1 Isostratified spaces and isostratified simplicial sets

Let $\mathbb{J}$ be a poset. By a $\mathbb{J}$-stratified space, we shall mean a space X equipped with a partition

$$X = \coprod_{j\in\mathbb{J}} X_j$$

into locally closed subspaces, such that the closure of X_j only meets $X_{j'}$ for $j' > j$. We call X_j the *pure strata*, $X_{\geq j} := \cup_{j' \geq j} X_{j'}$ the *closed strata* and $X_{\leq j} := \cup_{j' \leq j} X_{j'}$ the *open strata* of X.

One can consider two kind of maps $X \to Y$ between $\mathbb{J}$-stratified spaces: those who send $X_{\geq j}$ to $Y_{\geq j}$, and those who send X_j to Y_j. We call the former *stratified maps* and the latter *isostratified maps*.

We shall denote by $\mathbb{J}$-strat the category whose objects are $\mathbb{J}$-stratified spaces, and morphisms are $\mathbb{J}$-stratified maps, and by $\mathbb{J}$-$\mathsf{isostrat}$ the category whose objects are $\mathbb{J}$-stratified spaces, and morphisms are $\mathbb{J}$-isostratified maps. When a $\mathbb{J}$-stratified space is viewed as an object in $\mathbb{J}$-$\mathsf{isostrat}$, we shall call it a $\mathbb{J}$-isostratified space.

We shall be more interested in $\mathbb{J}$-isostratified spaces.

Example 1 Let G be a compact group, and $\mathbb{J}$ be the poset of conjugacy classes of closed subgroups of G. Then there is a natural functor from the catogery of G-spaces to $\mathbb{J}$-strat.

Indeed, any G-space has a natural $\mathbb{J}$-stratification by stabilizer groups. Since G-equivariant maps never decrease the size of the stabilizer, this functor lands in $\mathbb{J}$-strat.

Example 2 Let $|N\mathbb{J}|$ be the realization of the nerve of $\mathbb{J}$. The category of spaces over $|N\mathbb{J}|$ then admits a natural functor to $\mathbb{J}$-$\mathsf{isostrat}$. enj

The space $|N\mathbb{J}|$ itself is $\mathbb{J}$-stratified by letting $|N\mathbb{J}|_{\geq j} = |N\mathbb{J}_{\geq j}|$, where $\mathbb{J}_{\geq j}$ denotes the subposet of elements $j' \geq j$. Given $s : X \to |N\mathbb{J}|$, we can then pullback the stratification of $|N\mathbb{J}|$ to a stratification on X. Namely, we let $X_j = s^{-1}(|N\mathbb{J}|_j)$. Any map of spaces over $|N\mathbb{J}|$ is then automatically isostratified.

Given an isotratified map $X \to Y$, we shall be interested to know when it admits sections. For this we would like to put a model structure on the category of isostratified spaces. But this is impossible because that category is very badly behaved (no good products, no good initial object, etc.).

For this purpose, we introduce the notion of $\mathbb{J}$-isostratified simplicial sets.

Definition 3 *A $\mathbb{J}$-isostratified simplicial set is a simplicial set over $N\mathbb{J}$. We denote the category of $\mathbb{J}$-isostratified simplicial sets by* $\mathsf{sSets}{\downarrow}N\mathbb{J}$.

Given an isostratified simplicial set X, its geometric realization $|X|$ comes with a map to $|N\mathbb{J}|$ and is thus isostratified by Example 2. The functor $X \to |X|$ from $\mathsf{sSets}{\downarrow}N\mathbb{J}$ to $\mathbb{J}$-$\mathsf{isostrat}$ then has a right adjoint $\mathrm{Sing}_{\mathbb{J}} : \mathbb{J}\text{-}\mathsf{isostrat} \to \mathsf{sSets}{\downarrow}N\mathbb{J}$ given by

$$\mathrm{Sing}_{\mathbb{J}}(X)_k = \bigsqcup_{s:\Delta[k]\to N\mathbb{J}} \mathrm{Hom}_{\mathbb{J}}\Big(\big|(\Delta[k], s)\big|,\, X\Big),$$

where $\mathrm{Hom}_{\mathbb{J}}$ denotes the set of isostratified maps. The map $\mathrm{Sing}_{\mathbb{J}}(X) \to N\mathbb{J}$ is given by $(f : |(\Delta[k], s)| \to X) \mapsto s \in \mathrm{Hom}(\Delta[k], N\mathbb{J}) = (N\mathbb{J})_k$.

This pair of adjoint functors is morally an equivalence of homotopy theories between isostratified spaces and isostratified simplicial sets. But we shall not try to make that precise.

1.1 A model structure on the category of isostratified simplicial sets

Let N be a simplicial set and $\mathcal{P}$ the poset of nondegenerate simplices of N, ordered under inclusion. Our example of interest being when N is the nerve of a poset, we shall make the following simplifying assumption, valid throughout this section.

Assumption 4 *For every non-degenerate simplex $\sigma \in N_n$, the corresponding map $\sigma : \Delta[n] \to N$ is injective.* Ang

Under the above assumption, we have for example

$$N = \operatorname*{colim}_{(\sigma:\Delta[n]\hookrightarrow N)\in\mathcal{P}} \Delta[n].$$

Let $\mathsf{sSets}{\downarrow}N$ denote the category of simplicial sets over N, namely the category whose objects are simplicial sets equipped with a map to N, and whose morphisms are the obvious commutative triangles. An object $(X, s : X \to N)$ will sometimes be abreviated X. Given an object $X = (X, s) \in \mathsf{sSets}{\downarrow}N$ and a simplex $\sigma \in \mathcal{P}$, we shall denote by $X(\sigma)$ the simplicial set of lifts

$$\begin{array}{ccc} & & X \\ & \nearrow & \downarrow{\scriptstyle s} \\ \Delta[n] & \xrightarrow{\ \sigma\ } & N \end{array}$$

namely, the fiber over σ of the map $s \circ - : \mathrm{Map}(\Delta[n], X) \longrightarrow \mathrm{Map}(\Delta[n], N)$. For each $\tau < \sigma$ in $\mathcal{P}$, we have a restriction map $X(\sigma) \to X(\tau)$. Letting $\mathcal{P}^\circ$ be the opposite poset to $\mathcal{P}$, the assignement $\sigma \mapsto X(\sigma)$ then defines a functor $\mathcal{P}^\circ \to \mathsf{sSets}$. So we get a functor fcR fcL

$$\begin{aligned} R : \mathsf{sSets}{\downarrow}N &\longrightarrow \mathsf{sSets}^{\mathcal{P}^\circ} \\ X &\mapsto \big(\sigma \mapsto X(\sigma)\big). \end{aligned} \tag{1}$$

The functor R has a left adjoint

$$\begin{aligned} L : \mathsf{sSets}^{\mathcal{P}^\circ} &\longrightarrow \mathsf{sSets}{\downarrow}N \\ F &\mapsto \coprod_{\sigma:\Delta[n]\hookrightarrow N} F(\sigma) \times \Delta[n] \,/\, \sim \end{aligned} \tag{2}$$

sending a functor $F \in \mathsf{sSets}^{\mathcal{P}^\circ}$ to (a particular model for) its homotopy colimit. More precisely, $L(F)$ is the coequalizer of the two maps

$$\coprod_{\Delta[m]\overset{v}{\hookrightarrow}\Delta[n]\overset{\sigma}{\hookrightarrow}N} F(\sigma) \times \Delta[m] \ \underset{\coprod 1\times v}{\overset{\coprod F(v)\times 1}{\rightrightarrows}} \ \coprod_{\Delta[n]\overset{\sigma}{\hookrightarrow}N} F(\sigma) \times \Delta[n],$$

and the map $L(F) \to N$ is induced by the maps $F(\sigma) \times \Delta[n] \twoheadrightarrow \Delta[n] \overset{\sigma}{\to} N$. The goal of this section is to introduce an appropriate model structure for the category $\mathsf{sSets}{\downarrow}N$ such that the above functors become a Quillen equivalence.

Let $\partial^i\Delta[n]$ denote the facet of $\Delta[n]$ opposite to the ith vertex, and let

$$\Lambda[n, j] := \bigcup_{i\neq j} \partial^i\Delta[n]$$

be the jth horn.

Theorem 5 *There is a model structure on the category $\mathsf{sSets}{\downarrow}N$ where:* th:md

- *A map $(X,s) \to (Y,t)$ is a weak equivalence if for every $\sigma \in \mathcal{P}$, the induced map $X(\sigma) \to Y(\sigma)$ is a weak equivalence of simplicial sets.*
- *A map $(X,s) \to (Y,t)$ is a cofibration if it's a monomorphism.*
- *A map $(X,s) \to (Y,t)$ is a fibration if it satisfies the right lifting property with respect to all morphisms of the form*

$$\begin{array}{ccc} \Lambda[k,j] & \hookrightarrow & \Delta[k] \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle\beta} & \\ & N & \end{array} ,$$

whith β in the image of one of the two degeneracy maps $s_j, s_{j-1} : N_{k-1} \to N_k$.

In order to prove Theorem 5, we recall the machinery of cofibrantly generated model structures [1, Chapters 10 and 11]. Let $\mathcal{C}$ be a cocomplete category whose objects are small, and let I be a set of maps in $\mathcal{C}$. A map is called *I-injective* if it satisfies the right lifting property with respect to all elements of I. A map is called *I-cofibration* if it satisfies the left lifting property with respect to all I-injective maps. As a subclass of I-cofibrations, we have the *relative I-cell complexes*, which are the maps that can be written as a (transfinite) composition of maps of the form $X \to \operatorname{colim}(X \leftarrow A \xrightarrow{i} B)$ for $i \in I$. The I-cofibration are charachterized in therms of the relative I-cell complexes as follows:

Lemma 6 ([1, Corollary 10.5.2]) *Let $\mathcal{C}$, I be as above. Then a map $A \to B$ is an I-cofibrations if and only if it is a retract of a relative I-cell complex $A \to B'$.* □ uch

The following theorem of D. Kan is useful for recognizing model category structures:

Theorem 7 ([1, Theorem 11.3.1]) *Let $\mathcal{C}$ be cocomplete category whose objects are all small. Let I, J be two sets of morphisms in $\mathcal{C}$, and let W be a class of morphisms in $\mathcal{C}$ satisfying the "2 out of 3" axiom. If* rcg

(*i*). *Every J-cofibration is both an I-cofibration and an element of W.* it1:rcg

(*ii*). *Every I-injective is both J-injective and an element of W.* it2:rcg

(*iii*). *If an element of W is an I-cofibration, it is then also a J-cofibration.* it3:rcg

then there is a model category structure on $\mathcal{C}$ whose weak equivalences are W, whose cofibrations are the I-cofibrations, and whose fibrations are the J-injectives. □

If the hypotheses of Theorem 7 are satisfied, one says that $\mathcal{C}$ is a cofibrantly generated model category, one calls I the set of *generating cofibrations*, and J the set of *generating acyclic cofibrations*.

We now go back to our situation $\mathcal{C} = \mathsf{sSets}\downarrow N$. The maps in W are those whose image in $\mathsf{sSets}^{\mathcal{P}^\circ}$ is an objectwise weak equivalence. The set I consists of all maps of the form $(\partial\Delta[n], \alpha) \hookrightarrow (\Delta[n], \beta)$. And the set J consists of those maps $(\Lambda[n,j], \alpha) \hookrightarrow (\Delta[n], \beta)$ such that $\beta \in \operatorname{im}(s_{j'})$ for some $j' \in \{j-1, j\}$.

Proof of condition (*i*).

The class of I-cofibration is equal to the class of injective morphisms. Relative J-cell complexes are injective and by Lemma 6, so are the J-cofibrations. Since W is stable under pushouts, colimits, and retracts, to verify that all J-cofibrations are in W, it's enough to check that $J \subset W$. That's the content of Lemma 11. We begin by a few lemmas.

Lemma 8 *Let s be one of the following: (a) a surjective map $s : \Delta[i] \to \Delta[n]$, (b) the projection $s : \Delta[i] \times \Delta[n] \to \Delta[n]$, then the simplicial set of sections of s is contractible.*

Proof. (a) The simplicial set of sections is the product of all the preimages of vertices of $\Delta[n]$.
(b) The simplicial set of sections is $\mathrm{Map}(\Delta[n], \Delta[i])$, which is contractible because $\Delta[i]$ is a cone. $\square$

Let (X, s) be an object of $\mathsf{sSets}\!\downarrow\! N$. Given a non-degenerate simplex $\sigma : \Delta[n] \to N$, and a point p in the relative interior of $|\operatorname{im}(\sigma)|$, the evaluation $ev : X(\sigma) \times \Delta[n] \to X$ induces a map ajo

$$ev_p : |X(\sigma)| = |X(\sigma)| \times \{p\} \longrightarrow |s|^{-1}(p) \tag{3}$$

by restriction.

Lemma 9 *Let X, s, σ, p be as above, then (3) is a homotopy equivalence. Moreover, the inverse map $|s|^{-1}(p) \to |X(\sigma)|$, and the homotopies between the composites and the respective identity maps can be chosen naturally in X.* fbc

Proof. The natural inverse $\psi : |s|^{-1}(p) \to |X(\sigma)|$, and natural homotopies $h : |X(\sigma)| \times [0,1] \to |X(\sigma)|$, $\quad k : (|s|^{-1}(p)) \times [0,1] \to |s|^{-1}(p)$ are constructed inductively on skeleta, by use of obstruction theory.

Assume that ψ has been defined on the intersection of $|s|^{-1}(p)$ with the $(i-1)$-skeleton of $|X|$. We then want to extend it to the intersection with a given i-simplex of $|X|$. By naturality, it's enough to do the case $X = \Delta[i]$. Since p is in the interior of $|\operatorname{im}(\sigma)|$, then either $\operatorname{im}(s : \Delta[i] \to N) = \operatorname{im}(\sigma)$, or $|s|^{-1}(p)$ is empty. In the first case, $|X(\sigma)|$ is contractible by part (a) of Lemma 8, and so there are no obstructions. In the second case $|s|^{-1}(p) = \emptyset$, and there's nothing to extend.

Assume that h is defined on the $(i-1)$-skeleton of $|X(\sigma)|$ and that we want to extend it to a given i-simplex. Once again, there is a univeral example that implies all the other ones by naturality. It is given by $X = \Delta[i] \times \Delta[n]$ with $s = \sigma \circ \mathrm{pr}_2$, and by the i-simplex of $X(\sigma) \subset \mathrm{Map}(\Delta[n], \Delta[i] \times \Delta[n])$ that corresponds to the identity of $\Delta[i] \times \Delta[n]$. The space $|X(\sigma)|$ is contractible by part (b) of Lemma 8, so there are no obstructions to extending h.

The case of k is treated similarly, using the fact that the fibers of a surjective map $\Delta^i \to \Delta^n$ are contractible. $\square$

We can now give the following alternative description of weak equivalences in terms of the geometric realizations:

Proposition 10 *A map $f : (X, s) \to (Y, t)$ is a weak equivalence iff for every point $p \in |N|$, the induced map $|f| : |s|^{-1}(p) \to |t|^{-1}(p)$ is a homotopy equivalence.* gdw

Proof. Suppose f is a weak equivalence, and let $p \in |N|$ be a point. Let σ be the smallest non-degenerate simplex of N such that $p \in |\operatorname{im}(\sigma)|$. Then p is in the relative interior of $|\operatorname{im}(\sigma)|$. By Lemma 9, we have a commutative square

$$\begin{array}{ccc} |X(\sigma)| & \xrightarrow{\sim} & |s|^{-1}(p) \\ \downarrow & & \downarrow{\scriptstyle |f|} \\ |Y(\sigma)| & \xrightarrow{\sim} & |t|^{-1}(p) \end{array}$$

whose rows are homotopy equivalences. The left vertical map is a homotopy equivalence by assumption, and thus so is the right vertical one.

Conversely, if $|f| : |s|^{-1}(p) \to |t|^{-1}(p)$ is an equivalence for every $p \in |N|$, then by the same argument, $|X(\sigma)| \to |Y(\sigma)|$ is an equivalence for every $\sigma \in \mathcal{P}$. It follows that f is a weak equivalence. $\square$

We now show that $J \subset W$.

Lemma 11 *If $\beta \in \mathrm{im}(s_{j'})$ for some $j' \in \{j, j-1\}$, then the morphism* gqj

$$\begin{array}{ccc} \Lambda[k,j] & \hookrightarrow & \Delta[k] \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle\beta} & \\ & N & \end{array} \tag{4}$$

is a weak equivalence.

Proof. By Proposition 10, we need to check that $|\alpha|^{-1}(p) \to |\beta|^{-1}(p)$ is a homotopy equivalence for all $p \in |N|$.

Write $\beta = s_{j'}(\tau)$ for some $\tau \in N_{k-1}$ and $j' \in \{j-1, j\}$. By the simplicial identity $d_j \circ s_{j'} = \mathrm{Id}$, we have a commutative diagram

$$\begin{array}{ccccc} & \Delta[k] & \xrightarrow{\quad\beta\quad} & & N, \\ {\scriptstyle d^j}\nearrow & & \searrow{\scriptstyle s^{j'}} & \nearrow{\scriptstyle\tau} & \\ \Delta[k-1] & \xrightarrow{\quad\mathrm{Id}\quad} & \Delta[k-1] & & \end{array}$$

and so the images of β and $\beta \circ d^j$ agree. Let $F := |\beta|(\Delta^k) = |\beta|(\partial^j \Delta^k)$. Recall that

$$|\Lambda[k,j]| = \Delta^k - (\text{interior of } \Delta^k) - (\text{interior of } \partial^j \Delta^k),$$

and note that the interiors of Δ^k and $\partial^j \Delta^k$ both map to the interior of F. If p is not in the interior of F, then $|\alpha|^{-1}(p) \to |\beta|^{-1}(p)$ is an isomorphism. If p is in the interior of F, then $P := |\beta|^{-1}(p)$ is a polytope, and $P \cap \partial^j \Delta^k$ is one of its facets. We then have

$$\begin{aligned} |\alpha|^{-1}(p) &= P - (\text{interior of } \Delta^k) - (\text{interior of } \partial^j \Delta^k) \\ &= P - (\text{interior of } P) - (\text{interior of one of the facets of } P). \end{aligned}$$

In all cases, $|\alpha|^{-1}(p) \to |\beta|^{-1}(p)$ is a homotopy equivalence, therefore (4) is a weak equivalence. □

Proof of condition *(ii)***.**

The I-injectives are the trivial Kan fibrations. It is well known that trivial Kan fibrations satisfy the right lifting property with respect to all monomorphisms. In particular, they satisfy the right lifting property with respect to J. Given a trivial Kan fibration $f : X \to Y$, then (X, f) is homotopy equivalent to (Y, Id_Y) in $\mathsf{sSets} \downarrow Y$. The above homotopy equivalence then induces homotopy equivalences $X(\sigma) \simeq Y(\sigma)$ for all σ.

Proof of condition *(iii)***.**

We'll need to know which morphisms of the form

$$\begin{array}{ccc} \bigcup\big(\text{some facets of } \Delta[k]\big) & \hookrightarrow & \Delta[k] \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle\beta} & \\ & N & \end{array}$$

are relative J-cell complexes. Given a subset $\mathbf{j} \subset \{0, \ldots, k\}$, we let $\Lambda[k, \mathbf{j}] \subset \Delta[k]$ be the following union of facets

$$\Lambda[k, \mathbf{j}] := \bigcup_{i \notin \mathbf{j}} \partial^i \Delta[k].$$

For example, we have $\Lambda[k, \{j\}] = \Lambda[k, j]$.

Lemma 12 *Let $(\Delta[k], \beta)$ be an object of $\mathsf{sSets}\downarrow N$, and let α be the restriction of β to $\Lambda[k, \mathbf{j}]$.* woi
Let $v_0, \ldots, v_k$ be the vertices of $\Delta[k]$. If there exist $j, j' \in \{0, \ldots, k\}$, $j \in \mathbf{j}$, $j' \notin \mathbf{j}$ such that $\beta(v_j) = \beta(v_{j'})$, then the inclusion ash

$$\begin{array}{ccc} \Lambda[k, \mathbf{j}] & \hookrightarrow & \Delta[k] \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle\beta} & \\ & N & \end{array} \tag{5}$$

is a relative J-cell complex.

Proof. We prove this by induction on the size of $\mathbf{j}$. If $\#\mathbf{j} = 1$, then $\Lambda[k, \mathbf{j}] = \Lambda[k, j]$ for some j. The set of j' such that $\beta(v_j) = \beta(v_{j'})$ forms a subinterval of $\{0, \ldots, k\}$ (this uses Assumption 4). By hypothesis, that interval contains more than just j, so we may pick some $j' = j \pm 1$ in it. If $j' = j - 1$ then $\beta \in \mathrm{im}(s_{j-1})$; if $j' = j + 1$ then $\beta \in \mathrm{im}(s_j)$. In all cases (5) is in J.

Now let's assume that $\#\mathbf{j} > 1$, and let j, j' be as in the hypothesis of the lemma. Given $i \in \mathbf{j}$, $i \neq j$, we may factor the inclusion (5) as ek-s

$$\Lambda[k, \mathbf{j}] \longrightarrow \Lambda[k, \mathbf{j} \setminus i] \longrightarrow \Delta[k]. \tag{6}$$

The map $\Lambda[k, \mathbf{j} \setminus i] \to \Delta[k]$ is a relative J-cell complex by induction.

The first map in equation (6) is a pushout along $\Lambda[k, \mathbf{j}] \cap \partial^i \Delta[k] \to \partial^i \Delta[k]$. The latter can be identified with e-ks

$$\Lambda[k-1, \mathbf{j}'] \longrightarrow \Delta[k-1], \tag{7}$$

where $\mathbf{j}'$ is the image of $\mathbf{j} \setminus i$ under the natural isomorphism $\{0, \ldots, k\} \setminus i \simeq \{0, \ldots, k-1\}$. The images of j, j' in $\{0, \ldots, k-1\}$ satisfy the hypothesis of the lemma. So by induction, the map (7) is a realtive J-cell complex. Both maps in (6) are realtive J-cell complexes, thus so is (5). $\square$

The following lemma allows us to recognize pushouts along maps of the form $\Lambda[n, \mathbf{j}] \to \Delta[n]$.

Lemma 13 *Let B be a simplicial set, let $\sigma : \Delta[n] \to B$ be a non-degenerate simplex, and let $A \subset B$ be a subset of B such that $B = A \cup \mathrm{im}(\sigma)$. Let $\mathbf{j}$ be the set of $j \in \{0, \ldots, n\}$ such that $\sigma(\partial^j \Delta[n]) \not\subset A$, and let $F := \bigcap_{j \in \mathbf{j}} \partial^j \Delta[n]$. Then if the image $\sigma(F)$ is not contained in A, we have* zbr

$$B = A \underset{\Lambda[n, \mathbf{j}]}{\cup} \Delta[n].$$

Proof. Clearly $B = A \cup_{\sigma^{-1}(A)} \Delta[n]$, so it's enough to show that $\Lambda[n, \mathbf{j}] = \sigma^{-1}(A)$. Since $\sigma(\partial^i \Delta[n]) \subset A$ for all $i \notin \mathbf{j}$, we have $\Lambda[n, \mathbf{j}] \subset \sigma^{-1}(A)$. A face $F' \subset \Delta[n]$ not contained in $\Lambda[n, \mathbf{j}]$ necessarily contains F. Since the latter in not in $\sigma^{-1}(A)$, neither can F' be. It follows that $\Lambda[n, \mathbf{j}] = \sigma^{-1}(A)$. $\square$

Recall the two functors R, L from (1), (2), and let Φ be the functor given by

$$\begin{aligned} \Phi : \mathsf{sSets}\downarrow N &\longrightarrow \quad \mathsf{sSets}\downarrow N \\ (X, s) &\mapsto \operatorname*{colim}_{\tau : \Delta[k] \to X} LR(\Delta[k], s \circ \tau). \end{aligned}$$

It is the unique colimit preserving functor that agrees with LR on objects of the form $(\Delta[k], s)$. One should think of Φ as some kind of "subdivision functor". The following lemma gives a concrete description of $\Phi(\Delta[k], s)$.

Lemma 14 *Let $(\Delta[k], s)$ be an object of $\mathsf{sSets}\downarrow N$. Let $F := \mathrm{im}(s)$, and let $v_0, \ldots, v_n$ be the vertices of F. Then $\Phi(\Delta[k], s) = LR(\Delta[k], s)$ can be identified with the join* fhk

$$\mathop{\bigstar}_{i=0}^{n} s^{-1}(v_i) = \Big(\Delta[n] \times \prod_{i=0}^{n} s^{-1}(v_i)\Big) \Big/ \sim . \tag{8}$$

Proof. The values of $R(\Delta[k], s)$ are given by

$$R(\Delta[k], s)(\sigma) = \begin{cases} \prod_{v \in \mathrm{im}(\sigma)} s^{-1}(v) & \text{if} \quad \mathrm{im}(\sigma) \subset F, \\ \emptyset & \text{otherwise.} \end{cases}$$

Putting them into (2) produces the very definition of the join. □

The counit map $LR(X) \to X$ induces a natural transformation $\varepsilon : \Phi X \to X$.

Lemma 15 *For any $X \in \mathsf{sSets}{\downarrow}N$ the map $\varepsilon : \Phi X \to X$ is a weak equivalence.*

Proof. By Lemma 14, it's enough to show that $|\varepsilon| : |\Phi X| \to | X|$ is a fiberwise homotopy equivalence. We show it by constructing an auxiliary natural transformation $\tilde{\varepsilon} : |\Phi X| \to | X|$ which is an isomorphism for every X, and which is fiberwise homotopic to $|\varepsilon|$.

Clearly, as long as it's done in a functorial way, it's enough to construct $\tilde{\varepsilon}$ and the homotopy on objects of the form $\Delta[k] \to N$. Letting $v_0, \ldots, v_n$ be the vertices of $\mathrm{im}(s)$, the map $\tilde{\varepsilon}$ is given by

$$\tilde{\varepsilon} : \left| \bigstar_{i=0}^{n} s^{-1}(v_i) \right| \longrightarrow \Delta^k$$

$$(t, x_0, \ldots, x_n) \mapsto \sum_{i=0}^{n} t_i x_i$$

where $t = (t_0, \ldots, t_n)$ is a point of Δ^n, and $x_i \in |s|^{-1}(v_i) \subset \Delta^k$. The fiberwise homotopy between $\tilde{\varepsilon}$ and $|\varepsilon|$ is the straight line homotopy. □

Notation 16 Given a simplicial set X, and a top dimensional non-degenerate simplex F of X, we shall denote by $X \backslash F \subset X$ the union of all non-degenerate simplices F' of X, $F' \neq F$. not-

Now let's assume that we are given a map $f : A \to B$ which is both an I-cofibration and an element of W. We want to show that f is a J-cofibration. Since f is a retract of the corresponding map $A \cup_\varepsilon (\Phi A \times \Delta[1]) \to B \cup_\varepsilon (\Phi B \times \Delta[1])$, it's enough to show the following lemma.

Lemma 17 *If $f : A \to B$ in $\mathsf{sSets}{\downarrow}N$ is both injective and in W, then each one of the following two inclusions is a J-cofibration:* l:nc

$$A \cup (\Phi A \times \Delta[1]) \xrightarrow{(a)} A \cup (\Phi A \times \Delta[1]) \cup \Phi B \xrightarrow{(b)} B \cup (\Phi B \times \Delta[1]).$$

Proof. (a). It's enough to show that $\Phi f : \Phi A \to \Phi B$ is a J-cofibration. Since ΦB is the colimit of $\Phi A \hookrightarrow \Phi A \cup (\Phi B|_{\mathrm{sk}_0 N}) \hookrightarrow \Phi A \cup (\Phi B|_{\mathrm{sk}_1 N}) \hookrightarrow \cdots$, it's enough to show that each inclusion wab

$$\Phi A \cup (\Phi B|_{\mathrm{sk}_{n-1} N}) \longrightarrow \Phi A \cup (\Phi B|_{\mathrm{sk}_n N}) \tag{9}$$

is a J-cofibration. At this point, we assume without loss of generality that $N = \Delta[n]$. Because of the special form of our functor Φ, we can rewrite (9) as suj

$$E \cup (\Delta[n] \times C) \longrightarrow E \cup (\Delta[n] \times D), \tag{10}$$

where E lives over $\partial\Delta[n]$, and $C \to D$ is a cofibration of simplicial sets. Moreover, by combining Lemma 15 and Proposition 10, we see that $C \to D$ has to be a weak equivalence. Applying Lemma 6 to sSets, we get a new inclusion $C \to D'$, which is a relative $\{(\Delta[k], \Lambda[k, j])\}$-cell complex, and of which $C \to D$ is a retract. The map (10) is therefore a retract of $E \cup (\Delta[n] \times C) \to E \cup (\Delta[n] \times D')$. The latter being obtained by successive pushouts along maps of the form sin

$$(\partial\Delta[n]\times\Delta[k])\cup(\Delta[n]\times\Lambda[k,j])\longrightarrow\Delta[n]\times\Delta[k], \tag{11}$$

it's enough to show that (11) is a relative J-cell complex. This done in part (a) of Lemma 18.

(b). By induction on the simplices of B, it's enough to show that the inclusion pdmu

$$\partial\Delta[k]\cup(\Phi\partial\Delta[k]\times\Delta[1])\cup\Phi\Delta[k]\longrightarrow\Delta[k]\cup(\Phi\Delta[k]\times\Delta[1]) \tag{12}$$

is a relative J-cell complex. Let v_i be as in (8), and let k_i be the dimension of the simplex $s^{-1}(v_i)$. Only one of the non-degenerate k-simplices of $\Phi\Delta[k]$ hits the non-degenerate k-simplex of $\Delta[k]$. The pullback of (12) under the projection prqa

$$\Delta[n]\times\prod_{i=0}^{n}\Delta[k_i]\times\Delta[1]\longrightarrow\Delta[k]\cup\big(\Phi\Delta[k]\times\Delta[1]\big) \tag{13}$$

can thus be written in the form $\partial(\Delta[n]\times\prod_{i=0}^{n}\Delta[k_i]\times\Delta[1])\backslash F$, for an appropriate simplex F of $\Delta[n]\times\prod\Delta[k_i]\times\Lambda[1,0]$. If two distinct simplices of $\Delta[n]\times\prod\Delta[k_i]\times\Delta[1]$ have the same image under (13), they necessarily land in the image of (12). So the pullback square dsp

$$\begin{array}{ccc}
\partial\big(\Delta[n]\times\prod\Delta[k_i]\times\Delta[1]\big)\backslash F & \longrightarrow & \Delta[n]\times\prod\Delta[k_i]\times\Delta[1] \\
\downarrow & & \downarrow \\
\partial\Delta[k]\cup\big(\Phi\partial\Delta[k]\times\Delta[1]\big)\cup\Phi\Delta[k] & \longrightarrow & \Delta[k]\cup\big(\Phi\Delta[k]\times\Delta[1]\big)
\end{array} \tag{14}$$

is also a pushout square. To show that (12) is a relative J-cell complex, it's therefore enough to show that the top row of (14) is one. This is done in part (b) of Lemma 18. □

Lemma 18 *Let $N=\Delta[n]$. Then the following maps in* $\mathsf{sSets}{\downarrow}N$ *are relative J-cell complexes:* nex ih4 ih5

$$(a)\quad(\partial\Delta[n]\times\Delta[k])\cup(\Delta[n]\times\Lambda[k,i])\longrightarrow\Delta[n]\times\Delta[k] \tag{15}$$

where the map to N is the projection.

$$(b)\quad\partial\Big(\Delta[n]\times\prod_{i=1}^{r-1}\Delta[k_i]\times\Delta[1]\Big)\backslash D\longrightarrow\Delta[n]\times\prod_{i=1}^{r-1}\Delta[k_i]\times\Delta[1], \tag{16}$$

where the map to N is the projection, and D is a top dimensional simplex of $\Delta[n]\times\prod\Delta[k_i]\times\Lambda[1,0]$.

Proof. (a) Let us first assume that $i<k$. The top dimensional simplices of $\Delta[n]\times\Delta[k]$ are naturally indexed by the set $\mathbf{w}$ of words in a, b that contain n times the letter a and k times the letter b. Equip $\mathbf{w}$ with the lexicographical order. Given a word $w=w_1w_2\dots w_{n+k}\in\mathbf{w}$, let $F_w\subset\Delta[n]\times\Delta[k]$ denote the corresponding simplex. Let then $S_w\subset\Delta[n]\times\Delta[k]$ be the subset

$$S_w:=\big(\partial\Delta[n]\times\Delta[k]\big)\cup\big(\Delta[n]\times\Lambda[k,i]\big)\cup\textstyle\bigcup_{w'<w}F_{w'}.$$

The map (15) being a composite of inclusions $S_w\hookrightarrow S_w\cup F_w$, it's enough to show that each one of them is a relative J-cell complex. Given $j\in\{0,\dots,n+k\}$, we have

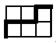
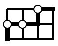
$$\begin{array}{lcl}
\partial^jF_w\subset\partial^r\Delta[n]\times\Delta[k] & \Leftrightarrow & w_j \text{ is the } r\text{th}^1\ a \text{ of } w, \text{ and } w_{j+1} \text{ is the } (r+1)\text{st}^2\ a \text{ of } w.\\
\partial^jF_w\subset\Delta[n]\times\partial^r\Delta[k] & \Leftrightarrow & w_j \text{ is the } r\text{th}^1\ b \text{ of } w, \text{ and } w_{j+1} \text{ is the } (r+1)\text{st}^3\ b \text{ of } w.\\
\exists\, w'<w:\ \partial^jF_w\subset F_{w'} & \Leftrightarrow & j\neq0,\ j\neq n+k,\ w_j=b,\ w_{j+1}=a.
\end{array}$$

([1]Omit this condition if $r=j=0$. [2]Omit this condition if $r=n$, $j=\Sigma k_i$. [3]Omit this condition if $r=k$, $j=\Sigma k_i$.)

The set $\mathbf{j}$ of elements $j \in \{0, \ldots, n+k\}$ such that $\partial^j F_w \not\subset S_w$ is therefore given by

$$\begin{aligned} \mathbf{j} &= \big\{ j \neq 0, n+k \,\big|\, w_j = a, w_{j+1} = b \big\} \cup \big\{ j \,\big|\, \partial^j F_w \subset \Delta[n] \times \partial^i \Delta[k] \big\}, \\ &= \big\{ j \neq 0, n+k \,\big|\, w_j = a, w_{j+1} = b \big\} \cup \left\{ j \,\middle|\, \begin{array}{l} w_j \text{ is the } i\text{th } b \text{ of } w \ \text{(Omit this condition if } i=j=0) \\ w_{j+1} \text{ is the } (i+1)\text{st } b \text{ of } w. \end{array} \right\}, \end{aligned}$$

where the second term has at most one element.

Example: *If $w = abbab$, represented pictorially by* *, and $i = 1$, then $\mathbf{j} = \{1, 2, 4\}$, represented pictorially by* *.*

The non-degenerate simplices of $\Delta[n] \times \Delta[k]$ are naturally indexed by chains in the poset $\{0, \ldots, n\} \times \{0, \ldots, k\}$. For a simplex not to be contained in $(\partial\Delta[n] \times \Delta[k]) \cup (\Delta[n] \times \Lambda[k, i])$, it is necessary and sufficient that the corresponding chain $c \subset \{0, \ldots, n\} \times \{0, \ldots, k\}$ satisfiy $\mathrm{pr}_1(c) = \{0, \ldots, n\}$ and $\mathrm{pr}_2(c) \cup \{i\} = \{0, \ldots, k\}$, where pr_1 and pr_2 are the projections. The chain corresponding to $\bigcap_{j\in\mathbf{j}} \partial^j F_w$ is obtained from that of F_w by deleting the jth elements for all $j \in \mathbf{j}$. It satisfies the above condition, and so

$$\textstyle\bigcap_{j\in\mathbf{j}} \partial^j F_w \not\subset (\partial\Delta[n] \times \Delta[k]) \cup (\Delta[n] \times \Lambda[k, i])$$

The intersection $\bigcap_{j\in\mathbf{j}} \partial^j F_w$ is also not contained in $F_{w'}$ for any $w' < w$, from which it follows that $\bigcap_{j\in\mathbf{j}} \partial^j F_w \not\subset S_w$. By Lemma 13, the inclusion $S_w \hookrightarrow S_w \cup F_w$ is therefore a pushout along mcr

$$\Lambda[n+k, \mathbf{j}] \to \Delta[n+k], \tag{17}$$

and we are reduced to checking that (17) is a relative J-cell complex.

If $w = b^k a^n$, then $i \in \mathbf{j}$, $n \notin \mathbf{j}$, and the corresponding vertices of $\Delta[n+k]$ both map to the 0th vertex of $\Delta[n]$. If $w \neq b^k a^n$, write it as $w = b^r a^s b^t w'$ for some $r \geq 0$ and $s, t > 0$, and some word w' which is either empty or starts by an a. We have $r + s \in \mathbf{j}$, $r + s + t \notin \mathbf{j}$, and the corresponding two vertices of $\Delta[n+k]$ both map to the rth vertex of $\Delta[n+k]$. In all cases, the hypothesis of Lemma 12 is satisfied, and so (17) is a relative J-cell complex.

The case $i = k$ is treated similarly, by using $\big(\partial\Delta[n] \times \Delta[k]\big) \cup \big(\Delta[n] \times \Lambda[k, i]\big) \cup \bigcup_{w' > w} F_{w'}$ instead of S_w. One can also derive it formally from the $i = 0$ case, by using the involution on the category of simplicial sets that exchanges $d^\alpha : \Delta[m-1] \to \Delta[m]$ and $d^{m-\alpha} : \Delta[m-1] \to \Delta[m]$.

(*b*) Letting $k_0 = n$ and $k_r = 1$, we can rewrite (16) as ih6

$$\partial\Big(\prod_{i=0}^{r} \Delta[k_i]\Big)\Big\backslash D \longrightarrow \prod_{i=0}^{r} \Delta[k_i]. \tag{18}$$

The top dimensional simplices of $\prod \Delta[k_i]$ are naturally indexed by the set $\mathbf{w}$ of words in the alphabet $\{a_0, a_1, \ldots, a_r\}$ such that, for each i, the letter a_i occurs exactly k_i times. Equipping that alphabet with the order $a_0 < a_1 < \cdots < a_r$, we get a corresponding lexicographical order on $\mathbf{w}$, denoted again $<$. We also define partial order $\prec$ on $\mathbf{w}$ as follows. Given two words $w, w' \in \mathbf{w}$, let $\pi_{w,w'}$ be the permutation on $\sum k_i$ elements given by

$$\left.\begin{array}{l} w_j \text{ is the } \ell\text{th occurrence of the letter } a_i \text{ in } w \\ w'_{j'} \text{ is the } \ell\text{th occurrence of the letter } a_i \text{ in } w' \end{array}\right\} \Rightarrow \pi_{ww'} : j \mapsto j'.$$

We then have $w \prec w' \overset{\text{def}}{\Longleftrightarrow} \big[\text{if } (i,j) \text{ is an inversion of } \pi_{ww'}, \text{ then } w(i) < w(j)\big]$, where by convention, (i, j) is an inversion of π if $i < j$ and $\pi(i) > \pi(j)$. Observe that $w \prec w'$ implies $w < w'$.

Given $w \in \mathbf{w}$, let $F_w \subset \prod \Delta[k_i]$ denote the corresponding simplex, and let w_0 be the unique element such that F_{w_0} contains D in its boundary. The map (16) can be factored as a composite of inclusions ahtz fsgf

$$\Big(\partial\big(\prod\Delta[k_i]\big)\backslash D\Big)\cup\bigcup_{\substack{v<w\\ v\not\succcurlyeq w_0}}F_v\hookrightarrow\Big(\partial\big(\prod\Delta[k_i]\big)\backslash D\Big)\cup\bigcup_{\substack{v\le w\\ v\not\succcurlyeq w_0}}F_v\qquad(\text{for } w\not\succcurlyeq w_0)\tag{19}$$

$$\Big(\big(\partial\prod\Delta[k_i]\backslash D\big)\cup\bigcup_{v\not\succcurlyeq w_0}F_v\Big)\cup\bigcup_{v>w}F_v\hookrightarrow\Big(\big(\partial\prod\Delta[k_i]\backslash D\big)\cup\bigcup_{v\not\succcurlyeq w_0}F_v\Big)\cup\bigcup_{v\ge w}F_v\tag{20}$$

$$(\text{for } w\succcurlyeq w_0).$$

It's therefore enough to show that each one of them is a relative J-cell complex. The right hand sides of (19), (20) are obtained from the corresponding left hand sides by adding one simplex F_w. To show that (19), (20) are relative J-cell complexes, we'll thus use Lemma 13 to write them as pushouts along maps gjkl

$$\Lambda[\textstyle\sum k_i,\mathbf{j}]\to\Delta[\sum k_i],\tag{21}$$

and then Lemma 12 to show that the latter are relative J-cell complexes.

We first treat the case (19). Since $w\neq w_0$, the simplex D is not contained in F_w. An element $j\in\{0,\dots,\sum k_i\}$ therefore satisfies $\partial^j F_w\subset\partial(\prod\Delta[k_i])\backslash D$ if and ony if $j=0$, $j=\sum k_i$, or $w_j=w_{j+1}$. The face $\partial^j F_w$ is contained in $F_{w'}$ for some $w'<w$ if and only if $w_j>w_{j+1}$. It follows that $\mathbf{j}:=\{j\,|\,\partial^j F_w\not\subset \text{LHS of (19)}\}$ is given by

$$\mathbf{j}=\{j\neq 0,\textstyle\sum k_i\,|\,w_j<w_{j+1}\}.$$

The intersection $\bigcap_{j\in\mathbf{j}}\partial^j F_w$ is not contained in $\partial(\prod\Delta[k_i])$, and not contained in any F_v for $v<w$. It is therefore not contained in the LHS of (19), and by Lemma 13, (19) is a pushout along (21). Clearly, $\mathbf{j}$ is not empty. Pick $j\in\mathbf{j}$, and let $j'>j$ be the smallest value such that $w_{j'+1}=a_0$; if no such value exists, set $j'=\sum k_i$. We have $j\in\mathbf{j}$, $j'\notin\mathbf{j}$, and the corresponding two vertices of F_w map to the same vertex of $\Delta[n]$. By Lemma 12, the map (21) is thus a relative J-cell complex, and therefore so is (19).

We now show that (20) is a relative J-cell complex. Let's first assume that $w\neq w_0$. Once again, $\partial^j F_w\subset\partial(\prod\Delta[k_i])\backslash D$ iff $j=0$, $j=\sum k_i$, or $w_j=w_{j+1}$. And $\partial^j F_w\subset F_{w'}$ for some $w'>w$ iff $w_j<w_{j+1}$. Given such a j, the only other simplex containing $\partial^j F_w$ is $F_{w'}$ for $w'=w_1\dots w_{j-1}w_{j+1}w_jw_{j+2}\dots$ The latter satisfies $F_{w'}\subset$ (LHS of (20)) iff $w'\not\succcurlyeq w_0$, which happens iff $\pi_{w'w_0}(j)<\pi_{w'w_0}(j+1)$, or equivalently $\pi_{ww_0}(j)>\pi_{ww_0}(j+1)$. Since $w\succ w_0$, the condition $\pi_{ww_0}(j)>\pi_{ww_0}(j+1)$ implies $w_j>w_{j+1}$, and so we can write

$$\mathbf{j}=\{j\neq 0,\textstyle\sum k_i\,|\,\pi_{ww_0}(j)>\pi_{ww_0}(j+1)\}.$$

Let $\mathbf{j}':=\{j\neq 0,\sum k_i\,|\,w_j>w_{j+1}\}$. Then $\bigcap_{j\in\mathbf{j}'}\partial^j F_w$ is not contained in $\partial(\prod\Delta[k_i])$, and also not in any F_v for $v>w$. Moreover, since $\mathbf{j}\subset\mathbf{j}'$, the same holds for $\bigcap_{j\in\mathbf{j}}\partial^j F_w$. A simplex F_v containing $\bigcap_{j\in\mathbf{j}}\partial^j F_w$ necessarily satisfies $v\not\succcurlyeq w_0$. It follows that $\bigcap_{j\in\mathbf{j}}\partial^j F_w\not\subset$ (LHS of (20)), and so by Lemma 13, (20) is a pushout along (21). Since $w\neq w_0$, the permutation π_{ww_0} is not the identity, and thus $\mathbf{j}\neq\emptyset$. Pick $j\in\mathbf{j}$, and let $j'<j$ be the largest value such that $w_{j'+1}=a_0$ (if no such value exists, set $j'=0$). We have $j\in\mathbf{j}$, $j'\notin\mathbf{j}$, and the corresponding two vertices of F_w map to the same vertex of $\Delta[n]$. It follows from Lemma 12 that (21) is a relative J-cell complex, and therefore so is (20).

Finally, we treat the case $w=w_0$. The left hand side of (20) can then be rewriten as $(\prod\Delta[k_i]\backslash F_w)\backslash D$, and the right hand side is simply $\prod\Delta[k_i]$. Since $D\subset\prod_{i=0}^{r-1}\Delta[k_i]\times\Lambda[1,0]$, the word w_0 necessarily starts by a_r. We have $D=\partial^0 F_{w_0}$, and so

$$\prod_{i=0}^{r}\Delta[k_i]=\Big(\big(\prod_{i=0}^{r}\Delta[k_i]\backslash F_w\big)\backslash D\Big)\underset{\Lambda[\Sigma k_i,0]}{\cup}\Delta[\textstyle\sum k_i].$$

The simplex $\Delta[\sum k_i]\simeq F_{w_0}\to N$ is in the image of the 0th degeneracy map, and so (20) is a relative J-cell complex. □

This finishes the proof of Theorem 5. □

We now show that the adjoint functors nqy

$$L: \ \mathsf{sSets}^{\mathcal{P}^\circ} \rightleftarrows \mathsf{sSets}{\downarrow}N \ : R \tag{22}$$

introduced in (1), (2) form a Quillen equivalence. The model structure on $\mathsf{sSets}^{\mathcal{P}^\circ}$ is the Bousfield-Kan model structure, where both fibrations and weak equivalences are checked objectwise, and the one on $\mathsf{sSets}{\downarrow}N$ is given by Theorem 5.

Proposition 19 *The adjunction (22) is a Quillen pair. Namely, L preserves cofibrations and acyclic cofibrations, and R preserves fibrations and acyclic fibrations.*

Proof. The condition that L preserve cofibrations and acyclic cofibrations is equivalent to R preserving fibrations and acyclic fibrations. We check the latter.

Let $X \to Y$ be a fibration (an acyclic fibration) in the category $\mathsf{sSets}{\downarrow}N$. Given $\sigma : \Delta[n] \hookrightarrow n$, consider the pullback squares

$$\begin{array}{ccc} X(\sigma) & \longrightarrow & \mathrm{Map}(\Delta[n], X) \\ \downarrow & & \downarrow \\ Y(\sigma) & \longrightarrow & \mathrm{Map}(\Delta[n], Y) \\ \downarrow & & \downarrow \\ \{\sigma\} & \longrightarrow & \mathrm{Map}(\Delta[n], N) \end{array}$$

defining $X(\sigma)$ and $Y(\sigma)$. The top right vertical map is a fibration (an acyclic fibration), and therefore so is the map $X(\sigma) \to Y(\sigma)$. It follows that $R(X) \to R(Y)$ is a fibration (an acyclic fibration). □

Theorem 20 *Given two objects $F \in \mathsf{sSets}^{\mathcal{P}^\circ}$, $X \in \mathsf{sSets}{\downarrow}N$, and a pair of adjoint morphisms $f : F \to R(X)$, $g : L(F) \to X$, we have* auk

$$f \text{ is a weak equivalence in } \mathsf{sSets}^{\mathcal{P}^\circ} \iff g \text{ is a weak equivalence in } \mathsf{sSets}{\downarrow}N. \tag{23}$$

In particular, the pair (22) is a Quillen equivalence.

Proof. By definition, g is a weak equivalence in $\mathsf{sSets}{\downarrow}N$ if and only if $R(g) : RL(F) \to R(X)$ is a weak equivalence in $\mathsf{sSets}^{\mathcal{P}^\circ}$. Letting $\eta : 1 \to RL$ be the unit of the adjunction, we can factor f as

$$F \xrightarrow{\eta_F} RL(F) \xrightarrow{R(g)} R(X).$$

If we new that η_F is a weak equivalence, we would have equivalences (f is a weak equivalence) $\Leftrightarrow$ ($R(g)$ is a weak equivalence) $\Leftrightarrow$ (g is a weak equivalence), as desired. So it's enough to show that η_F is a weak equivalence.

Given $\sigma : \Delta[n] \hookrightarrow N$, we need to show that fnl

$$\eta_F(\sigma) : F(\sigma) \longrightarrow RL(F)(\sigma) = \Big(\coprod_{\tau : \Delta[k] \hookrightarrow N} F(\tau) \times \Delta[k] \,/\!\sim \Big)(\sigma) \tag{24}$$

is a weak equivalence of simplicial sets. Letting Y be the object $(F(\sigma) \times \Delta[n], \sigma \circ \mathrm{pr}_2) \in \mathsf{sSets}{\downarrow}N$, there's an obvious inclusion $Y \to L(F)$, inducing a map $\iota : Y(\sigma) = \mathrm{Map}(\Delta[n], F(\sigma)) \to RL(F)(\sigma)$. The map (24) is then the composite of the "constant function" map $cst : F(\sigma) \to \mathrm{Map}(\Delta[n], F(\sigma))$ with ι

$$\eta_F(\sigma) = \iota \circ cst : |F(\sigma)| \longrightarrow |\,\mathrm{Map}(\Delta[n], F(\sigma))| \longrightarrow |RL(F)(\sigma)|.$$

Let $s : \coprod F(\tau) \times \Delta[k] \,/\!\sim\, \to N$ be the structure map of $L(F)$, and p be point in the relative interior of $|\operatorname{im}(\sigma)|$. Replacing X with $L(F)$ in equation (3), we get the "evaluation at p" map

$$ev_p : \Big| \Big(\coprod_{\tau : \Delta[k] \hookrightarrow N} F(\tau) \times \Delta[k] \,/\!\sim \Big)(\sigma) \Big| \longrightarrow |s|^{-1}(p) = |F(\sigma)| \times \{p\},$$

which is a homotopy equivalence by Lemma 9. The composite

$$|F(\sigma)| \xrightarrow{cst} |\operatorname{Map}(\Delta[n], F(\sigma))| \xrightarrow{\iota} |RL(F)(\sigma)| \xrightarrow{ev_p} |F(\sigma)|$$

is the identity map on $|F(\sigma)|$, and thus (24) is a weak equivalence. □

References

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