RESPONSE TO ANDRE HENRIQUES STATEMENTS ABOUT OUR PAPERS [AU1, AU2, AU3, AU4]

JØRGEN ELLEGAARD ANDERSEN AND KENJI UENO

First of all, Andre Henriques claims that, we use that our definition of a modular functor is equivalent to Turaev's. This is completely wrong. We state in all the papers [AU2, AU3, AU4] that our definition of a modular functor is different from in particular the one in Turaev's book [T]. Here is the statement from [AU4], which is the first paragraph of section two of that paper:

"We shall in this section give the axioms for a modular functor. These are due to Segal and appeared first in [46]. We present them here in a topological form, which is due to Walker [57]. See also [28,34]. We note that similar, but different, axioms for a modular functor are given in [49] and in [22]. The authors are not aware of a proof of the equivalence of these definitions of a modular functor. However, we will not need it in this paper." (For the convenience of the reader we recall that [49] is Turaev's book [T].)

From [AU2] it follows that we get a modular functor (in our version of Walker's axioms) for any simple Lie algebra and a choice of isomorphisms from $(V_\lambda)^*$ to $V_{\lambda^\dagger}$ (here λ's are highest weight vectors and V_λ the associated irreducible representation). This construction is based on the conformal field theory constructions in [AU1] and [TUY]. This modular functor does of course depend on this choice of these isomorphisms and they are not all isomorphic, that is clear. The point is that one can only scale these choices via isomorphims of modular functors in something which is symmetric in the labels, due to compatibility with morphisms which interchanges the points. Hence the isomorphism class is precisely determined by the composition of the two isomorphism, the one for λ and the transposed of the one for $\lambda^\dagger$ for all λ.

If one starts with a modular tensor category, it is totally trivial to check based on what is in Turaev's book that one gets by restricting Turaev's definition of his modular functor an assignment of vector spaces to label marked surfaces, where the labels are chosen from the set of simple objects (all with plus signs), which are part of the data for a modular tensor category. For each choice of isomorphism $(V_\lambda)^*$ to $V_{\lambda^\dagger}$ where these are now the simple objects, one can construct the needed glueing isomorphism by combining Turaev's construction with these isomorphism. This follows directly from Lemma 5.8.1 in [T] which gives a natural isomorphism between the vector spaces when one marking $+V^*$ is replaced with $-V$ and that the constructions in the proof of this same lemma also provides isomorphism (induced from an isomorphism from $q : U \to V$) between the vector spaces when one replaces U by V in markings. The same remark as in the previous paragraph precisely determines the isomorphism class of the resulting modular functor. Although all details needed for this are to be found in Turaev's book, a detailed account of this will be posted on the arXiv jointly by JEA's and his PhD student William Elbæk Pedersen, since it appears there is a need for such a note providing assistance for readers of Turaev's beautiful book [T].

As it is clear and explained in the above two paragraphs, we remark that the same choices of isomorphisms from $(V_\lambda)^*$ to $V_{\lambda^\dagger}$ for all simples occur on both sides of the isomorphism presented in [AU4], namely for modular functor coming from the conformal field theory and for the modular functor coming from Blanchet's skein theory construction of the modular tensor category for the $SU(n)$ theories [B].

The isomorphism from [AU4] proves that for any set of choices on one side, say on the Skein modular functor side, there is a corresponding set of choice on the conformal field theory such that the construction in [AU4] is an isomorphism. The same is the case if one makes any choice on the conformal field theory side first. Our construction is simply not sensitive to this choice of normalisation. They way the normalisation passes form one side to the other is via the very heart of our construction, namely in genus zero with only box labels, where we line both sides up with Wenzl's construction of his representations and their inclusions into each other, when one increases the number of points. This passes a set of normalisation on one side to a set of normalisation on the other by the very way the glueing morphism is involved in this inclusion process.

The explicit construction of a duality for any modular functor coming from a modular tensor category comes from Turaev construction of duality for his associated TQFT combined with the label change isomorphism mentioned above as done in [T]. This duality axiom and the construction of such a structure for any modular tensor category will also be spelled out in complete details in the above mentioned note by JEA and William Elbæk Pedersen.

Now, if one wants the normalisation which gives the Reshetikhin-Turaev TQFT, so not just modular functors, but the whole TQFT via a modular functor with **duality**, one simply just have to choose the above scalers such that the pairing for λ composed with that for $\lambda^\dagger$ is the quantum dimension of λ on both sides, e.g. on the conformal field theory and on the skein theory side. These two are mapped to each other by our isomorphism.

We are sorry, but we see no reason to change the definition of modular functors. In fact our defintion seems optimal in order to establish that the modular functor which comes from conformal field theory is isomorphic to the one which comes from the corresponding modular tensor categories, which is indeed our main theorem for the $\mathrm{SU}(n)$ series stated and proved in [AU4].

REFERENCES

[AU1] J. E. Andersen & K. Ueno, Abelian Conformal Field theories and Determinant Bundles, *International Journal of Mathematics* **18**, 919 - 993 (2007) .

[AU2] J. E. Andersen & K. Ueno, Geometric Construction of Modular Functors from Conformal Field Theory, *Journal of Knot theory and its Ramifications* **16**, 127 – 202 (2007).

[AU3] JJ.E. Andersen & K. Ueno, "Modular functors are determined by their genus zero data", Quantum Topology **3** (2012), 255–291.

[AU4] J.E. Andersen & K. Ueno, "Construction of the Witten-Reshetikhin-Turaev TQFT from conformal field theory". Invent. Math. **201 (2)** (2015), 519–559.

[B] C. Blanchet, Hecke algebras, modular categories and 3-manifolds quantum invariants, *Topology* **39** no. 1, 193–223 (2000).

[TUY] A. Tsuchiya, K. Ueno and Y. Yamada", Conformal Field Theory on Universal Family of Stable Curves with Gauge Symmetries, *Advanced Studies in Pure Mathematics* **19**, 459–566 (1989).

[T] V. Turaev, Quantum invariants of knots and 3-manifolds, W. de Gruyter, Berlin (1994).

CENTER FOR QUANTUM GEOMETRY OF MODULI SPACES, UNIVERSITY OF AARHUS, DK-8000, DENMARK

E-mail address: andersen@imf.au.dk

DEPARTMENT OF INDUSTRIAL AND SYSTEM ENGINEERING, FACULTY OF SCIENCE AND ENGINEERING, HOSEI UNIVERSITY, KOGANEI, TOKYO, 184-0002 JAPAN

E-mail address: ueno@math.kyoto-u.ac.jp