# The *576*-fold Bott periodicity of the Majorana fermions

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<u>*Bott* (1957):</u>

topology of $U(\infty)$ and $O(\infty)$

$K$-theory:

complex: period 2
real : period 8

Bott periodicity

Clifford algebras

classification of topological insulators

***complex:*** $\boxed{0, \mathbb{Z}}\ 0, \mathbb{Z}, 0, \mathbb{Z}, 0, \mathbb{Z}, 0, \mathbb{Z}, 0, \mathbb{Z}, 0, \mathbb{Z}, 0, \mathbb{Z}, 0, \mathbb{Z}, 0$

***real:*** $\boxed{\mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}}\ \mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}, \mathbb{Z}_2$

## K-theory:

It assigns to every space $X$ a sequence of abelian groups $K^0(X)$, $K^1(X)$, $K^2(X)$, $\ldots$

*Bott periodicity:*

*complex:*

$$K^n(X) = K^{n+2}(X)$$

*real:*

$$KO^n(X) = KO^{n+8}(X)$$

## Elliptic cohomology:

*Slogan:*

$$Ell^n(X) \approx K^n_{S^1}(LX)$$

$K$-theory is to quantum mechanics as Elliptic cohomology is to quantum field theory

*Periodicity:*

$$Ell^n(X) = Ell^{n+576}(X)$$

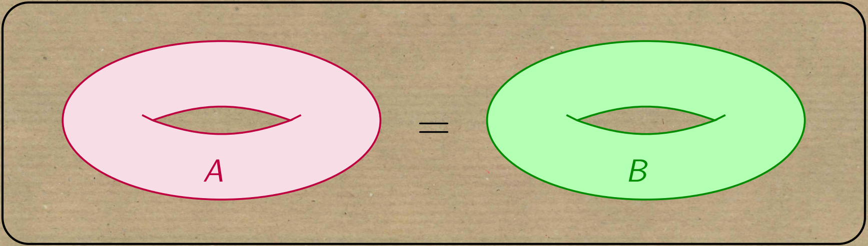
# Abelian Chern-Simons theory

3 dimensional TQFT with action functional $S[A] = e^{iCS(A)}$
where $$CS(A) = \frac{1}{4\pi}\int_{M_3} K_{ij}\, A^i \wedge dA^j$$

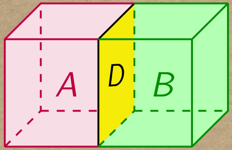
$T$-connection

- At the *classical level*, abelian Chern-Simons theories are classified by the lattice $\Lambda := \ker(\exp : \mathfrak{t} \to T)$.

- At the *quantum level*, theories can become equivalent.




Two theories $A$ and $B$ are equivalent if there exists an *invertible* defect $D$




- At the *quantum level*, theories can become equivalent.

At the classical level, Chern-Simons theories with abelian gauge group $T$ are classified by a lattice $\Lambda = \ker(\exp : \mathfrak{t} \to T)$.

At the quantum level, two such theories can become equivalent.

invariants:

$\Lambda \to$ rank $r \in$ $\begin{cases} \mathbb{Z}_{24} & \text{according to Belov–Moore} \\ \mathbb{Z} & \text{according to Kapustin–Saulina} \end{cases}$

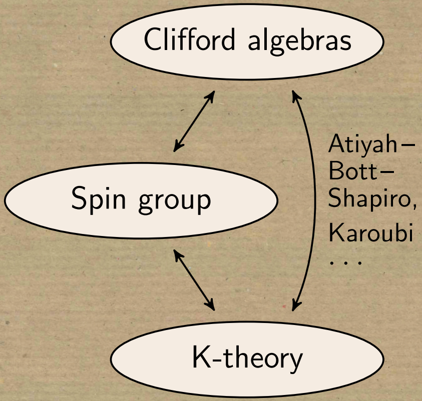
$\Lambda \to$ discriminant group $D = \Lambda^*/\Lambda$

$\Lambda \to$ quadratic form $D \to \mathbb{Q}/\mathbb{Z}$

?

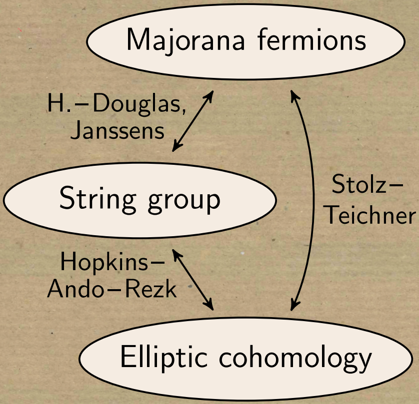
## K-theory

**period 2 / period 8**



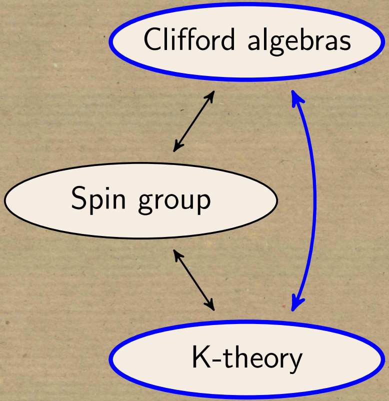

## Elliptic cohomology

**period 576**



$$\mathit{Cliff}_{\mathbb{C}}(n) \approx_{\mathrm{M}} \mathit{Cliff}_{\mathbb{C}}(n+2)$$
$$\mathit{Cliff}_{\mathbb{R}}(n) \approx_{\mathrm{M}} \mathit{Cliff}_{\mathbb{R}}(n+8)$$

Morita equivalence

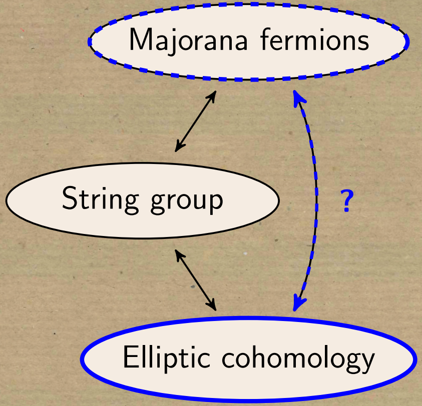



**period 2 / period 8**

$$\mathit{Fer}(n) \approx \mathit{Fer}(n+576)$$

Conjecture

equivalence relation on 2d CFTs




**period 576 = $24^2$**

**invariants:**

$\Lambda \to$ rank $r \in$ **?**

$\Lambda \to$ discriminant group $D = \Lambda^*/\Lambda$

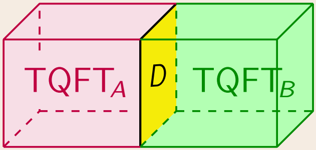
$\Lambda \to$ quadratic form $D \to \mathbb{Q}/\mathbb{Z}$

$$Fer(n) \approx Fer(n+576)$$

Conjecture

We have $\text{CFT}_A \approx \text{CFT}_B$

if $\exists$ an invertible defect




$Fer(n)$:

$$[\psi_i(z), \psi_j(w)]_+ = \delta(z-w)\delta_{ij}$$

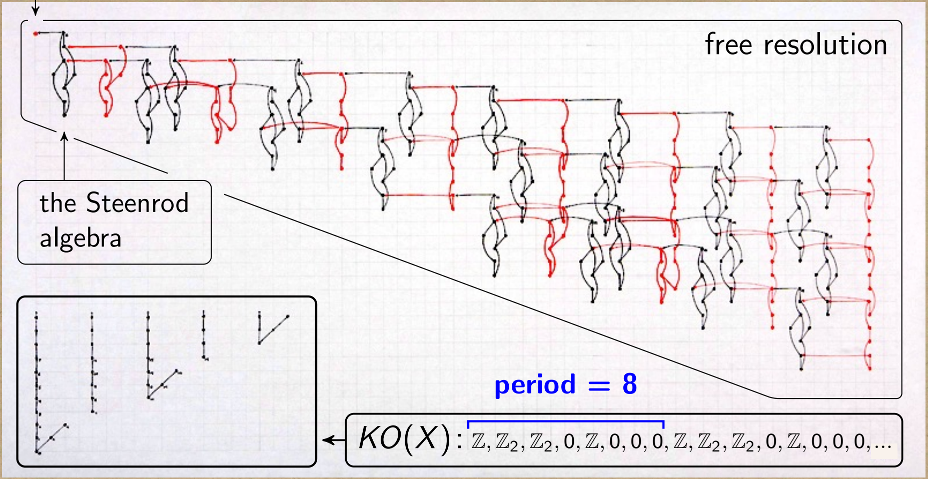
$$c = \tfrac{n}{2} \qquad i, j = 1 \ldots n$$

$Fer(2) \leftrightarrow CS$ for $U(1)$

$Fer(2n) \leftrightarrow CS$ for $U(1)^n$

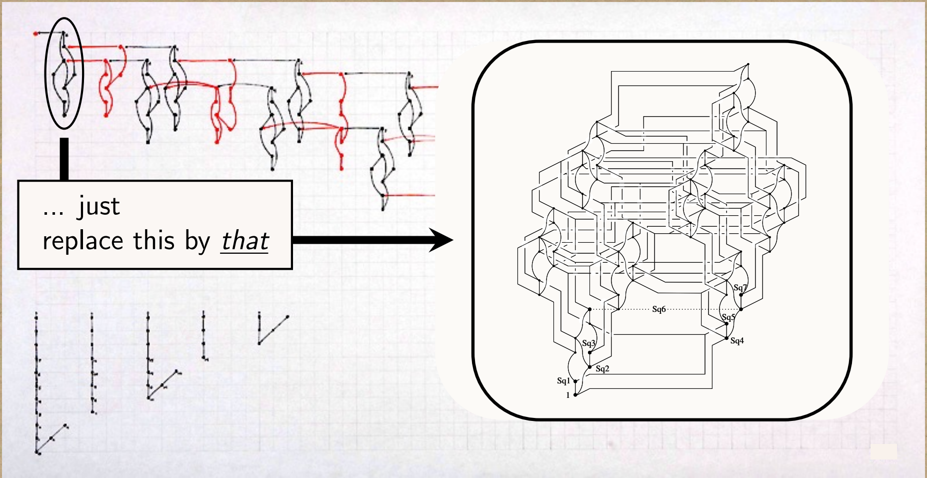
$Fer(576) \leftrightarrow CS$ for $U(1)^{288}$

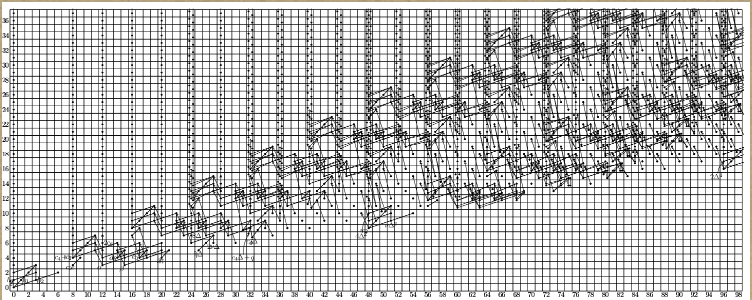
the space $X$ for which we want to compute $KO(X)$




(Picture by Robert Bruner)

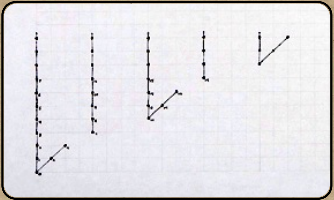
# To compute $Ell(X)$, proceed identically…





**period = 576**

The analog of:



**period = 8**

is then given by...

## Clifford algebras

$$\mathit{Cliff}(n) = \mathit{Cliff}(1)^{\otimes n} = \langle e_1, \ldots, e_n \mid e_i^2 = 1, e_i e_j = -e_j e_i \rangle$$

**Bott:** $\boxed{\mathit{Cliff}_{\mathbb{R}}(n) \approx_{\mathrm{M}} \mathit{Cliff}_{\mathbb{R}}(n+8)}$

Morita equivalence

$A \approx_{\mathrm{M}} \mathit{Mat}_{k\times k}(A)$

$\mathit{Cliff}_{\mathbb{R}}(n+8)$ is equivalent to matrices of size $16 \times 16$ with entries in $\mathit{Cliff}_{\mathbb{R}}(n)$.

## Proof of Bott periodicity for Clifford algebras

$$\begin{aligned} \mathit{Cliff}(1) &= \langle e \mid e^2 = 1\rangle \\ \mathit{Cliff}(n) &= \mathit{Cliff}(1)^{\otimes n} \\ \mathit{Cliff}(-1) &= \langle f \mid f^2 = -1\rangle \\ \mathit{Cliff}(-n) &= \mathit{Cliff}(-1)^{\otimes n} \end{aligned}$$

$$\boxed{\mathit{Cliff}(n) \approx_{\mathrm{M}} \mathit{Cliff}(n+8)}$$

$\mathit{Cliff}(1)$ and $\mathit{Cliff}(-1)$ are each other's inverse up to $\approx_{\mathrm{M}}$ :

$$\left\{ \begin{array}{l} \mathit{Cliff}(1) \otimes \mathit{Cliff}(-1) \;=\; \mathit{Mat}_{2\times 2}(\mathbb{R}) \\[2ex] e \otimes 1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad 1 \otimes f \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{array} \right.$$

QED

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$$\boxed{Cliff(n) \approx_{\mathrm{M}} Cliff(n+8)}$$

We have $\mathbb{H} \otimes Cliff(1) = Cliff(-3)$ : $\begin{aligned} i \otimes e &\mapsto f_1 \\ j \otimes e &\mapsto f_2 \\ k \otimes e &\mapsto f_3 \end{aligned}$

and

$\mathbb{H} \otimes Cliff(-1) = Cliff(3)$ : $\begin{aligned} i \otimes f &\mapsto e_1 \\ j \otimes f &\mapsto e_2 \\ k \otimes f &\mapsto e_3 \end{aligned}$

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$$\boxed{\mathit{Cliff}(n) \approx_{\mathrm{M}} \mathit{Cliff}(n+8)}$$

$$\begin{aligned} \mathit{Cliff}(-1) &\approx_{\mathrm{M}} \mathit{Cliff}(1)^{\otimes -1} \\ \mathbb{H} \otimes \mathit{Cliff}(1) &= \mathit{Cliff}(-3) \\ \mathbb{H} \otimes \mathit{Cliff}(-1) &= \mathit{Cliff}(3) \end{aligned}$$

Proof:

$$\begin{tikzcd} \mathit{Cliff}(-1) \arrow[d] & \mathbb{R} \arrow[l] \arrow[r] & \mathit{Cliff}(1) \arrow[d] \\ \mathit{Cliff}(-2) \arrow[d] & & \mathit{Cliff}(2) \arrow[d] \\ \mathit{Cliff}(-3) & \mathbb{H} \arrow[l] \arrow[r] & \mathit{Cliff}(3) \end{tikzcd}$$

QED

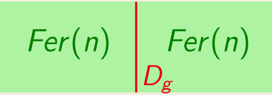
## The String group

$$\begin{tikzcd}
\textit{Orthogonal group} & O(n) \arrow[d, "\boxed{\text{replace by something connected}}"'] \\
& SO(n) \arrow[d, "\boxed{\text{replace by something simply connected}}"'] \\
& Spin(n) \arrow[d, "\boxed{\text{kill } \pi_3}"'] \\
\textit{infinite dimensional} \arrow[r] & String(n)
\end{tikzcd}$$

# The String group

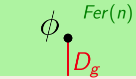
The orthogonal group acts of $Fer(n)$ by permuting the $\psi_i$'s.

Every $g \in O(n)$ yields a *defect line* where the $\psi_i$ are discontinuous and transform under $g$.




**Theorem:** (H.–Douglas; Janssens)

$$\left\{ (g, \phi) \;\middle|\; \begin{array}{l} g \in O(n), \\ \phi \text{ is a twist field:} \end{array} \right.$$




$$\left. \vphantom{\begin{array}{l} a \\ b \end{array}} \right\}$$

is a model for the string group.

Upon replacing

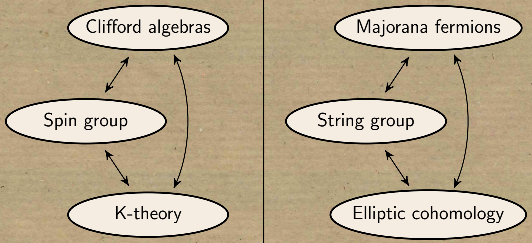
$Fer(n) \to Cliff(n)$

'defect' $\to$ 'bimodule'

'field' $\to$ 'linear map'

one gets a model for the Spin group.

| **K-theory** | **Elliptic cohomology** |
|---|---|
| period 2 / period 8 | period 576 |

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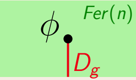
$$\left\{ (g,\phi) \;\middle|\; \begin{array}{l} g \in O(n), \\ \phi \text{ is a twist field:} \end{array} \right.$$  $\}$



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