Examples of Stolz–Teichner cocycles

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Disclaimer: The ideas and statements in this note are informal and speculative. We present an exploration of what might, one day, become interesting examples of Stolz–Teichner cocycles.

According to the Stolz–Teichner conjecture [5, 6], the moduli space of $(0,1)$-supersymmetric, $\mathit{Fer}(n)$-twisted, extended, 2-dimensional euclidian field theories has the homotopy type of the $(-n)^{\text{th}}$ space of the spectrum of topological modular forms:

$$TMF_{-n} \;\approx\; \left\{ \begin{array}{c} \text{Moduli space of } (0,1)\text{-susy} \\ \mathit{Fer}(n)\text{-twisted extended } 2d \text{ EFTs} \end{array} \right\}. \tag{1}$$

Given a space X, a "degree $-n$ Stolz–Teichner cocycle" over X is, roughly speaking, a map from X to the right hand side of (1). Such a thing should represent a class in $TMF^{-n}(X)$. In the special case when X is a point, this is the same thing as an element of $\pi_n(TMF)$, the n^{th} homotopy group of the spectrum of topological modular forms.

We now explain, at a very rough level, the terms that appear in the right hand side of (1). First of all, in Stolz–Teichner's language, the term "*euclidian field theory*" means essentially the same thing as "quantum field theory on flat space-time". A theory is *extended* if it is defined not on Minkowski/Euclidean space, but on compact (flat) 2-manifolds with boundary, on 1-manifolds with boundary (the theory assigns Hilbert spaces to 1-manifolds), and on points (the theory assigns von Neumann algebras to points – see [2] for an explanation of this concept in the context of CFTs). A theory is $(0,1)$-*supersymmetric* if the infinitesimal generator $\bar{L}_0$ of anti-holomorphic translations (on the Hilbert space associated to a circle, say) has an odd square root.

There is an obvious forgetful map from CFTs to EFTs: if a theory knows how to assign values to conformal surfaces, then in particular it knows how to assign values to flat surfaces. We shall be interested in the following question: how much of the right hand side of (1) can one see using only conformal field theories? Can one get interesting elements of $\pi_n(TMF)$ that way?

Let us first go back to (1) and discuss the meaning of "$\mathit{Fer}(n)$-twisted". We concentrate on the case $n \geq 0$. Here, $\mathit{Fer}(n)$ refers to the conformal field theory of n real chiral free fermions (with central charge $c = n/2$). We'll say that a conformal field theory is "$\mathit{Fer}(n)$-twisted" if it contains a copy of $\mathit{Fer}(n)$. (If n is negative, then the theory should instead contain $|n|$ antichiral fermions.) Given a conformal field theory Z, let us write $\chi(Z)$ for the maximal chiral sub-theory of Z, and $\bar{\chi}(Z)$ for the maximal anti-chiral sub-theory. Thus, we say that Z is "$\mathit{Fer}(n)$-twisted" if it comes equipped with an embedding $\mathit{Fer}(n) \hookrightarrow \chi(Z)$.

Finally, Z is $(0,1)$-supersymmetric if the canonical Virasoro $\overline{Vir} \subset \bar{\chi}(Z)$ comes equipped with an extension to an $N = 1$ super-Virasoro:

$$\begin{tikzcd} \overline{sVir} \arrow[drr, dashed] & & \\ \cup & & \\ \overline{Vir} \arrow[rr, hook] & & \bar{\chi}(Z) \end{tikzcd}$$

Recapitulating, we are looking at full CFTs with an embedding of $Fer(n)$ in the chiral part, and an $N = 1$ supersymmetric structure on the anti-chiral part. The easiest full CFT that satisfies the first requirement is $Fer(n) \otimes \overline{Fer(n)}$. We now need a supersymmetric extension of the standard Virasoro of $\overline{Fer(n)}$

$$\bar{T}(\bar{z}) = \tfrac{1}{2} \sum_{i=1}^{n} : \partial\bar{\psi}_i(\bar{z})\bar{\psi}_i(\bar{z}) :$$

This is the datum of a primary field $\bar{G}(\bar{z})$ of conformal dimension $\frac{3}{2}$, satisfying certain commutation relations. Fields of dimension $\frac{3}{2}$ are linear combinations of $: \bar{\psi}_i\bar{\psi}_j\bar{\psi}_k :$ and of $\partial\bar{\psi}_i$. The latter cannot occur in $\bar{G}(\bar{z})$ since it otherwise wouldn't be primary, so the most general form is the following:

$$\bar{G}(\bar{z}) = \tfrac{1}{6} \sum f^{ijk} : \bar{\psi}_i(\bar{z})\bar{\psi}_j(\bar{z})\bar{\psi}_k(\bar{z}) :$$

It turns out that the above field satisfies the required commutation relations if and only if f^{ijk} are the structure constants of a Lie algebra on $\mathbb{R}^n$ whose Killing form agrees with the standard inner product on $\mathbb{R}^n$ [1, 4]. In other words, there are exactly as many $N = 1$ supersymmetric structures on $\overline{Fer(n)}$ as there are n-dimensional semi-simple Lie algebras of compact type. All in all, this provides (a sketch of) a construction

$$(2) \qquad \left\{ \begin{array}{c} n\text{-dimensional semi-simple} \\ \text{Lie algebras of compact type} \end{array} \right\} \quad \to \quad \left\{ \begin{array}{c} \text{degree } -n \text{ Stolz–Teichner} \\ \text{cocycles (over a point)} \end{array} \right\}$$

At this point, the natural question is: are there elements in $\pi_n(TMF)$ that are naturally parametrised by n-dimensional semi-simple Lie algebras of compact type, and that one could reasonably conjecture are the images of the above construction? The answer turns out to be yes. The simply connected Lie group integrating a Lie algebra is a framed manifold, when equipped with its left invariant framing. It therefore represents a class in the n-dimensional framed bordism group. The latter is isomorphic to the n^{th} stable homotopy group of spheres $\pi_n^{stable}(S^0) = \pi_n(\mathbb{S})$ by the Pontrjagin-Thom isomorphism (here $\mathbb{S}$ denotes the sphere spectrum). Being a ring spectrum, TMF admits a unit map from the sphere spectrum, which induces a map $\pi_n(\mathbb{S}) \to \pi_n(TMF)$ at the level of homotopy groups. As mentioned in [3], many interesting classes in the homotopy groups of TMF are represented by compact Lie groups with their left invariant framing, and we conjecture that the construction (2) hits exactly those classes.

References

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