F_x \times (\{0\} \times [0,1] \cup [0,1] \times \{0,1\}) \arrow[rr, "\ell \cup \ell'"] \arrow[d, hook] & & E \arrow[d] \\ F_x \times [0,1]^2 \arrow[r, "\mathrm{pr}_2"] \arrow[urr, dotted] & {[0,1]^2} \arrow[r, "h"] & X \end{tikzcd} \tag{4.44}$$

with the inclusion of $F_x \times \{1\} \times [0,1]$ provides the desired homotopy between $\nabla_\gamma$ and $\nabla_{\gamma'}$. This also shows that the lift $\ell$ is well defined up to homotopy. $\square$

**Corollary 4.24** *Let $E \to X$ be a stratified fibration. Then the homotopy type of the fibers is constant on each connected component of each stratum of $X$.*

**Lemma 4.25** *Let $X$ be a stratified space and $\mathcal{T}$ a compatible oriented triangulation. Let $X^{op}$ be the opposite of $X$ defined with respect to $\mathcal{T}$ (see Definition 4.7). Then we have an equivalence of categories between $\Pi_1(X^{op})$ and $\Pi_1(X)^{op}$.*

*Proof.* Given a point $x \in X$, let $\sigma : \Delta^n \to X$ be the smallest non-degenerate simplex of $\mathcal{T}$ in the image of which $x$ lies. Let $e_0 \in \Delta^n$ be the 0th vertex. Since $\mathcal{T}$ is compatible with the stratification, $x$ is in the same stratum as $\sigma(e_0)$. This shows that for every point $x \in X$, there exists a vertex of $\mathcal{T}$ which is in the same connected component of the same stratum.

Two points in the same connected component of the same stratum are isomorphic as objects of $\Pi_1(X)$. Therefore the full subcategory $\Pi_1'(X)$ whose objects are the vertices of $\mathcal{T}$ is equivalent to $\Pi_1(X)$.

Any directed path between objects of $\Pi_1'(X)$ can be homotoped to a path that follows the edges of the triangulation. Similarly, any homotopy can be homotoped to one that remains in the 2-skeleton on $X$. So we get the following presentation of $\Pi_1'(X)$. We have a generator for each edge of $\mathcal{T}$. If an edge has a constant stratification, then we also have an inverse to the above generator. Each 2-simplex of $\mathcal{T}$ gives a relation.

By the same argument, $\Pi_1(X^{op})$ is equivalent to a category $\Pi_1'(X^{op})$ on the same set of objects as $\Pi_1'(X)$, and with the opposite presentation. We conclude that $\Pi_1'(X^{op}) = \Pi_1'(X)^{op}$ and therefore $\Pi_1(X^{op}) \simeq \Pi_1(X)^{op}$. $\square$

## 4.6 Obstructions to lifting

This section sets up an obstruction theory for lifting maps across stratified fibrations.

If $p : E \to X$ is a usual fibration, the obstructions to finding a lift

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dotted] & X \end{tikzcd}$$

live in the relative cohomology groups $H^*(B, A; \pi_*(F))$, where $F$ denotes the fiber of $p$. It might happen that $\pi_1(B)$ acts non-trivially on $\pi_*(F)$. In that case, the above groups should be understood as the cohomology of a locally constant sheaf on $B$.

If $E \to X$ is a stratified fibration, then $\pi_*(F)$ still makes sense, but it will fail in general to be locally constant. It will instead be a constructible cosheaf.

**Definition 4.26** *A constructible sheaf is a contravariant functor from the fundamental category $\Pi_1(X)$ to the category of abelian groups. A constructible cosheaf on a stratified space $X$ is a covariant functor from $\Pi_1(X)$ to abelian groups.*

By proposition 4.25, a constructible cosheaf $\mathcal{A}$ on $X$ is equivalent to a constructible sheaf $\mathcal{A}^{op}$ on $X^{op}$. Dually, a constructible sheaf on $X$ is equivalent to a constructible cosheaf on $X^{op}$.

**Example 4.27** Let $p : E \to X$ be a stratified fibration. Suppose that all the fibers $p^{-1}(x)$ are connected and have trivial $\pi_1$-action on $\pi_k$. Then composing the functor $\nabla$ given by Lemma 4.23 with the $k$-th homotopy group functor produces a constructible cosheaf. We denote it by $\pi_k(F)$, the letter $F$ standing for "a fiber".

There is a well known notion of cohomology of a space with coefficients in a sheaf $\mathcal{F}$. Similarly, we can take homology with coefficients in a cosheaf $\mathcal{A}$. In our setting, the easiest ways to define them is to introduce the simplicial chain and cochain complexes $C_*(X; \mathcal{A})$ and $C^*(X; \mathcal{F})$ given by

$$C_k = \bigoplus_{\substack{k\text{-simplices} \\ \sigma}} \mathcal{A}(\sigma) \qquad \text{and} \qquad C^k = \prod_{\substack{k\text{-simplices} \\ \sigma}} \mathcal{F}(\sigma). \tag{4.45}$$

Here $\mathcal{A}(\sigma)$ and $\mathcal{F}(\sigma)$ denote the values of $\mathcal{A}$ and $\mathcal{F}$ at the 0th vertex of $\sigma$ (or equivalently, at any point in the interior of $\sigma$). The differential in $C_*$ and $C^*$ is the usual alternating sum of (co)face maps.

We define the cohomology of a constructible cosheaf by using the opposite constructible sheaf.

**Definition 4.28** *Given a constructible cosheaf $\mathcal{A}$ on a space $X$, we define its cohomology to be*

$$H^*(X; \mathcal{A}) := H^*(X; \mathcal{A}^{op}).$$

*Similarly, for a pair $(X, Y)$, we let $H^*(X, Y; \mathcal{A}) := H^*(X, Y; \mathcal{A}^{op})$.*

Given a simplex $\sigma$ of $X$ with corresponding simplex $\sigma^{op}$ of $X^{op}$, the first vertex of $\sigma$ is the last vertex of $\sigma^{op}$, and vice versa. So the complex $C^*(X;\mathcal{A})$ computing cosheaf cohomology now looks like

$$C^*(X;\mathcal{A}) = \prod_{\substack{k\text{-simplices}\\ \sigma}} \mathcal{A}(x_\sigma), \tag{4.46}$$

where $x_\sigma$ denotes the last vertex of $\sigma$. Given a cochain $c \in C^n(X;\mathcal{A})$, with components $c(\sigma) \in \mathcal{A}(x_\sigma)$, its differential is then given by

$$\delta c(\sigma) = \sum_{i=0}^{n-1} (-1)^i c(d_i\sigma) + (-1)^n \mathcal{A}(\gamma) c(d_n\sigma), \tag{4.47}$$

where $d_i$ are the usual face maps, $\gamma$ is the edge from the $(n-1)$st to the $n$th vertex of $\sigma$, and $\mathcal{A}(\gamma)$ is the map from the value of $\mathcal{A}$ at the $(n-1)$st vertex to the value of $\mathcal{A}$ at the $n$th vertex. The relative cochain complex $C^*(X,Y;\mathcal{A}) \subset C^*(X,\mathcal{A})$ is the subcomplex of cochains vanishing on the simplices of $Y$.

We can now state the main theorem of this section:

**Theorem 4.29** *Let $E \to X$ be a stratified fibration, whose fibers are connected and have trivial action of $\pi_1$ on $\pi_*$. Let $A \hookrightarrow B$ be a cofibration of stratified spaces and let $A \to E$ and $B \to X$ be stratified maps making the following diagram commute:*

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & E \arrow[d] \\ B \arrow[r] \arrow[ur, dotted] & X. \end{tikzcd} \tag{4.48}$$

*Then the obstructions to finding a lift $\ell : B \to E$ making both triangles commute live in the cosheaf cohomology groups $H^{k+1}(B, A; \pi_k(F))$.*

*Proof.* We build the lift $B \to E$ by induction on the skeleta of $B$.

Suppose that we have a lift $\ell : A \cup B^{(k)} \to E$. Such a lift defines a cocycle $c = c_\ell \in C^{k+1}(B, A; \pi_k(F))$ as follows. Let $i_0, i_1 : \Delta^{k+1} \to \Delta^{k+1} \times [0,1]$ be the two inclusions, and $r$ the retraction to the last vertex of $\Delta^{k+1}$. Then given a simplex $\sigma : \Delta^{k+1} \to B$, we pick a lift $m$ of

$$\begin{tikzcd} \partial\Delta^{k+1} \arrow[r, "\sigma"] \arrow[d, hook, "i_0"'] & X^{(k)} \arrow[r, "\ell"] & E \arrow[d] \\ \partial\Delta^{k+1} \times [0,1] \arrow[r, "r"] \arrow[urr, dotted, "m"] & \Delta^{k+1} \arrow[r, "\sigma"] & X \end{tikzcd} \tag{4.49}$$

and let $c(\sigma)$ be the map $m \circ i_1 : \partial\Delta^{k+1} \to \Delta^{k+1} \times [0,1] \to F \subset E$, where $F = p^{-1}(x_\sigma)$ is the fiber over the last vertex of $\sigma$.

The values of $c$ are well defined elements of $\pi_k(F)$. Suppose that we have two solutions $m$ and $m'$ of (4.49), giving two elements $c(\sigma), c(\sigma)' \in \pi_k(F)$. We can then find a lift $n$ of

$$\begin{tikzcd}
\partial\Delta^{k+1} \times \big(\{0\} \times [0,1] \cup [0,1] \times \{0,1\}\big) \arrow[rrr, "(\ell\times[0,1])\cup m\cup m'"] \arrow[d, hook] & & & E \arrow[d, two heads] \\
\partial\Delta^{k+1} \times [0,1]^2 \arrow[rrru, dotted, "n"] \arrow[r, "1\times \mathrm{pr}_1"] & \partial\Delta^{k+1} \times [0,1] \arrow[rr, "\sigma\circ r"] & & X.
\end{tikzcd} \tag{4.50}$$

The map $n \circ (i_1 \times 1) : \partial\Delta^{k+1} \times [0,1] \to \partial\Delta^{k+1} \times [0,1]^2 \to F \subset E$ then provides a homotopy between $c(\sigma)$ and $c(\sigma)'$, thus proving their equality in $\pi_k(F)$.

The lift $\ell$ extends to $B^{(k+1)}$ if and only if $c = 0$. Indeed, if $\ell$ extends, we can use that extension to produce a solution of (4.49). The resulting map $c(\sigma) : \partial\Delta^{k+1} \to F$ will then be constant, hence trivial in $\pi_k(F)$. Conversely, suppose that $c = 0$ in $\pi_k(F)$. The homotopy $m$ defining $c$ and the disk bounding $c$ assemble to a disk $D^{k+1}$ bounding $\ell \circ \sigma|_{\partial\Delta^{k+1}}$. That disk satisfies the hypothesis of Lemma 4.17 so we get our desired lift.

We now show that $c$ is closed and therefore defines an element in $H^{k+1}(B, A; \pi_k(F))$. Let $i_0, i_1 : \Delta^{k+2} \to \Delta^{k+2} \times [0,1]$ be the two inclusions, and $r$ the retraction to the last vertex of $\Delta^{k+2}$. Given a simplex $\tau : \Delta^{k+2} \to B$, we let $s$ be a lift of

$$\begin{tikzcd}
sk_k(\Delta^{k+2}) \arrow[r, "\tau"] \arrow[d, hook, "i_0"'] & X^{(k)} \arrow[r, "\ell"] & E \arrow[d, two heads] \\
sk_k(\Delta^{k+2}) \times [0,1] \arrow[rru, dotted, "s"] \arrow[r, "r"] & \Delta^{k+2} \arrow[r, "\tau"] & X,
\end{tikzcd} \tag{4.51}$$

and consider the map $f := s \circ i_1 : \mathrm{sk}_k(\Delta^{k+2}) \to \mathrm{sk}_k(\Delta^{k+2}) \times [0,1] \to F \subset E$. Precomposing $f$ with the coface maps $d^i : \partial\Delta^{k+1} \to \mathrm{sk}_k(\Delta^{k+2})$ produces the $k+3$ terms in (4.47). For $i < k+2$, the map $f \circ d^i$ agrees with the definition of $c(d_i\tau)$. To see that the remaining map $f \circ d^{k+2}$ represents $\mathcal{A}(\gamma)c(d_{k+2}\tau)$, we use a diagram similar to (4.50). The top map is instead assembled from the the restriction to $d^{k+2}(\partial\Delta k + 1)$ of the map $s \circ i_1$ given in (4.51), and from the composite of the homotopies used to define $c$ and $\mathcal{A}(\gamma)$. The $k+3$ summands in (4.47) assemble to a map $sk_k(\Delta^{k+2}) \to F$. Therefore $\delta c(\tau)$ is zero in $\pi_k(F)$.

Assuming now that $[c] = [c_\ell] = 0 \in H^{k+1}(B, A; \pi_k(F))$ we produce a new lift $\ell' : B^{(k)} \to E$ which agrees with $\ell$ on $B^{(k-1)}$. The new obstruction cocycle $c_{\ell'}$ then vanishes identically and $\ell'$ thus extends to the $(k+1)$-skeleton. To build $\ell'$, we write $c = \delta b$ for some $b \in C^k(B, A; \pi_k(F))$ and geometrically subtract $b$ from $\ell$. More concretely, given a simplex $\rho : \Delta^k \to B$, we assemble $\ell \circ \rho$ and $-b(\rho)$ to a map $D^k \to E$ bounding $\ell \circ \rho|_{\partial\Delta^k}$. We then apply Lemma 4.17 to produce the new lift $\ell'$.

At last, we verify that $c_{\ell'} = 0$. Let $\sigma : \Delta^{k+1}B$ be a simplex. It is enough by Lemma 4.17 to build an appropriate disk $D^{k+1}$ bounding $\ell' \circ \sigma|_{\partial\Delta^{k+1}}$. Such a disk can be obtained by assembling the homotopy $m$ of (4.49), the $k+2$ mapping cylinders used in the construction of $\ell'$ for each facet of $\sigma$ (see proof of Lemma 4.17), the homotopy used in the definition of $\mathcal{A}(\gamma)c_\ell(d_{k+1}\sigma)$ for the cosheaf $\mathcal{A} = \pi_k(F)$ (see Example 4.27 and Lemma 4.23), and the homotopy $S^k \times [0,1] \to F$ proving the equality $c_\ell(\sigma) = \delta b(\sigma)$ in $\pi_k(F)$. $\square$

A startling consequence of Theorem 4.29 is that the property of being a stratified fibration is in some sense independent of the stratification of the base.

**Proposition 4.30** *Let $(X, s)$ be a $\mathbb{J}'$-stratified space, $f : \mathbb{J}' \to \mathbb{J}$ a map of posets, and $s := f \circ s'$ the corresponding $\mathbb{J}$-stratification. Let $L_f : \Pi_1(X, s') \to \Pi_1(X, s)$ be the map on fundamental categories induced by the identity $X \to X$. Let $p : E \to X$ be a stratified fibration with respect to $s'$.*

*If the functor $\nabla_{s'} : \Pi_1(X, s') \to Ho(\mathsf{spaces})$ factors through $L_f$ then $E \to X$ is also a stratified fibration with respect to $s$.*

*Proof.* Let $\iota : A \hookrightarrow B$ be a cofibration, and consider our usual lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook, "\iota"'] & E \arrow[d] \\ B \arrow[r, "\beta"] \arrow[ur, dotted] & X. \end{tikzcd}$$

By Theorem 4.29, the obstructions to finding a lift $B \to E$ lie in $H^{k+1}(B, A; \pi_k(F))$. Suppose that $\iota$ is directed, with respect to $s \circ beta$, we want to show that the obstruction groups vanish. By the long exact sequence in cohomology, it's enough to show that the restriction maps $H^*(B; \pi_k(F)) \to H^*(A; \pi_k(F))$ are isomorphisms. Without loss of generality, we can assume that $B$ is compact, and therefore that $\mathbb{J} = \{0, 1, \ldots, k\}$.

Let $\mathcal{F}$ be the sheaf $(\pi_k(F))^{op}$ used to define $H^*(-; \pi_k(F))$. We show by induction on $j \in \mathbb{J}$, that

$$H^*(B^{op}_{\leq j}; \mathcal{F}) \to H^*(A^{op}_{\leq j}; \mathcal{F}) \tag{4.52}$$

are isomorphisms. So let's assume that (4.52) is an isomorphism. In order to show that the same thing holds for $j + 1$, it's enough by the five lemma to show that

$$\widetilde{H}^*(B^{op}_{\leq j+1}/B^{op}_{\leq j'}; \mathcal{F}) \to \widetilde{H}^*(A^{op}_{\leq j+1}/A^{op}_{\leq j}; \mathcal{F}) \tag{4.53}$$

is an isomorphism. Now $A^{op}_{\leq j+1}/A^{op}_{\leq j} \hookrightarrow B^{op}_{\leq j+1}/B^{op}_{\leq j}$ is a homotopy equivalence because $\iota$ was assumed to be a directed cofibration with respect to $s$. And the sheaf $\mathcal{F}$ is locally constant on $B^{op}_{\leq j+1} \setminus B^{op}_{\leq j}$ because of the factorization of $\nabla_{s'}$. It follows that (4.53) are isomorphisms. The maps (4.52) are also isomorphisms, the obstruction groups vanish, and so our lift exists. $\square$

## 4.7 Stratified simplicial sets

Unfortunately, applying simple categorical constructions to $\mathbb{J}$-stratified spaces quickly produces pathological spaces. For example, the terminal object is the set $\mathbb{J}$ equipped with the topology generated by the subsets $\{j \in \mathbb{J} \,|\, j \leq j_0\}$. In particular, we have not been able to find a convenient model category structure on $\mathbb{J}$-stratified spaces. Because of this problem, we introduce an analogue of stratifications in the world of

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