Construction of primary fields

The irreduible positive energy representations H_i of $LSU(2)$ are classified by their level ℓ and the lowest energy space $V_i = H_i(0)$s, an irreducible representation of the constant loops $SU(2)$ of spin i, a half integer with $0 \leq i \leq \ell/2$. If V is any irreducible representation of $G = SU(2)$, then $\mathcal{V} = C^\infty(S^1, V)$ has an action of $LG \rtimes \mathrm{Rot}\, S^1$ with LG acting by multiplication and $\mathrm{Rot}\, S^1$ by rotation, $r_\alpha f(\theta) = f(\theta + \alpha)$. There is a corresponding infinitesimal action of $L^0\mathbf{g} \rtimes \mathbb{R}$ which leaves invariant the finite energy subspace $\mathcal{V}^0$. We can write $\mathcal{V}^0 = \sum \mathcal{V}()$ where $\mathcal{V}(n) = z^- \otimes V$. Set $v_n = z^n v$ for $v \in V$. Thus $dv_n = -nv_n$ (so that $\frac{d=-id}{d\theta}$) and $X_n v_m = (Xv)_{m+n}$. Let H_i and H_j be irreducible positive eergy representations at level ℓ. A map $phi : \mathcal{V}^o timsH_j^0 \rightarrow H_i^0 0$ commuting with the action of $L^0\mathbf{g} \rtimes \mathrm{Rot}\, S^1$ is called a *primary field with charge* V. For $v \subset V$, we define $\phi(v, n) = \phi(v_n) : H_j^0 \rightarrow H_i^0$. These are called the *modes* of ϕ. The intertwining property of Φ is expressed in terms of the modes though the commutation relations:

$$[X(n), \phi(v, m)] = \phi(X \cdot v, m + n), \;\; [D, \phi(v, m)] = -m\phi(v, m).$$

Uniqueness theorem. *If ϕ is a primary field, then ϕ restricts to a G–invariant map ϕ_0 of $V \otimes H_j(0)$ into $H_i(0)$. Moreover ϕ is uniquely determined by ϕ_0, the initial term of ϕ.*

We shall only require primary fields of charge 1/2 and 1. Let L_m be the Virasoro operators given by the Segal–Sugawara construction as quadratic expressions in the $X(n)$'s. In particular $L_0 = D + h_j$ on H_j where $h_j = (j^2 + j)/(\ell + 2)$. For $v \in V_k$ define

$$\Phi(v, z) = \sum \phi(v, n) z^{-n-\delta},$$

where $\delta = h_i + h_k - h_j$ is the *conformsl anomoly* of the field. Then

$$[L_0, \Phi(v, z)] = \{z\frac{d}{dz} + h_j\}\Phi(v, z).$$

Using the Segal–Sugawara formula for L_0 and L_m together with the covariance relation

$$[X(n)\Phi(v, z)] = z^n \Phi(X \cdot v, z),$$

the following covariance relation can be deduced (cf Tsuchiya–Kanie, page 320):

$$[L_n, \Phi(v, z)] = z^n\{z\frac{d}{dz} + (n + 1)h_k\}\Phi(v, z).$$

In my Inventiones paper, I establish that the spin 1/2 primary fields are compressions of complex fermions, so in particular satisfy an L^2 bound $\|\phi(v, f)\| \leq K\|f\|_2$. The same is true for spin 1 primary fields which occur as soon as $\ell \geq 2$. This will emerge below from the coset construction, but we give a direct proof here. For level 2, the result follows because the irreducible positive energy representations have spin 0, 1/2 and 1. The real Neveu–Schwartz fermions with values in $V_1 = \mathbf{g}$ give a primary field of spin 1 on the corresponding Fock space $\mathcal{F}_{\mathrm{NS}}$, which splits as $H_0 \oplus H_1$ as representation of $LSU(2)$. The real Ramond fermions with values in V give a primary field of spin 1 on the corresponding Fock space $\mathcal{F}_{\mathrm{R}}$, which is isomorphic to $H_{1/2}$. Thus at level 2 the spin 1 primary fields are given by real fermions, so in particular automatically satisfy an L^2 bound. To prove the same result for $\ell \geq 3$, there are three types of primary field.

(1) $\Phi(v, z) : H_i \rightarrow H_{i-1}$ for $1 \leq i \leq \ell/2$. In this case the compression of $\Phi(v, z) \otimes I : H_1 \otimes H_{i-1} \rightarrow H_0 \otimes H_{i-1}$ to a map from H_i to H_{i-1} gives the primary field. Here and below the first factors are at level 2 and the second at level ℓ.

(2) $\Phi(v, z) : H_i \rightarrow H_{i+1}$ for $0 \leq i \leq \ell/2 - 1$. In this case the compression of $\Phi(v, z) \otimes I : H_0 \otimes H_i \rightarrow H_1 \otimes H_i$ to a map from H_i to H_{i+1} gives the primary field.

(3) $\Phi(v, z) : H_i \rightarrow H_i$ for $1/2 \leq i \leq \ell/2 - 1/2$. In this case the compression of $\Phi(v, z) \otimes I : H_0 \otimes H_i \rightarrow H_1 \otimes H_i$ to a map from H_i to H_i gives the primary field. (In fact, identifying V_1 with $\mathbf{g}$, the initial term is just $X \otimes v_i \mapsto Xv_i$.)

In Tsuchiya–Kanie and in my Inventiones paper, the braiding properties of spin 1/2 primary fields was established using the the reduced 4–point function, a function of a complex variable with values in an auxiliary finite–dimensional space. We now recall this theory and its relation to products $\Phi(u,z)\Phi(v,w)$.

At level L (either ℓ or $\ell+2$), given spin 1/2 primary fields $\Phi_{ji}(v,w) : H_i \to H_j$, $\Phi_{kj}(u,z) : H_j \to H_k$, we may form the 4–point function $F_j(z,w) = (\Phi(u,z)\Phi(v,w)\xi,\eta)$ for lowest energy vectors ξ and η. Then

$$F_j(z,w) = \sum_{m\geq 0} (\Phi(u,m)\Phi(v,-m)\xi,\eta) z^{-m} z^{-\delta} w^m w^{-\delta'} = f_j(\zeta) z^{-\delta} w^{-\delta'},$$

where

$$f_j(\zeta) = \sum_{m\geq 0} = (\Phi(u,m)\Phi(v,-m)\xi,\eta)\zeta^m.$$

The functions $f_j(\zeta)$ are holomorphic functions for $|\zeta|<1$ with values in $W = \mathrm{Hom}_G(V_{1/2}\otimes V_{1/2}\otimes V_i, V_k)$ with $G = SU(2)$. The value at $\zeta = 0$ gives $(\Phi(u,0)\Phi(v,0)\xi,\eta)$, i.e. the map factorising through $V_{1/2}\otimes V_i \to V_j$, $V_{1/2}\otimes V_j \to V_k$. These functions and the related functions

$$\overline{f}_j(\zeta) = \zeta^{\lambda_j} f_j(\zeta),$$

where $\lambda_j = (j^2+j-i^2-i-3/4)/(L+2)$, are called reduced four point functions. The $\overline{f}_j(z)$ are defined on the unit disc with $[0,1)$ removed. They satisfy the Knizhnik–Zamolodchikov ordinary differential equation, which in this case is equivalent to Gauss' hypergeometric equation. This equation has the form

$$\overline{f}'(z) = A(z)\overline{f}(z),$$

where

$$A(z) = \frac{P}{z} + \frac{Q}{1-z},$$

for P,Q operators on W. As shown in section 19 of the Inventiones article, there is a holomorphic gauge transformation $g : \mathbb{C}\backslash[1,\infty)$ such that

$$g^{-1}Ag - g^{-1}g' = P/z.$$

The solutions of the ODE are then $\overline{f}(z) = g(z)z^P T$ where T are eigenvectors of P. It follows that the columns of $g(z)$ are just the solutions $f_i(z)$. On the other hand we can similarly choose $h(z)$ with $h(0) = I$ and

$$hAh^{-1} - h'h^{-1} = -P/z.$$

(This corresponds to replacing $A(z)$ by $-A(z)^t$.) Then

$$(hg)' = [P, hg]/z.$$

The only formal power series solution of this ODE equal to I at 0 is the constant solution I, so that $h(z) = g(z)^{-1}$. Taking transposes, the rows of $g(z)^{-1}$, i.e. columns of $(g(z)^{-1})^t$ are up to powers of z the fundamental solutions of $k'(z) = -A(z)^t k(z)$. The transport matrix for this ODE is just the inverse of that for the original one. The fundamental solutions at ∞ give a gauge transformation equal to I at ∞, transforming A to the singular in $A(z)$ at ∞. The computation in the Inventiones paper shows that all the transport coefficients are non–zero. Note that the analysis there applies equally well to the case of braiding between a spin 1/2 and spin 1 primary field.

In particular the functions $\overline{f}_j(z)$ extend analytically to single–valued holomorphic functions on $\mathbb{C}\backslash[0,\infty)$. Now the functions $\overline{f}_j(z^{-1})$ satisfy the same ordinary differential equation in the same domain, so there is a transport relation

$$\overline{f}_j(z) = \sum_m c_{jm}\overline{f}_m(z^{-1}).$$

The coefficients have been computed by Tsuchiya–Kanie and me. They are always non–zero and give the braiding relations. For primary fields smeared in disjoint intervals, this gives a relation

$$(\Phi_{kj}(u,f)\Phi_{ji}(v,g)\xi,\eta)=\sum_m c_{jm}(\Phi_{km}(v,e_\mu g)\Phi_{mi}(u,e_{-\mu}f)\xi,\eta),$$

where

$$e_\mu(\theta)=e^{i\theta(\lambda_j-\lambda_m)}.$$

This braiding relation extends immediately to any finite energy vectors ξ and η and hence arbitrary vectors, so that

$$\Phi_{kj}(u,f)\Phi_{ji}(v,g)\sum_m c_{jm}(\Phi_{km}(v,e_\mu g)\Phi_{mi}(u,e_{-\mu}f).$$

Similarly the relation implies that in the sense of analytic continuation

$$(\Phi_{kj}(u,z)\Phi_{ji}(v,w)\xi,\eta)=\sum_m c_{jm}(\Phi_{km}(v,w)\Phi_{mi}(u,z)\xi,\eta),$$

where both sides are defined for z, w amd z/w in $\mathbb{C}\backslash[0,\infty)$. The same method of proof for smeared primary fields shows that this relation holds for arbitrary finite energy vectors ξ and η and thus it can be written simply as

$$\Phi_{kj}(u,z)\Phi_{ji}(v,w)\sum_m c_{jm}\Phi_{km}(v,w)\Phi_{mi}(u,z),$$

in the sense of analytic continuation.

Construction of charge $(1/2,1/2)$ primary fields. Consider $\Phi^{1/2}_{ji}(v,z)\otimes I: H_i\otimes\mathcal{F}H_j\otimes\mathcal{F}$. By the coset construction

$$H_i\otimes\mathcal{F}=\bigoplus_a H_a\otimes H_{i,a},\ \ H_j\otimes\mathcal{F}=\bigoplus_b H_b\otimes H_{j,b}.$$

Let P_a and P_b be the projection onto the summands $H_a\otimes H_{i,a}$ and $H_j\otimes H_{j,b}$. Fix L_0–eigenvectors $\xi\in H_{i,a}$ and $\eta\in H_{j,b}$ and consider

$$\mathrm{id}\otimes\omega_{xi,\eta}(P_b\Phi(v,z)P_b):H_a\to H_b$$

where $\omega_{\xi,\eta}(T)=(T\xi,\eta)$. By the uniqueness theorems for primary fields of $SU(2)$, this is a rational power of z times a spin $1/2$ primary field $\phi^{1/2}_{ba}(v,z)$ for $LSU(2)$ at level $\ell+2$. The precise power of z can be determined from the following formula for the difference of the conformal anomolies of the spin $1/2$ primary fields $\Phi(v,z)$ and $\phi(v,z)$.

Lemma. $\Delta^{1/2}_{ji}(\ell)=\Delta^{1/2}_{ba}(\ell+2)+\Delta^{(1/2,1/2)}_{(j,b),(i,a)}+((j-b)^2-(i-a)^2)/2.$

Proof. This is a special case of the identity

$$\Delta^k_{ji}(\ell)=\Delta^c_{ba}(\ell+2)+\Delta^{(k,c}_{(j,b),(i,a)})+((j-b)^2-(i-a)^2-(k-c)^2)/2, \qquad (*)$$

which is easy to verify directly.

(Note that the version of formula $(*)$ by Palcoux has two sign errors.) It follows that

$$P_b\Phi(v,z)P_a=\phi(v,z)\otimes\theta_{(j,b),(i,a)}(z)z^m,$$

where $m=m_{i,j,a,b}$ is an integer and

$$\theta(z)=\sum\theta(n)z^{-n-\delta}$$

is a vertex operator between $H_{i,a}$ and $H_{j,b}$. It is easy to check that $\theta(z)$ satisfies the covariance relation for a charge half primary field for the coset Virasoro generators. Likewise, using the GKO formula for the G_r's, the compatibility relations with the G_r's are immediately verified, so that $\theta(z)$ is the ordinary part of a primary field of charge $(1/2,1/2)$ of the Neveu–Schwartz algebra, possibly zero. Moreover the value of σ for this primary field is as predicted.

Now

$$[X(n),\Phi(v,z)] = z^n\Phi(X\cdot v,z)],\ [X(n),\phi(v,z)] = z^n\phi(X\cdot v,z)]$$

and by construction

$$[L_0,\Phi(v,z)] = (z\frac{d}{dz}+\frac{3}{4(\ell+2)})\Phi(v,z),\ [L_0,\phi(v,z)] = (z\frac{d}{dz}+\frac{3}{4(\ell+4)})\phi(v,z).$$

Thus by uniqueness $\phi(v,z)$ is proportional to the primary field of spin $1/2$ between H_a and H_b (if one exists). But then

$$[L_n,\Phi(v,z)] = z^n(z\frac{d}{dz}+(n+1)\frac{3}{4(\ell+2)})\Phi(v,z),\ [L_n,\phi(v,z)] = z^n(z\frac{d}{dz}+(n+1)\frac{3}{4(\ell+4)})\phi(v,z).$$

It follows that

$$[L_n,\theta(z)] = z^n(z\frac{d}{dz}+(n+1)\frac{3}{2(\ell+4)(\ell+2)})\phi(v,z).$$

Since

$$h_{1/2,1/2} = \frac{(16-4)}{8(\ell+2)(\ell+4)} = \frac{3}{2(\ell+4)(\ell+2)}$$

and the compatibility relation

$$[G_{-1/2},\theta(z)] = z^{-r-1/2}[G_r,\theta(z)]$$

can be verified using the explicit formula for G_r in the GKO construction, $\theta(z)$ is the ordinary part of a charge $(1/2,1/2)$ primary field between $H_{i,a}$ and $H_{j,b}$. The value of σ follows from the lemma above.

We now use a simplification of Loke's method to deduce that all the primary fields are non–zero. His method was inspired by the analysis of Tsuchiya and Nakanishi which related the primary fields of $LSU(NM)$ at level 1 and their braiding coefficients with those of $LSU(N)$ at level M and $LSU(M)$ at level N. This simplified analysis applies equally to the coset construction of the Virasoro algebra, permitting monodromy properties of 4–point functions to be deduced indirectly via the coset construction without using an ordinary differential equation with regular singular points at 0, 1 and ∞.

We start by showing that when non–zero the 4–point functions

$$F(z,w) = (\theta_{(k,c),(j,b)}(z)\theta_{(j,b),(i,a)}(w)\xi,\eta)$$

have the expected properties. Thus if ξ,η are finite energy vectors, these should be a function of the form $z^\alpha w^\beta f(z/w)$ where $f(\zeta)$ is holomorphic function in $\mathbb{C}\backslash[0,\infty)$. . For the primary fields of $LSU(2)$ this is known from their work when one of the fields has spin $1/2$ and the other is arbitrary (we shall only need spin $1/2$ or 1), because it follows from well–known properties of Gauss' hypergeometric function. An important additional property is the non–vanishing of the hypergeometric functions $F(\alpha,\ beta,\gamma;z)$ on $\mathbb{C}\backslash[1,\infty)$ and their two limits on the cut $(1,\infty)$. For the particular values of α,β,γ that arise in the work of Tsuchiya and Kanie, classical results of Hurwitz and Van Vleck, simplified by Runckel in 1971 (Mathematische Annalen, 191, 53-58), guarantee these coefficients do not vanish. We will in fact not use this property, but instead a holomorphic gauge transformation explained in the Inventiones paper.

We shall derive in detail these properties for charge $(1/2,1/2)$ primary fields. The same techniques can be used to proved the existence, GKO construction of charge $(0,1)$ primary fields and their braiding with spin $(1/2,1/2)$ primary fields. Since the whole pattern of the proof and the results used are practically identical, we shall only give an outline of the details.

The method of Loke rests on showing that if (i,a), (j,b) occurs with $|i-j|=1/2$ and $|a-b|=1/2$, then $\phi^{1/2}_{ba}(v,z)$ necessarily occurs, i.e. $\theta_{(j,b),(i,a)}(z)\neq 0$. We may assume $j=i+1/2$ (because of the behavious of spin $1/2$ primary fields under taking adjoints).

If neither of $\phi^{1/2}_{b,a}(v,z)$ with $b=a\pm 1/2$ occurs, then $(\Phi(v,z)\otimes I)u=0$ for a finite energy vector u in $H_i\otimes\mathcal{F}$. The commutation relations of $\Phi(v,z)$ with operator $X(n)$ imply that $(\Phi(v,z)\otimes I)u'=0$ if u' is obtained by applying operators $X(n)\otimes I$ or $I\otimes\psi(x,n)$ to u. Since u is cyclic for these operators, it follows that $\Phi(v,z)\equiv 0$, a contradiction. So $\phi_{ba}^{1/2}(v,z)$ occurs where $b=a-1/2$ or $b=i-1/2$.

We write

$$\Phi_{ji}(v,z)=\sum\phi_{ba}(v,z)\otimes\theta_{(j,b),(i,a)}(z),$$

where the second factor is a multiple of a charge $(1/2,1/2)$ primary field, possibly zero. Now in general

$$(\Phi_{kj}(u,z)\Phi_{ji}(v,w)\xi_a\otimes\xi_{k,c},\eta_a\otimes\xi_{i,a})=\sum_b(\phi_{cb}(u,z)\phi_{ba}(v,w)\xi_a,\eta_c)(\theta_{(k,c),(j,b)}(z)\theta_{(j,b),(i,a)}(w)\xi_{i,a},\xi_{k,c}).$$

Each term has values in $W=\mathrm{Hom}_G(V_{1/2}\otimes V_{1/2}\otimes V_i,V_k)$. We can write this as a relation for reduced 4–point functions

$$F_j(\zeta)=\sum f_b(\zeta)h_{j,b}(\zeta).$$

Here *a priori* $h_{jb}(\zeta)$ is a formal power series convergent in $|\zeta|<1$ times a rational power of ζ. F_j and f_b are holomorphic functions from $\mathbb{C}\backslash[0,\infty)$ into W. If we apply $g(zeta)^{-1}$ we get

$$g(\zeta)^{-1}F_j(\zeta)=\sum\zeta^{-\mu_b}\phi_{cb}(0)\phi_{ba}(0)h_{j,b}(\zeta),$$

for constants μ_b. It follows immediately that the $h_{j,b}$'s are holomorphic on $\mathbb{C}\backslash[0,\infty)$. Thus all 4–point functions in the θ's, if they are arise through the coset construction, are holomorphic for $z,w,z/w$ in $\mathbb{C}\backslash[0,\infty)$. In fact we get a formula for $h_{j,b}(\zeta)$:

$$h_{j,b}(\zeta)=C\zeta^{\mu_b}(g(\zeta)^{-1}F_j(\zeta),\phi_{cb}(0)\phi_{ba}(0)),\qquad(**)$$

for some constant $C\neq 0$.

This formula immediately implies an analogue of the duality for braiding established by Tsuchiya and Nakanishi. The fields charge (1/2,1/2) $\theta_{(j,b),i,a}(z)$ have braiding coefficients given by the product of the level ℓ braiding coefficients for spin 1/2 and the inverse of the level $\ell+2$ braiding coefficients for spin 1/2. All these coefficients are non–zero, even if some of the terms $\theta_{(j,b),i,a}(z)$ vanish.

Resuming the proof of constructibility, we have to show that this can never happen. Suppose that $\theta_{(i+1/2,b),(i,a)}(z)=0$. Then

$$0=\theta_{(i,a),(i+1/2,b)}(z)\theta_{(i+1/2,b),(i,a)}(w)\sim\sum\mu_{j,c}\theta_{(i,a),(j,c)}(w)\theta_{(j,c)}(z),$$

where all the coefficients are non–zero. This is a contradiction, since the right hand side is not identically zero. It follows that if $|a-b|=1/2$, then the term $\phi_{ba}(z)$ appears and hence also $\theta_{(i+1/2,b),(i,a)}(z)$, i.e. all charge $(1/2,1/2)$ primary fields can be constructed by the coset construction.

A clearer way to preseent this is to note that if $\phi_{ba}(z)$ and $\theta_{((j,b),(i,a)}(z)$ appear, then by taking adjoints so does $\theta_{(i,a),(j,b)}(z)$. But then using the formula $(**)$ for the reduced 4–point function associated with $\theta_{(i,a),(j,b)}(z)\theta_{(j,b),(i,a)}\xi,\eta)$, we see that all possible terms appear on braiding. Consequently if $\phi_{ba}(z)$ occurs with $b=a\pm 1/2$ so does $a\mp 1/2$, as required.

For charge $(0,1)$ primary fields, we consider this time the compressions of the Neveu–Schwarz fermion field $I\otimes\psi(v,z)$. It commutes with the field $\Phi(u,z)\otimes I$. The same principles can be used to compute the braiding with charge $(1/2,1/2)$ primary fields and to give formulas for the reduced 4–point function

$$(\theta_{(k,c),(j,b)}(z)^{(1/2,1/2)}\theta^{(0,1)}_{(j,b),(i,a)}(w)\xi,\eta).$$

Again all the braiding coefficients are non–zero and, if some $\theta^{(0,1)}_{(j,b),(i,a)}(z)$ vanishes, as before the braiding relation for the above product gives a contradiction.

This completes the proof of the existence of charge $(1/2,1/2)$ and $(0,1)$ primary fields. The braiding coefficients betwen a charge $(1/2,1/2)$ primary field and one of these two types of field are always non–zero. The charge $(1/2,1/2)$ fields are compressions of spin 1/2 primary fields of level ℓ, and there satisfy L^2 bounds. The charge $(0,1)$ fields are compressions Neveu–Schwartz fermions and there satisfy L^2 bounds.

The braiding properties of smeared field follow immediately by the method of convolution and the transport properties of reduced 4–point functions.

Irreducibility for $LSU(2)$. We have already seen that in the vacuum sector the local algebra $\pi_0(L_IG)''$ is the weak operator linear span of chains of products of spin 1/2 primary fields concentrated in I. If I_1 and I_2 are the intervals obtained by removing an internal point from I, then each spin 1/2 primary field can be written as the sum of two spin 1/2 primary fields concentrated in I_1 and I_2. Thus the chain of primary fields is a linear combination of a product of primary fields concentrated in either I_1 or I_2. But fields concentrated in I_1 satisfy braiding relations with fields concentrated in I_2. Thus $\pi_0(L_IG)''$ is the weak operator linear span of products

$$a_{0i_1}a_{i_1i_2}\cdots a_{i_mi}b_{ij_1}b_{j_1j_2}\cdots b_{j_n0}, \qquad (*)$$

with a_{pq} concentrated in I_1 and b_{rs} in I_2.

We may assume without loss of generality that the internal point removed is 1 and that $-1\notin I$. Thus the modular group U_t leaves I invariant for $t>0$ and as $t\to+\infty$ contracts I to the point 1. Let $a=a_{i,i-1/2}$ be a field concentrated in I_1. Applying a rotation, the field

$$b=b_{i,i-1/2}=R_\theta a_{i,i-1/2}R_\theta^*$$

is concentrated in I_2. The following is a weak form of the operator product expansion of primary fields:

Theorem. *After a possible adjustment in θ, $U_tab^*U_t^*$ tends weakly to a positive multiple of the identity.*

Proof. We start by proving two simpler weak forms of operator product expansions.

Lemma 1 (case $i=0$ of theorem). *Let $a=a_{0,1/2}$ and $b=R_\theta a_{0,1/2}R_\theta^*$, so that a is concentrated in I_1 and b in I_2. Then by the Reeh–Schlieder argument, a small variation of θ will guarantee that $(ab^*\Omega,\Omega)\neq 0$, in which case $U_tab^*U_t^*$ tends weakly to a positive multiple of the identity, namely $(ab^*\Omega,\Omega)$.*
Proof. In fact let

$$F(e^{i\varphi})=(aR_\varphi b^*R_\varphi\Omega,\Omega)=(aR_\varphi b^*\Omega,\Omega).$$

By the positive energy condition the right hand side is the continuous boundary value of a holomorphic function on the open unit disc. On the other hand if F vanishes on a small arc on the unit circle near $\varphi=\theta$, it must vanish identically (by an application of the Schwarz reflection principle). But then it vanishes for $\varphi=0$, so that $(aa^*\Omega,\Omega)=0$. Hence $a^*\Omega=0$. Since a^* commutes with $L_{I^c}G$ and the vacuum vector is a cyclic vector for this subgroup, it follows that $a^*=0$, so that $a=0$, a contradiction.

The operators $U_tab^*U_t^*$ are uniformly bounded. Let (t_n) be a sequence with $t_n\to\infty$. Passing to a subsequence (s_n), we may assume that $U_{s_n}ab^*U_{s_n}^*$ has a weak limit T. If J_n and K_n are intervals increasing to the two halves J and J^c of $S^1\backslash\{\pm1\}$, then T evidently commutes with $L_{J_n}G$ and $L_{K_n}G$. Hence it commutes with $M=\pi_0(L_JG)''$ and $N=\pi_0(L_{J^c}G)''$. So

$$T\in M'\cap N'=M'\cap M=\mathbb{C},$$

by Haag duality and factoriality of M. Thus T is a scalar. On the other hand

$$(U_tab^*U_t\Omega,\Omega)=(ab^*\Omega,\Omega)\neq 0,$$

since U_t fixes the vacuum vector Ω. Hence $T=(ab^*\Omega,\Omega)$. Since this limit os independent of the subsequence (t_n), the lemma follows.

Lemma 2. *If $a=a_{1,1/2}$ is concentrated in I_1 and $b=b_{1/2,0}$ in I_2, then $U_ta_{1,1/2}b_{1/2,0}U_t^*$ tends weakly to zero.*

Proof. Again we take a sequence (t_n) with $t_n\to+\infty$ and pass to a subsequence s_n with $X_n=U_{s_n}abU_{s_n}^*$ weakly convergent to V say. If ξ is any finite energy vector in H_1, then

$$(X_n\Omega,\xi)=(U_{s_n}\Omega,\xi)\to 0.$$

Indeed H_1 breaks up as a direct sum of positive energy irreducible representations of a central cyclic extension $\mathcal{G}$ of $SU(1,1)$; each is a discrete series representation of $\mathcal{G}$, so can be realised as a subrepresentation of $L^2(\mathcal{G})$ and thus has matrix coefficients tending to zero at ∞.

In particular $V\Omega = 0$ and by the same argument as in Lemma 1, this implies that $V = 0$. Since this limit is independent of the subsequence, the result follows.

Remark. 1. The technique used here is a particular case of a representation theoretic result in ergodic theory going back to Gelfand–Fomin, Mautner, Moore and Howe: if a unitary representation of a connected semisimple group with finite centre has no fixed vectors, then its matrix coefficients vanish at ∞. (A proof using direct integral decompositions can be found in Zimmer's book and an elementary proof has been given by Scott Adams.)

2. Both of these lemmas could be proved directly using 2–point and 3–point functions which are easy to determine without using differential equations.

We now claim that linear span of chains

$$c_{i,i-1/2}c_{i-1/2,i-1}\cdots c_{1/2,0}\pi_0(g)\Omega$$

with the spin 1/2 primary fields c_{pq}'s and g localised in I^c is dense in H_i. In fact, by the covariance relation, these vector have the form

$$\pi_i(g)c_{i,i-1/2}c_{i-1/2,i-1}\cdots c_{1/2,0}\Omega.$$

A simple Reeh–Schlieder argument shows that the linear span of the vectors $c_{i,i-1/2}c_{i-1/2,i-1}\cdots c_{1/2,0}\Omega$ is the same as that of the vectors without restriction on the support. Taking the product of zero modes yields a vector in $H_i(0)$, on which the $\pi_i(g)$'s act cyclically by another application of the Reeh–Schlieder argument.

Now $a = a_{i-1/2,i}$ be a field concentrated in I_1 and $b = b_{i-1/2,i} = R_\theta a_{i-1/2,i}R_\theta^*$ be concentrated in I_2. Set $d_{i,i-1/2} = a^*$, also a spin 1/2 primary field supported in I_1. By the braiding relations

$$\begin{aligned}U_t(a^*b)U_t^* & c_{i,i-1/2}c_{i-1/2,i-1}\cdots c_{1/2,0}\pi_0(g)\Omega \\ &= U_t(d_{i,i-1/2}b_{i-1/2,i})U_t^*c_{i,i-1/2}c_{i-1/2,i-1}\cdots c_{1/2,0}\pi_0(g)\Omega \\ &= \lambda c_{i,i-1/2}c_{i-1/2,i-1}\cdots c_{1/2,0}\pi_0(g)U_t(d_{0,1/2}b_{1/2,0})U_t^*\Omega \\ &\quad + (\sum \mu_j c_{i,j_1}c_{j_1,j_2}\cdots c_{j_{2i},1})U_t(d_{1,1/2}b_{1/2,0})U_t^*\Omega.\end{aligned}$$

For any sequence (t_n) such that $U_{t_n}(a^*b)U_{t_n}^*$ converges weakly to T as $t_n \to \infty$, we have by Lemmas 1 and 2

$$\begin{aligned}Tc_{i,i-1/2}c_{i-1/2,i-1}&\cdots c_{1/2,0}\pi_0(g)\Omega \\ &= kc_{i,i-1/2}c_{i-1/2,i-1}\cdots c_{1/2,0}\pi_0(g)\Omega,\end{aligned}$$

for some constant $k \neq 0$. Since the linear span of the vectors $c_{i,i-1/2}c_{i-1/2,i-1}\cdots c_{1/2,0}\pi_0(g)\Omega$ is dense, it follows that $T = kI$. Since k is independent of the sequence (t_n), it follows that $U_t(a^*b)U_t^* \to kI$ weakly as $t \to \infty$.

Corollary (von Neumann density). $\pi_j(L_IG)'' = \pi_j(L_{I_1}G)'' \vee \pi_j(L_{I_2}G)''$.

Proof. By local equivalence, it suffices to prove this when $j = 0$, i.e. for the vacuum representation. A typical element has the form $(*)$. By the theorem this can be approximated by a weak operator limit of elements

$$a_{0i_1}a_{i_1i_2}\cdots a_{i_mi}a'_{i,i-1/2}b'_{i-1/2,i}b_{ij_1}b_{j_1j_2}\cdots b_{j_n0},$$

so that the index i has been changed to $i-1/2$. Continuing in this way we can in turn reduce $i-1/2$ to $i-1$ and so on, until eventually the approximation is by chains with $i=0$. But when $i=0$, the chain is a product of two chains for I_1 and I_2, each starting and ending at the vacuum representation. This is the product of an element of $\pi_0(L_{I_1}G)''$ and an element of $\pi_j(L_{I_2}G)''$. The result follows.

Corollary (irreducibility). *The subfactor given by the failure of Haag duality in H_i is irreducible.*