# Operator Algebras and Conformal Field Theory of the Discrete Series Representations of Diff($S^1$)

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## Preface

The contents of this dissertation are original, except where explicit reference is made to the work of others.

The original material is the result of my own work and includes nothing which is the outcome of work done in collaboration.

No part of this dissertation has been or is being concurrently submitted for a degree, diploma or qualification at this or any other University.

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## Contents

# Introduction

In this dissertation, we are concerned with a class of continuous projective unitary representations of the group of orientation-preserving diffeomorphisms of the circle $\mathrm{Diff}^+S^1$, and with the associated structures. These representations satisfy a certain *positive energy* condition, which requires that the set of eigenvalues of the infinitesimal generator of the rotation subgroup be bounded below, and that the corresponding eigenspaces be finite-dimensional. Positive energy representations of $\mathrm{Diff}^+S^1$, as well as of loop groups $LG$, are of intrinsic interest by virtue of the fact that no systematic theory of representations exists for infinite-dimensional Lie groups, but are in fact much richer in content because of their close associations with conformal field theory, i.e. conformally-invariant quantum field theory in two dimensions. They occur by "integrating" the unitary highest weight representations of certain infinite dimensional Lie algebras, viz. the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n},$$

and affine Kac-Moody algebras in the case of loop groups. Positive energy representations of $\mathrm{Diff}^+S^1$ are completely reducible and the irreducible ones are characterised by a pair of non-negative numbers, i.e. the highest weight $(h, c)$, where $c$ is the central charge of the corresponding Virasoro algebra, and $h$ is the smallest eigenvalue of the diagonal operator $L_0$. We shall only be concerned with the *discrete series* representations; these are precisely those with central charge $0 < c < 1$, in fact $c = 1 - 6/m(m+1)$, $m = 3, 4, \ldots$, and are distinguished by the fact that, at a fixed central charge $c$, only a finite number of distinct irreducible representations exist. Our dissertation studies the construction of the quantum field theory associated to the discrete series representations at a fixed central charge $c$, and the algebraic structure of the corresponding category of positive energy representations.

The theory of von Neumann algebras — von Neumann's *rings of operators* — was invented in part to provide a framework to study quantum mechanics and group representation theory. Since the foundational work of Murray and von Neumann, the impetus has come from the modular theory of Tomita and Takesaki, the Connes theory of injective factors and, most recently, the subfactor theory of Jones.

It is well-known that von Neumann subfactors can be defined using the Yang-Baxter braiding in certain critical lattice models. The continuum limit of such a model is believed

to be described by a conformal field theory, with the Yang-Baxter braiding preserved in the braiding relations of primary fields in the chiral components of the corresponding conformal field theory; and this has been verified by Tsuchiya and Kanie for a class of models corresponding to the conformal field theories associated to the positive energy representations of the loop group $LG$ with $G = SU(N)$. Following from this, Jones and Wassermann have constructed von Neumann subfactors directly from positive energy representations of loop groups. In our dissertation, we develop the corresponding theory for the discrete series representations of $\mathrm{Diff}^+S^1$.

Originating in attempts to reconcile quantum mechanics with special relativity, quantum field theory has developed in several different ways. Specific models have been extensively studied using path integral techniques and other deep but mathematically ambiguous ideas. At the same time, attempts at a rigorous *constructive* quantum field theory have had only limited success. In a different vein, ***axiomatic*** and *algebraic* quantum field theory study the consequences of a set of axioms — the Wightman and Haag-Kastler axioms respectively — hypothesising the properties of a quantum field theory. Although not without its successes, they suffer from a scarcity of specific models that can be shown to satisfy the axioms. Since few results are model-independent, they have largely remained as languages and frameworks without being actual theories. Historically, the most important models of quantum field theory are defined on Minkowski space-time $\mathbb{R}^{d-1,1}$, or Euclidean space $\mathbb{R}^d$, particularly with $d = 4$, the dimension of physical space-time. Some of the difficulties in a constructive approach to quantum field theory are alleviated by special phenomena in lower dimensions, and by the simplifying features of particular models that only exist in lower dimensions. At the same time, Euclidean space can be replaced by an arbitrary Riemannian manifold. This leads to the study of "toy models" of quantum field theory that illustrate the general principles but side-step the problems. Some of these theories are interesting in their own right, and may model, say, lower-dimensional systems that occur in solid-state physics. Conformal field theories in particular occur as models of two dimensional statistical systems at criticality, and in the string theories of high energy physics.

In the 1980s, a large class of models of conformal field theory were discovered by Belavin, Polyakov and Zamolodchikov, and by Knizhnik and Zamolodchikov, that are closely related to the representation theory of the Virasoro and affine Kac-Moody algebras. Roughly speaking, these models have a factorisation property into left/right chiral components: the "physical space" decomposes as a finite sum of tensor products of unitary highest weight representations $H = \oplus_i V_i \otimes W_i$, and there is an analogous decomposition for the "quantum fields". The work of Jones and Wassermann, and that in our disserta. . is concerned with quantum field theories associated to positive energy representations, and therefore with the chiral components of these conformal field theories (but they are theories in their own right). The transition from what is the purely algebraic representation theory of the Virasoro or affine Kac-Moody algebras to continuous group representations of Diff$^+S^1$ and loop groups $LG$ on Hilbert spaces introduces the ideas and techniques of operator algebras to the study of conformal field theory.

The construction of subfactors fits neatly into the framework of algebraic quantum field theory, especially as developed by Doplicher, Haag and Roberts, but with the circle and Möbius group in place of Minkowski space and the Poincaré group, and the braid group in place of the symmetric group. This is to say that mathematical structures can be constructed from positive energy representations of loop groups and the discrete series representations of Diff$^+S^1$ which satisfy the basic postulates, and constitute concrete models, of Doplicher-Haag-Roberts theory, albeit in the modified context. The appearance of the braid group is accompanied by new phenomena and reflects a deeper connection to the theory of quantum invariants of knots. Proving the basic postulates as theorems involves a range of techniques from conformal field theory, especially the hierarchy of conformal inclusions that realises one conformal field theory as a subtheory of another. In particular, we make essential use of the Goddard-Kent-Olive coset construction of the discrete series representations, and a substantial part of our dissertation is devoted to an analogous construction of the primary fields of the theory. The hierarchical structure is fundamental to the methods developed by Wassermann, which uses a result of Takesaki that relates the modular theory of a von Neumann algebra to that of a subalgebra which is preserved by the modular automorphism group.

The primary fields that we construct are the "quantum fields" of the theory, i.e. they are certain operator-valued distributions. The latter are the basic objects of axiomatic quantum field theory and, in fact, these theories also satisfy the Wightman axioms. Primary fields are characterised in group-theoretic terms and have been classified (in the case of the discrete series representations) by Feigin and Fuchs; our work provides an alternative construction. The unitary highest weight representations of the Virasoro and affine Kac-Moody algebras can be realised in two complementary ways: as the quotient of a Verma module by its maximal submodule — the method of Feigin and Fuchs — and using conformal inclusions. An analogous statement holds for the corresponding primary fields. This dual approach is also found in the representation theory of the compact Lie groups, where Borel-Weil theory provides a uniform construction that works for all representations, and Weyl theory constructs all the representations from a handful of "simpler" ones by the decomposition of tensor products of the latter.

A substantial part of the study of low-dimensional topology is devoted to the construction of invariants of knots and 3-manifolds, using the techniques of Chern-Simons field theory, conformal field theory and quantum groups. In the latter approach, a basic notion is that of a modular category. This is a monoidal category that is further endowed with a braiding, a twist — generalising the notion of commutativity of the usual tensor product — and a compatible duality. In addition, it has the property of finite decompositions: for us, this means that the category is semisimple with a finite number of simple objects, and semisimple objects decompose as a finite sum of simple ones. The prime examples of modular categories come from the representation theory of quantum groups — a class of Hopf algebras — with the deformation parameter $q$ equal to a root of unity. However, Wassermann has defined a tensor product operation on the abelian category of positive energy representations of loop groups at a fixed level (and, with our results, also of discrete series representations of Diff$^+S^1$ at a fixed central charge). More precisely, the positive energy representations have to be regarded as $(M_I, M_I)$-bimodules, where $M_I$ is the von Neumann algebra generated by a local loop group $L_IG$ (or local diffeomorphism group Diff$_I S^1$). This operation of *Connes fusion* originates in Connes' theory of bimodules and fusion, where, exploiting a correspondence between $(A, B)$-bimodules and homomorphisms $B \to A$, a tensor product operation is defined on bimodules ($A$, $B$ are von Neumann algebras). This associates to an $(A, B)$-bimodule $H_1$ and a $(B, C)$-bimodule $H_2$, an $(A, C)$-bimodule $H_1 \boxtimes H_2$. When $M$ is a Type III factor, we show that the category of $(M, M)$-bimodules is a $C^*$ monoidal category. When $M = M_I$, the irreducible positive energy representations are the simple objects of a semisimple monoidal subcategory. In fact, this is a modular category. Part of our dissertation is concerned with the details of this construction for the discrete series representations.

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In Chapter I we introduce positive energy representations of the group of orientation-preserving diffeomorphisms of the circle and obtain their basic analytic properties. These are a class of continuous projective unitary representations that satisfy a certain *positive energy* condition, introduced by Segal; and have been constructed by Goodman and Wallach by integrating unitary highest weight representations of the Virasoro algebra. We show that the irreducible positive energy representations always arise in this manner. Their classification — uniqueness and existence — is therefore determined by the corresponding Lie algebraic classification due to Friedan. Qiu and Shenker; and Goddard. Kent and Olive.

In Chapter II we introduce the primary fields of Belavin. Polyakov and Zamolodchikov associated to the discrete series representations of the Virasoro algebra. and briefly sketch their classification in the Verma module approach of Feigin and Fuchs. The basic concepts of conformal field theory — state-field correspondence. correlation functions. braiding properties, operator product expansions — are developed. We compute some braiding coefficients.

In Chapter III we give a new construction and existence proof of the primary fields associated to the discrete series representations by exploiting the coset construction of Goddard. Kent and Olive. This can be regarded. especially in view of the state-field correspondence. as the natural counterpart of the result for representations. The construction makes manifest certain properties of primary fields that are hard to establish, even mysterious. in the Verma module approach.

In Chapter IV we apply the construction of discrete series primary fields in Chapter III to establish Sobolev inequalities for these operators. These inequalities extend a primary field $\phi : H_1^{fin} \otimes V_{\lambda,\mu}^{fin} \rightarrow H_2^{fin}$ to a jointly continuous linear map $H_1^\infty \otimes V_{\lambda,\mu} \rightarrow H_2^\infty$. The smeared primary field $\phi(f)$ is a densely-defined. closeable operator. At least when $\phi$ has conformal dimension $h_{1,2}$ or $h_{2,2}$. it has bounded closure and satisfies a stronger $L^2$-inequality. We describe the construction of localised fields by smearing with bump functions, and obtain the braiding relations they satisfy when they have disjoint support.

In Chapter V we give a brief exposition of some results of Wassermann's on the von Neumann algebras generated by local diffeomorphism groups acting on the discrete series representations. Together with other results. they imply the construction of quantum field theories satisfying the axioms of Doplicher-Haag-Roberts theory. The method is by descent from tensor products of the $LSU(2)$ theories to the discrete series theories, which are realised as sub-theories by the GKO construction. A key tool is the Tomita-Takesaki-Connes theory of modular operators and Takesaki devissage.

In Chapter VI we introduce Connes fusion of bimodules over a Type III factor $M$. The category $\mathbf{Bimod}_M$ of $(M, M)$-bimodules is a $C^*$ monoidal category. A discrete series representation $H_{h,c}$ can be regarded as an $(M_I, M_I)$-bimodule, where $M_I = \pi_0(\mathrm{Diff}_I S^1)''$. With $M = M_I$, the discrete series representations at a fixed central charge are the simple objects of a semi-simple subcategory $\mathbf{Pos}_c$ of $\mathbf{Bimod}_M$, closed under the tensor product operation. The subcategory $\mathbf{Pos}_c$ has considerable more structure; in fact, it is a modular category. A key ingredient is the construction, from localised fields, of bounded intertwiners

that satisfy braiding relations, following a general prescription due to Wassermann. We also compute the representation ring associated to Connes fusion of the discrete series representations.

## Chapter I

# Positive energy representations of the diffeomorphism group of the circle

We introduce positive energy representations of the group of orientation-preserving diffeomorphisms of the circle, which form the basic objects of study in this dissertation. These are a class of continuous projective unitary representations that satisfy a certain *positive energy* condition, introduced by Segal [Seg]; and have been constructed by Goodman and Wallach [GW] by integrating unitary highest weight representations of the Virasoro algebra. We show that the irreducible positive energy representations always arise in this manner. Their classification — uniqueness and existence — is determined by the corresponding Lie algebraic classification due to Friedan, Qiu and Shenker [FQS]; and Goddard, Kent and Olive [GKO].

## 1. Positive energy representations of Diff$^+S^1$

**1.1. Diffeomorphisms and vector fields.** The group of diffeomorphisms of the circle Diff$S^1$ is topologised as an open subset of $C^\infty(S^1, S^1)$, the smooth maps from the circle to itself, endowed with the $C^\infty$ topology. Taking $S^1 = \mathbb{T} \subset \mathbb{C}$, the complex numbers of unit modulus, this is in turn regarded as a closed subspace of $C^\infty(S^1, \mathbb{R}^2)$. With this topology, Diff$S^1$ is a separable, metrisable topological group. The smooth vector fields on the circle Vect$S^1$ is identified with $C^\infty(S^1, \mathbb{R})$, also with the $C^\infty$ topology. Diff$S^1$ has the structure of a regular, infinite-dimensional Lie group modelled on the Fréchet space Vect$S^1$. It has two connected pieces and the identity component is the group of orientation-preserving diffeomorphisms Diff$^+S^1$. The fundamental group of Diff$^+S^1$ is the integers $\mathbb{Z}$; its universal covering group $\overline{\mathrm{Diff}^+S^1}$ can be identified with the subgroup of diffeomorphisms $\phi : \mathbb{R} \to \mathbb{R}$ of the real line satisfying $\phi(x + 2\pi) = \phi(x) + 2\pi$; and Diff$^+S^1$ with the quotient by the equivalence relation $\phi \sim \phi + 2\pi$. (For details, see [Ham], [Mi], [PS].) We also note the result of Epstein, Herman and Thurston that Diff$^+S^1$ is a simple group; this implies that the image of the exponential map, which is not a local isomorphism, generates the whole group (for the references, see [Mi]).

Diff$^+S^1$ contains the Möbius subgroup

$$\mathrm{Mob} = \{\mathbb{T} \to \mathbb{T}, z \mapsto \frac{az + b}{\bar{b}z + \bar{a}}, |a|^2 - |b|^2 = 1\}, \tag{1.1.1}$$

which is therefore just the semi-simple Lie group $PSU(1,1)$ (this is conjugate to $PSL(2,\mathbb{R})$ in $PSL(2,\mathbb{C})$). Its Lie algebra is spanned by the vector fields $x = \sin\theta \frac{\partial}{\partial\theta}$, $y = \cos\theta \frac{\partial}{\partial\theta}$ and $h = \frac{\partial}{\partial\theta}$, with commutators

$$[h, x] = y, \quad [h, y] = -x, \quad [x, y] = -h. \tag{1.1.2}$$

In particular, Mob contains Rot$S^1$, the subgroup of rotations of the circle. We have $\pi_1(\mathrm{Mob}) = \pi_1(\mathrm{Rot}S^1) = \mathbb{Z}$; let $\overline{\mathrm{Mob}}$ be the universal covering group of Mob. More generally, for each integer $n \geq 1$, the vector fields

$$x = \sin n\theta \, \frac{1}{n} \frac{\partial}{\partial\theta}, \quad y = \cos n\theta \, \frac{1}{n} \frac{\partial}{\partial\theta}, \quad h = \frac{1}{n} \frac{\partial}{\partial\theta} \tag{1.1.3}$$

span an isomorphic Lie algebra. By regarding them as vector fields on $\mathbb{R}/\frac{2\pi}{n}\mathbb{Z}$ instead of $\mathbb{R}/2\pi\mathbb{Z}$, we can see that they generate a subgroup $\mathrm{Mob}_n \subset \mathrm{Diff}^+S^1$ that is isomorphic to an $n$-fold covering of Mob; in particular, $\mathrm{Mob}_n$ is semi-simple.

**1.2. Definition.** *A continuous unitary representation $\varphi : \mathbb{T} \to U(H)$ of the circle on a Hilbert space $H$ is positive energy if the set of eigenvalues of the infinitesimal generator is bounded below, and the corresponding eigenspaces are finite dimensional. A positive energy representation of* Diff$^+S^1$ *is a continuous projective unitary representation $\pi$ :* Diff$^+S^1$ $\to$ *$PU(H)$ that is positive energy as a representation of* Rot$S^1$*, the subgroup of rotations.*

We note the following. The group of unitary operators $U(H)$ on a separable Hilbert space $H$, endowed with the strong (equivalently, weak) operator topology, is a separable, metrisable topological group. Its center, the scalars of unit modulus $\mathbb{T}$, is a closed normal subgroup, and the projective unitary group $PU(H)$ is the quotient $U(H)/\mathbb{T}$.

Since $\mathrm{Mob}_n$ is a semisimple Lie group, $\pi|_{\mathrm{Mob}_n} : \mathrm{Mob}_n \to PU(H)$ admits a lifting to a continuous homomorphism $\sigma_n : \overline{\mathrm{Mob}}_n \to U(H)$ [Ba2]. In particular, $\pi|_{\mathrm{Rot}S^1}$ lifts to a continuous homomorphism $\mathbb{R} \to U(H)$; by Stone's theorem, this is given by $t \mapsto e^{itL_0}$, with $L_0$ self-adjoint. We necessarily have $e^{2\pi i L_0} = e^{2\pi i h}$ for some $h \in \mathbb{R}$ (modulo the integers), since the left-hand-side is mapped to the identity element by the covering homomorphism $p : U(H) \to PU(H)$. Then $e^{it} \mapsto e^{it(L_0 - h)}$ is a continuous homomorphism from $\mathbb{T}$ into $U(H)$, whence $L_0 - h$ is diagonal with integral eigenvalues; the positive energy condition requires that the spectrum of eigenvalues of $L_0$ be bounded below, and that the corresponding eigenspaces be finite-dimensional.

We note that a closed invariant subspace of a positive energy representation $H$ is also a positive energy representation, as is its orthogonal complement. By finite-dimensionality of the $L_0$-eigenspaces, $H$ is completely reducible to a (possibly infinite) direct sum of irreducible pieces.

**1.3. Action of the sl(2, R) subalgebras.** Let $X, Y$ and $-iL_0$ be the infinitesimal generators of the one-parameter subgroups of $\overline{\text{Mob}}$, corresponding respectively to the vector fields $\sin\theta\frac{\partial}{\partial\theta}$, $\cos\theta\frac{\partial}{\partial\theta}$ and $\frac{\partial}{\partial\theta}$. On $C^\infty(\sigma_1)$, the dense subspace of smooth vectors for $\sigma_1$ (see, for example, [Kn]), we have

$$[-iL_0, X] = Y, \quad [-iL_0, Y] = -X, \quad [X, Y] = iL_0. \tag{1.3.1}$$

$C^\infty(\sigma_1)$ contains the Gårding subspace for $\sigma_1$ and is therefore a common core for the infinitesimal generators of $\overline{\text{Mob}}$ (Theorem 3.1 of [Se]). With $L_0$ thus fixed, we also take $h$ to be the smallest eigenvalue of $L_0$. Now $H$ is also completely reducible as a unitary representation of $\overline{\text{Mob}}$ to a direct sum of irreducibles. The irreducible unitary representations of the universal covering group of $SL(2,\mathbb{R})$ have been classified by Pukánszky [Pu]. Since the $L_0$-spectrum is bounded below, the only irreducible representations of $\overline{\text{Mob}}$ that can occur are the trivial one; and the $D_l^+$ $(l > 0)$ representations, which have $L_0$-spectrum $\{l + j : j = 0, 1, 2, \ldots\}$. It follows that $h \geq 0$.

More generally, let $\frac{1}{n}X_n$, $\frac{1}{n}Y_n$ and $-\frac{i}{n}(L_0 + c_n)$, $c_n \in \mathbb{R}$, be the infinitesimal generators of the one-parameter subgroups of $\overline{\text{Mob}}_n$ corresponding to the Lie algebra elements (1.1.3). On $C^\infty(\sigma_n)$, the smooth vectors for $\sigma_n$ and a common core for the infinitesimal generators of $\overline{\text{Mob}}_n$,

$$[-iL_0, X_n] = nY_n, \quad [-iL_0, Y_n] = -nX_n, \quad [X_n, Y_n] = n\,i(L_0 + c_n). \tag{1.3.2}$$

Reasoning as before, we have $h + c_n \geq 0$. Let $L_n = iY_n - X_n$ and $L_{-n} = iY_n + X_n$ on $C^\infty(\sigma_n)$; then $[L_0, L_{\pm n}] = \mp nL_{\pm n}$ and $[L_n, L_{-n}] = 2n(L_0 + c_n)$. We also define $c_0 = 0$, and $c_{-n} = c_n$ for each $n \geq 1$.

**1.4. Remark.** We can pull back the circle extension $p : U(H) \to PU(H)$ by the continuous homomorphism $\pi : \text{Diff}^+S^1 \to PU(H)$ to obtain a topological circle extension of $\text{Diff}^+S^1$ by $\mathbb{T}$ given by

$$\pi^* U(H) = \text{Diff}^+S^1 \times U(H)\,|\,\text{Graph}(\pi) \tag{1.4.1}$$

and a commutative diagram of continuous homomorphisms of topological groups

$$\begin{tikzcd}
\pi^* U(H) \arrow[r, "\overline{\pi}"] \arrow[d, "\pi^* p"'] & U(H) \arrow[d, "p"] \\
\text{Diff}^+S^1 \arrow[r, "\pi"] & PU(H),
\end{tikzcd} \tag{1.4.2}$$

where $\pi^* p$, $\overline{\pi}$ are the projections onto the first and second factors of $\pi^* U(H)$, a closed subgroup of $\text{Diff}^+S^1 \times U(H)$. Since the bundle $p : U(H) \to PU(H)$ has continuous local sections [Ba2], the circle bundle $\pi^* p : \pi^* U(H) \to \text{Diff}^+S^1$ is locally trivial. In fact, if $s : U \to p^{-1}U$ is a local section, we have the homeomorphism $(\pi^* p)^{-1}\pi^{-1}U \to \pi^{-1}U \times \mathbb{T}$, $(g, u) \mapsto (g, u\,s(\pi(g))^{-1})$. The collection of such charts makes $\mathbb{T} \to \pi^* U(H) \to \text{Diff}^+S^1$ a topological circle extension.

**1.5. Sobolev spaces, smooth and finite energy vectors.** Let $A$ be a positive self-adjoint operator on a Hilbert space $H$. For $s \in \mathbb{R}$, let $H^s$ be the completion of $\mathcal{D}(A^s)$, the domain of $A^s$, with respect to the inner product $\langle \xi, \eta \rangle_s = \langle (1 + A)^s \xi, (1 + A)^s \eta \rangle$. For $s \geq 0$, $H^s = (1 + A)^{-s} H$. The spaces $H^s$, $s \in \mathbb{R}$, are the Sobolev spaces, or *scale*, associated to $A$. (See [Ne2].) Let $H^\infty = \cap_{s \geq 0} H^s$ be the corresponding Fréchet space of *smooth* vectors for $A$. If $A$ is diagonal, define $H^{fin}$, the subspace of *finite energy* vectors, as the algebraic direct sum of the eigenspaces. $H^{fin}$ is dense in $H^s$ for each $s \in \mathbb{R}$, and in $H^\infty$.

If $H = L^2(S^1)$, the square-integrable functions on the circle, and $R$ the infinitesimal generator of rotations, then $R$ is diagonal, its eigenvectors are the Fourier modes, and $|R|$ is positive diagonal. The finite energy vectors are trigonometric polynomials, the functions with finite Fourier series. However, the $C^\infty$ topology on the smooth functions $C^\infty(S^1)$ coincides with the Fréchet topology for the norms $||f||_s = \sum_n |f_n|(1 + |n|)^s$, $s \geq 0$, where the $f_n$ are the Fourier modes of $f$ (cf. above).

**1.6. Lemma.** *The subspace of finite energy vectors $H^{fin} \subset \cap_{n=1}^{\infty} C^\infty(\sigma_n)$, and is left invariant by the $L_n$'s.*

*Proof.* Given an integer $n \geq 1$ and $\xi \in H^{fin}$, we are required to show that $\xi \in C^\infty(\sigma_n)$. Let $Z$ be an infinitesimal generator of $\overline{\text{Mob}}_n$. The limit operator

$$\xi \mapsto Z\xi = \lim_{t \to 0} \frac{e^{tZ}\xi - \xi}{t} \tag{1.6.1}$$

is defined on $H^{fin}$ and leaves it invariant (see [Ba1], p. 601; or [Pu], p. 99-100). It follows that $H^{fin} \subset C^\infty(\sigma_n)$ (see [Kn], Lemma 3.1.3). □

**1.7. Lemma.** *For $\xi \in H^{fin}$ and $m \in \mathbf{Z}$,*

$$||L_m\xi|| \leq K(1+|m|)^{\frac{1}{2}}(1+|c_{|m|}|)^{\frac{1}{2}}||\xi||_1. \tag{1.7.1}$$

*$H^{fin}$ is a common core for $-iL_0, X_m, Y_m$ $(m \geq 1)$.*

*Proof.* We use some arguments of Goodman and Wallach [GW]. Fix an integer $n \geq 1$, and let $K$ be an irreducible sub-representation of $\sigma_n : \overline{\mathrm{Mob}}_n \to U(H)$. Then $K^{fin}$ is the algebraic direct sum of one-dimensional $L_0$-eigenspaces. On $K^{fin}$, let

$$Q = (L_0 + c_n)^2 - \frac{1}{2}(L_nL_{-n} + L_{-n}L_n) = qI \quad (q \in \mathbf{R}) \tag{1.7.2}$$

be the Casimir operator. Let $\eta \in K$ be a unit vector in the lowest energy eigenspace, with eigenvalue $\alpha$. Then $L_n\eta = 0$.

$$||L_{-n}\eta||^2 = \langle L_nL_{-n}\eta, \eta\rangle = 2n(\alpha + c_n), \tag{1.7.3}$$

and

$$\begin{aligned} q = \langle Q\eta, \eta\rangle &= ||(L_0 + c_n)\eta||^2 - \frac{1}{2}||L_{-n}\eta||^2 - \frac{1}{2}||L_n\eta||^2 \\ &= (\alpha + c_n)^2 - n(\alpha + c_n). \end{aligned} \tag{1.7.4}$$

Hence for each integer $N \geq 0$,

$$\begin{aligned} ||L_nL_{-n}^N\eta||^2 + ||L_{-n}L_{-n}^N\eta||^2 &= \langle (L_{-n}L_n + L_nL_{-n})\, L_{-n}^N\eta, L_{-n}^N\eta\rangle \\ &= \langle \{2(L_0 + c_n)^2 - 2Q\}\, L_{-n}^N\eta, L_{-n}^N\eta\rangle \\ &= \{2n(\alpha + c_n)(N+1) + (nN)^2\}\, ||L_{-n}^N\eta||^2 \\ &\leq 4(1+n)(1+|c_n|)\, ||L_{-n}^N\eta||_1^2. \end{aligned} \tag{1.7.5}$$

Since the $L_0$-eigenspaces are one-dimensional and mutually orthogonal, for all $\xi \in K^{fin}$,

$$||L_{\pm n}\xi|| \leq 2(1+n)^{\frac{1}{2}}(1+|c_n|)^{\frac{1}{2}}\, ||\xi||_1. \tag{1.7.6}$$

Since the constants in the inequality (1.7.6) are independent of $K$, this establishes the inequality (1.7.1). It follows from Nelson's commutator theorem [Ne1] that $H^{fin}$ is a core for the operators $X_n = \frac{1}{2}(L_{-n} - L_n)$ and $Y_n = \frac{1}{2i}(L_n + L_{-n})$. □

**1.8. Remark.** More generally, if $f$ is a vector field with finite Fourier series, define on $H^{fin}$ the operator $L(f) = \sum_n f_nL_n$. From (1.7.6),

$$||L(f)\,\xi||_s \leq C_s(f)\, ||\xi||_{s+1}, \tag{1.8.1}$$

where $C_s(f) = 2\sum_n |f_n|(1+|n|)^{|s|+\frac{1}{2}}(1+|c_{|n|}|)^{\frac{1}{2}}$, for all $\xi \in H^{fin}$ and $s \in \mathbf{R}$. Then Nelson's commutator theorem applies equally to $L(f)$, which is therefore essentially self-adjoint on $H^{fin}$. When there is no confusion, we use the same symbol for its self-adjoint closure. It also follows from (1.8.1) that the operators $L(f)$ extend to continuous linear operators $H^{s+1} \to H^s$ for each $s \in \mathbf{R}$, and hence also $H^\infty \to H^\infty$.

**1.9. Sums and Lie products of vector fields.** The following theorem of Nelson's [Ne1] is a key result that we shall need.

**1.9.1. Theorem (Nelson).** *Let $\phi_f(t)$, $\phi_g(t)$ be one-parameter subgroups generated by the vector fields $f, g$ respectively. There is an $\epsilon > 0$ such that*

$$\phi_{f+g}(t) = \lim_{n\to\infty}\left[\phi_f(\frac{t}{n})\,\phi_g(\frac{t}{n})\right]^n \tag{1.9.1.1}$$

*in the $C^\infty$ topology, and uniformly for $|t| \leq \epsilon$; and*

$$\phi_{[f,g]}(t) = \lim_{n\to\infty}\left[\phi_g(-\sqrt{\frac{t}{n}}\,)\,\phi_f(-\sqrt{\frac{t}{n}}\,)\,\phi_g(\sqrt{\frac{t}{n}}\,)\,\phi_f(\sqrt{\frac{t}{n}}\,)\right]^n \tag{1.9.1.2}$$

*in the $C^\infty$ topology, and uniformly for $0 \leq t \leq \epsilon$.*

*Proof.* This is a special case of Theorems 1 and 2 in Section 4 of [Ne1], which prove locally uniform convergence when $f, g$ are locally Lipschitz in the former and $C^2$ in the latter, and defined on an open set of a Banach space. For smooth vector fields on the circle, those proofs are easily modified to show convergence in the $C^\infty$ topology. □

**1.10. Proposition.** *Let $f$ be a vector field with finite Fourier series, and $\phi_f(t)$ the one-parameter subgroup it generates. Then $\pi(\phi_f(t)) = p(e^{itL(f)})$.*

*Proof.* The Proposition is, by definition, true for $f$ in the linear span of $\frac{\partial}{\partial\theta}$, $\sin n\theta\frac{\partial}{\partial\theta}$ and $\cos n\theta\frac{\partial}{\partial\theta}$, for each integer $n \geq 1$. It therefore suffices to show that if the Proposition holds for the vector fields $f$ and $g$, then it also holds for the vector field $f+g$. We recall that the homomorphisms $\pi$ and $p$ are continuous (see § 1.4). By the first assertion of Nelson's theorem (Theorem 1.9.1),

$$\begin{aligned} \pi(\phi_{f+g}(t)) &= \lim_{n\to\infty}\left[\pi(\phi_f(\frac{t}{n}))\,\pi(\phi_g(\frac{t}{n}))\right]^n \\ &= \lim_{n\to\infty}\left[p(e^{i\frac{t}{n}L(f)})\,p(e^{i\frac{t}{n}L(g)})\right]^n \\ &= \lim_{n\to\infty} p(\left[e^{i\frac{t}{n}L(f)}\,e^{i\frac{t}{n}L(g)}\right]^n). \end{aligned} \tag{1.10.1}$$

Since $H^{fin}$ is a core for $L(f)$, $L(g)$ and $L(f+g)$, on which $L(f+g) = L(f) + L(g)$, the Trotter product formula (see [Nel], Section 8, Theorem 6) applies and

$$\lim_{n\to\infty}\left[e^{i\frac{t}{n}L(f)}e^{i\frac{t}{n}L(g)}\right]^n = e^{itL(f+g)}. \tag{1.10.2}$$

The result follows by continuity of $p$. □

**1.11. Theorem.** *On $H^{fin}$,*

$$[L_m, L_n] = (m-n)\,L_{m+n} + \frac{c}{12}(m^3 - m)\,\delta_{m+n,0} \tag{1.11.1}$$

*for some $c \in \mathbb{R}$.*

*Proof.* Let $f, g$ be vector fields with finite Fourier series. By the second part of Nelson's theorem (Theorem 1.9.1) and the continuity of $\pi$,

$$\pi(\phi_{[f,g]}(t)) = \lim_{n\to\infty}\left[\pi(\phi_g(-\sqrt{\frac{t}{n}}))\,\pi(\phi_f(-\sqrt{\frac{t}{n}}))\,\pi(\phi_g(\sqrt{\frac{t}{n}}))\,\pi(\phi_f(\sqrt{\frac{t}{n}}))\right]^n. \tag{1.11.2}$$

By Proposition 1.10, this is

$$p(e^{itL([f,g])}) = \lim_{n\to\infty}\left[p(e^{-i\sqrt{\frac{t}{n}}L(g)})\,p(e^{-i\sqrt{\frac{t}{n}}L(f)})\,p(e^{i\sqrt{\frac{t}{n}}L(g)})\,p(e^{i\sqrt{\frac{t}{n}}L(f)})\right]^n. \tag{1.11.3}$$

So there exists a sequence $\mu_n(t) \in \mathbb{T}$ such that

$$\begin{aligned} e^{itL([f,g])} &= \lim_{n\to\infty}\ \mu_n(t)\left[e^{-i\sqrt{\frac{t}{n}}L(g)}\,e^{-i\sqrt{\frac{t}{n}}L(f)}\,e^{i\sqrt{\frac{t}{n}}L(g)}\,e^{i\sqrt{\frac{t}{n}}L(f)}\right]^n \\ &= \mu(t)\ \lim_{j\to\infty}\left[e^{-i\sqrt{\frac{t}{n_j}}L(g)}\,e^{-i\sqrt{\frac{t}{n_j}}L(f)}\,e^{i\sqrt{\frac{t}{n_j}}L(g)}\,e^{i\sqrt{\frac{t}{n_j}}L(f)}\right]^{n_j}, \end{aligned} \tag{1.11.4}$$

where we have passed to a convergent subsequence $\mu_{n_j}(t) \to \mu(t)$. We cannot apply the Lie product formula (see [Nel], Section 8, Theorem 7) to the right-hand-side since it is not known that $[L(f), L(g)]$ is essentially self-adjoint; instead, we expand each side in powers of $\sqrt{t}$. We note that, if $f$ is a vector field with finite Fourier series and $\xi \in H^{fin}$, the map $t \mapsto e^{itL(f)}\xi$ is smooth since $L(f)$ leaves $H^{fin}$ invariant. For $\xi \in H^{fin}$, the left-hand-side of (1.11.4) is

$$\xi + it\,L([f,g])\,\xi + o(t), \tag{1.11.5}$$

and the right-hand-side

$$\mu(t)\ \{\,\xi + t\,[iL(g), iL(f)]\,\xi + o(t)\,\}. \tag{1.11.6}$$

Here, we note that the equality (1.11.4) guarantees convergence of its right-hand-side. By moving $\mu(t)$ to the left-hand-side, and $e^{itL([f,g])}$ to the right-hand-side, of (1.11.4), we see that $\mu(t) = 1 + t\,\mu'(0) + o(t)$. We deduce that

$$[iL(g), iL(f)] = iL([f,g]) - \mu'(0) \tag{1.11.7}$$

on $H^{fin}$. The Theorem follows from the elementary classification [Seg] of 2-cocycles of the Witt algebra $[L_m, L_n] = (m-n)\,L_{m+n}$, the complexification of the Lie algebra of vector fields with finite Fourier series. □

**1.12. Remark.** It follows from the previous theorem that

$$2m\,c_m \ = \ \frac{c}{12}(m^3 - m) \tag{1.12.1}$$

for some $c \in \mathbb{R}$, and for each integer $m \geq 1$. Since $h + c_m \geq 0$ for all $m$, we necessarily have $c \geq 0$. Substituting (1.12.1) in (1.8.1), we obtain

$$\|L(f)\,\xi\|_s \leq K\,\|f\|_{|s|+\frac{3}{2}}\,\|\xi\|_{s+1} \tag{1.12.2}$$

for $\xi \in H^{fin}$ and $f$ a vector field with finite Fourier series. By continuity, we can define operators $L(f) = \sum_n f_n\,L_n$ on $H^{fin}$, for arbitrary vector fields $f$. Just as before, by Nelson's commutator theorem, $L(f)$ is essentially self-adjoint and we use the same symbol for its closure when there is no confusion. In particular, $L(f)$ extends by continuity to a linear map $H^\infty \to H^\infty$. The linear map $\mathrm{Vect}S^1 \otimes H^\infty \to H^\infty$, $f \otimes \xi \mapsto -iL(f)\,\xi$, is jointly continuous and defines a projective representation of $\mathrm{Vect}S^1$ on $H^\infty$.

**1.13. Proposition.** *If $\phi_f(t)$ is the one-parameter subgroup generated by a vector field $f$, then $\pi(\phi_f(t)) = p(e^{itL(f)})$.*

*Proof.* By Proposition 1.10, this holds for the dense subspace of vector fields with finite Fourier series. So let $f_n \to f$ be a convergent sequence of vector fields with finite Fourier series. For each $t \in \mathbb{R}$, the exponential map $\mathrm{Vect}S^1 \to \mathrm{Diff}^+S^1$, $g \mapsto \phi_g(t)$, is continuous (even smooth; see [Mi]), so that $\phi_{f_n}(t) \to \phi_f(t)$ in $\mathrm{Diff}^+S^1$. By (1.12.1),

$$\|L(f_n)\,\xi - L(f)\,\xi\| \leq K\,\|f_n - f\|_{\frac{3}{2}}\,\|\xi\|_1 \to 0 \tag{1.13.1}$$

for $\xi \in H^{fin}$. Since $H^{fin}$ is a common core for $L(f)$ and the $L(f_n)$, we have $L(f_n) \to L(f)$ in the strong resolvent sense ([RS], Theorem VIII.25). This is equivalent, by a theorem of Trotter ([RS], Theorem VIII.21), to the statement that $e^{itL(f_n)} \to e^{itL(f)}$ in $U(H)$. Hence

$$\pi(\phi_f(t)) = \lim_{n\to\infty}\,\pi(\phi_{f_n}(t)) = \lim_{n\to\infty}\,p(e^{itL(f_n)}) = p(e^{itL(f)}), \tag{1.13.2}$$

which proves the result. □

**1.14. Adjoint action of the diffeomorphism group.** If $\phi \in \mathrm{Diff}^+S^1$ and $f \in \mathrm{Vect}S^1$, let $Ad(\phi)f$ denote the adjoint action of $\phi$ on $f$.

$$Ad(\phi)f = (\phi^{-1})'^{-1} f \circ \phi^{-1}. \tag{1.14.1}$$

**1.14.1. Claim.** *For $\phi \in \mathrm{Diff}^+S^1$ and $f \in \mathrm{Vect}S^1$, we have the operator identity*

$$\pi(\phi)\overline{L(f)}\pi(\phi)^* = \overline{L(Ad(\phi)f)} + b(\phi, f) \tag{1.14.1.1}$$

*for some $b(\phi, f) \in \mathbb{R}$. If $\phi \in$ Mob. then $b(\phi, f) = 0$, i.e. the adjoint action of the Möbius subgroup is "anomaly-free".*

*Proof.* Since the image of the exponential map $\exp : \mathrm{Vect}S^1 \to \mathrm{Diff}^+S^1$ generates $\mathrm{Diff}^+S^1$, it is sufficient to prove the first assertion for $\phi = \phi_g(t)$, the one-parameter subgroup generated by a vector field $g$. Then

$$\begin{aligned} p(e^{itL(g)}\, e^{isL(f)}\, e^{-itL(g)}) &= \pi\,(\phi_g(t)\,\phi_f(s)\,\phi_g(-t)) \\ &= \pi(\phi_{Ad(\phi_g(t))f}(s)) \\ &= p(e^{isL(Ad(\phi_g(t))f)}). \end{aligned} \tag{1.14.1.2}$$

so that

$$e^{itL(g)}e^{isL(f)}e^{-itL(g)} = \lambda(s)\, e^{isL(Ad(\phi_g(t))f)} \tag{1.14.1.3}$$

for some $\lambda(s) \in \mathbb{T}$ depending on $tg$ and $f$. It is easy to see that $s \mapsto \lambda(s)$ is a continuous homomorphism $\mathbb{R} \to \mathbb{T}$, so that $\lambda(s) = e^{is\, c(tg, f)}$ for some $c(tg, f) = t\, c(g, f) \in \mathbb{R}$. The first claim follows from the 1-1 correspondence between self-adjoint operators and strongly continuous homomorphisms $\mathbb{R} \to U(H)$ in Stone's theorem. Now note that $c(g, f)$ is just the Lie algebra cocycle. Let $\xi \in H^\infty$ and expand each side of the equality

$$e^{itL(g)}\, L(f)\, e^{-itL(g)}\,\xi \;=\; L(Ad(\phi_g(t))f)\,\xi + t\, c(g, f)\,\xi \tag{1.14.1.4}$$

in powers of $t$, using $L(h)\, H^\infty \subset H^\infty$, $h \in \mathrm{Vect}S^1$. On the left, we have

$$L(f)\,\xi - it\,[L(f), L(g)]\,\xi + o(t); \tag{1.14.1.5}$$

and on the right,

$$L(f)\,\xi + t\, L([f, g])\,\xi + t\, c(g, f)\,\xi + o(t). \tag{1.14.1.6}$$

It follows that $[L(f), L(g)] = i\, L([f, g]) + i\, c(g, f)$, with

$$c(g, f) = \frac{c}{24\pi}\int_0^{2\pi} \{\, f''' + f' \,\}\, g \; d\theta. \tag{1.14.1.7}$$

In particular, $c(g, f) = 0$ when either of the vector fields $f, g$ is in the Lie algebra of the Möbius group. The second claim follows. □

**1.15. Proposition.** *For each integer $n \geq 0$ and $\phi \in \mathrm{Diff}^+S^1$, $\pi(\phi)\, H^n = H^n$. The map $\mathrm{Diff}^+S^1 \times H^n \to H^n/\mathbb{T}$ is jointly continuous.*

*Proof.* The proof proceeds as in the case of loop groups [Wa2]. The case $n = 0$ is clear, so let $n \geq 1$. The operator $[1 + L_0]^n$ is positive diagonal with finite-dimensional eigenspaces, and has domain $H^n = [1 + L_0]^{-n} H$. For $\phi \in \mathrm{Diff}^+S^1$, let $A = \pi(\phi)^*\, L_0\, \pi(\phi) = L(g) + a(\phi)$, where $g = \phi'^{-1}\frac{\partial}{\partial\theta}$ and $a(\phi) \in \mathbb{R}$. Then $[1 + A]^n$ is positive diagonal with finite-dimensional eigenspaces, and has domain

$$\pi(\phi)^*\, H^n = \{\xi \in H : \|[1 + A]^n\,\xi\| < \infty\}. \tag{1.15.1}$$

By Remark 1.8, $A$ restricts to a bounded operator from $H^s$ to $H^{s-1}$, for each $s \in \mathbb{R}$; hence $[1 + A]^n$ restricts to a bounded operator from $H^n$ to $H$, whence $H^n \subset \pi(\phi)^*\, H^n$. So $\pi(\phi)\, H^n = H^n$, proving the first assertion. Claim: the map $\phi \mapsto a(\phi)$ is continuous. Let $\phi_m \to \phi$ in $\mathrm{Diff}^+S^1$, and choose unitary operators such that $\pi(\phi_m) \to \pi(\phi)$ in $U(H)$; let $g_m = \phi_m'^{-1}\frac{\partial}{\partial\theta}$, so that $\pi(\phi_m)\, L_0\, \pi(\phi_m)^* = L(g_m) + a(\phi_m)$, and $g_m \to g$ in $\mathrm{Vect}S^1$. Let $\xi \in H^\infty$. Then the sequence of vectors

$$a(\phi_m)\,[1 + L_0]^{-1}\,\pi(\phi_m)\,\xi \;=\; \pi(\phi_m)\,\xi - [1 + L_0]^{-1}\,\pi(\phi_m)\,[1 + L(g_m)]\,\xi \tag{1.15.2}$$

in $H$ is norm-convergent to

$$a(\phi)\,[1 + L_0]^{-1}\,\pi(\phi)\,\xi \;=\; \pi(\phi)\,\xi - [1 + L_0]^{-1}\,\pi(\phi)\,[1 + L(g)]\,\xi; \tag{1.15.3}$$

with $[1 + L_0]^{-1}\,\pi(\phi_m)\,\xi$ norm-convergent to $[1 + L_0]^{-1}\,\pi(\phi)\,\xi$. It follows that $a(\phi_m) \to a(\phi)$, and proves the claim. For $\xi \in H^n$, we have

$$\begin{aligned} \|\pi(\phi)\,\xi\|_n &= \|\pi(\phi)^*\,[1 + L_0]^n\,\pi(\phi)\,\xi\| \\ &= \|[1 + A]^n\,\xi\| \\ &\leq \sum_{r=0}^{n} \binom{n}{r}\, \|A^r\,\xi\| \\ &\leq K(\phi)\,\|\xi\|_n, \end{aligned} \tag{1.15.4}$$

where $\phi \mapsto K(\phi)$ is continuous. Here we use the fact that, for $\eta \in H^{s+1}$,

$$\|A\,\eta\|_s \;=\; \|\,[L(g) + a(\phi)]\,\eta\|_s \;\leq\; K\,\|g\|_{|s|+\frac{3}{2}}\,\|\eta\|_{s+1} + |a(\phi)|\,\|\eta\|_s \tag{1.15.5}$$

and the continuity of $\phi \mapsto ||g||_{|s|+\frac{3}{2}}$ and $\phi \mapsto a(\phi)$. We now prove the second assertion. Let $\phi_m \to \phi$ in $\mathrm{Diff}^+S^1$, $\pi(\phi_m) \to \pi(\phi)$ in $U(H)$ as before: and let $\xi_k \to \xi$ in $H^n$. It is sufficient to consider the case when $\phi$ is the identity and $\pi(\phi) = I$, in which case $L(g) = L_0$ and $a(\phi) = 0$. We have

$$\begin{aligned} ||\pi(\phi_m)\xi_k - \xi||_n &\leq ||\pi(\phi_m)(\xi_k - \xi)||_n + ||\pi(\phi_m)\xi - \xi||_n \\ &\leq K(\phi_m)\,||\xi_k - \xi||_n + ||\pi(\phi_m)\xi - \xi||_n. \end{aligned} \tag{1.15.6}$$

Since $\phi_m \to \phi$, the $K(\phi_m)$ are uniformly bounded and $K(\phi_m)\,||\xi_k - \xi||_n \to 0$. Moreover,

$$\begin{aligned} ||\pi(\phi_m)\xi - \xi||_n &= ||\pi(\phi_m)^*[1 + L_0]^n \pi(\phi_m)\xi - \pi(\phi_m)^*[1 + L_0]^n \xi|| \\ &\leq ||[1 + L(g_m) + a(\phi_m)]^n \xi - [1 + L_0]^n \xi|| \\ &\quad + ||[1 - \pi(\phi_m)^*][1 + L_0]^n \xi||. \end{aligned} \tag{1.15.7}$$

Since $\pi(\phi_m) \to I$, the second term converges to zero. Since $a(\phi_m) \to 0$ and $g_m \to \frac{\partial}{\partial r}$, the first term also converges to zero. It follows that $||\pi(\phi_m)\xi_k - \pi(\phi)\xi||_n \to 0$. □

## 2. The discrete series of representations of Diff⁺S¹

**2.1. Representations of the Virasoro algebra.** A positive energy representation $\pi : \mathrm{Diff}^+S^1 \to PU(H)$ defines a pair of numbers $(h, c)$, where $h \geq 0$ is the lowest eigenvalue of $L_0$, and $c \geq 0$ is the central charge of the Virasoro algebra $\mathfrak{Vir} = \sum_{n\in\mathbb{Z}} \mathbb{C}L_n \oplus \mathbb{C}C$,

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{C}{12}(m^3 - m)\delta_{m+n,0}, \tag{2.1.1}$$

with $C$ a central element.

A representation $\varphi : \mathfrak{Vir} \to \mathrm{End}V$, if constructed from a positive energy representation of $\mathrm{Diff}^+S^1$ (Theorem 1.11), would have the following properties: (a) $V$ is a locally-finite graded vector space, i.e. $V = \sum_{n\in\mathbb{Z}} V(n)$ (algebraic direct sum) with each subspace $V(n)$ finite-dimensional; such that $V(n) = 0$ for $n > 0$ and $V(0) \neq 0$; (b) $\varphi$ is a graded homomorphism, with the natural grading on $\mathfrak{Vir}$; in particular, the $V(n)$ are eigenspaces for $L_0$ (with lowest eigenvalue $h \in \mathbb{R}$); (c) the central element $C \in \mathfrak{Vir}$ acts by scalar multiplication (by $c \in \mathbb{R}$, the central charge); (d) The representation is unitary in the sense that there is a contravariant inner product on $V$, i.e. an inner product $\langle\, .\, ,\, .\,\rangle$ satisfying $\langle L_n\xi, \eta\rangle = \langle \xi, L_{-n}\eta\rangle$. With these conditions, we necessarily have $h, c \geq 0$.

Let $V, W$ be representations of $\mathfrak{Vir}$ satisfying (a)–(d). They are isomorphic, $V \cong W$, if there is a linear isomorphism $T : V \to W$ that intertwines the action of $\mathfrak{Vir}$. The following are immediate: (i) Let $U$ be a sub-representation of $V$. Then $U$ satisfies the conditions (a)–(d). $U^\perp$, its orthogonal complement in $V$, is a sub-representation. If $U$ is irreducible, it is generated by a vector $\xi \in V$, an eigenvector of $L_0$, satisfying $L_n\xi = 0$ for all $n \geq 1$. Conversely, every such vector $\xi$ generates an irreducible sub-representation. In particular, if $V$ is irreducible, then $\dim V(0) = 1$. $V$ is completely reducible to a (possibly infinite) direct sum of (mutually orthogonal) irreducible sub-representations. (ii) $V, W$ are not isomorphic if they correspond to distinct values of $(h, c)$; (iii) For each pair $(h, c) \in \mathbb{R}^2_+$, there is up to isomorphism at most one irreducible representation satisfying (a)–(d) with lowest $L_0$-eigenvalue $h$ and central charge $c$; such a representation is called a *unitary highest weight* representation, and $(h, c)$ the corresponding highest weight. To be sure, we have the vector space decomposition $\mathfrak{Vir} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, where $\mathfrak{n}_\pm = \sum_{n\geq 1} \mathbb{C}L_{\pm n}$, $\mathfrak{h} = \mathbb{C}L_0 \oplus \mathbb{C}C$, and the Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$, with respect to which the lowest energy vectors of an irreducible representation satisfying (a)–(d) are precisely the primitive vectors.

**2.2. Lemma.** *Let $\pi : \mathrm{Diff}^+S^1 \to PU(H)$ be a positive energy representation. The following are equivalent:*

*(i) $H$ is irreducible;*

*(ii) $H^{fin}$ is irreducible as a representation of the Virasoro algebra.*

*Proof.* Clearly, (ii) ⇒ (i). We prove the contrapositive of (i) ⇒ (ii). Let $K^{fin}$ be a proper $\mathfrak{Vir}$-submodule of $H^{fin}$ and $L = K^{fin\perp}$, the proper closed subspace orthogonal to $K^{fin}$. To show that $\pi(\phi)L \subset L$ for all $\phi \in \mathrm{Diff}^+S^1$, it is sufficient to show it for $\phi = \phi_f(t)$, the one-parameter subgroup generated by a vector field $f$. From the representation theory of semisimple Lie groups, we know that $L$ must be invariant under $\mathrm{Mob}_n$, for each $n$. Equivalently,

$$\pi(\phi_f(t))\,L \subset L \tag{2.2.1}$$

if $f$ is in the Lie algebra of $\mathrm{Mob}_n$, for all $n$. Now $L$ is closed in $H$. So the first assertion of Nelson's theorem (Theorem 1.9.1) implies that (2.2.1) also holds for all vector fields $f$ with finite Fourier series. The latter are dense in $\mathrm{Vect}S^1$, and if $f_n \to f$ in $\mathrm{Vect}S^1$, then $\phi_{f_n}(t) \to \phi_f(t)$ in $\mathrm{Diff}^+S^1$. Hence (2.2.1) holds for all $f \in \mathrm{Vect}S^1$, and $L$ is a proper sub-representation of $H$. □

**2.3. Remark.** If $H, K$ are positive energy representations of $\mathrm{Diff}^+S^1$, we say that they are unitarily equivalent, $H \cong K$, if there is a unitary map $U : H \to K$ that intertwines the $\mathrm{Diff}^+S^1$-actions, by which we mean that $U e^{itL(f)} U^* = e^{itL(f)}$ for all $f \in \mathrm{Vect}S^1$. Claim:

a positive energy representation $H$ of $\mathrm{Diff}^+S^1$ is characterised up to unitary equivalence by the representation of $\mathfrak{Vir}$ on the subspace of finite energy vectors $H^{fin}$. By complete reducibility, it is sufficient to consider the case when $H$ is irreducible.

So suppose that $H, K$ are irreducible, and $T : H^{fin} \to K^{fin}$ is an isomorphism of representations of $\mathfrak{Vir}$. Up to a normalisation, $T$ must be an isometry (since, up to normalisation, there is a unique contravariant inner prduct on a unitary highest weight representation) and hence extends by continuity to a unitary map $T : H \to K$. It is easy to see that $T H^\infty = K^\infty$, and that $T$ intertwines the actions of $\mathrm{Vect}\,S^1$ on the smooth vectors. Hence, for $\xi \in H^\infty$,

$$\frac{d}{dt} e^{-itL(f)} T e^{itL(f)} \xi = 0. \qquad (2.3.1)$$

Then $T$ intertwines the $\mathrm{Diff}^+S^1$ actions, and provides a unitary equivalence $T : H \to K$. Conversely, if $T : H \to K$ is a unitary equivalence, its restriction to the finite energy vectors provides an isomorphism $T : H^{fin} \to K^{fin}$ of representations of $\mathfrak{Vir}$. In summary, $H \cong K$ if and only if $H^{fin} \cong K^{fin}$.

**2.4. Classification of the irreducible representations.** Claim: the classification of the irreducible positive energy representations of $\mathrm{Diff}^+S^1$ reduces to that of the unitary highest weight representations of the Virasoro algebra. The latter is a well-known result of Friedan, Qiu and Shenker [FQS]; and Goddard, Kent and Olive [GKO].

**2.4.1. Theorem (Friedan-Qiu-Shenker).** *If $V_{h,c}$ is a unitary highest weight representation of $\mathfrak{Vir}$ with highest weight $(h, c)$, we either have*

$$c \geq 1, \quad h \geq 0; \qquad (2.4.1.1)$$

*or there are integers $m = 3, 4, \ldots$, $p = 1, 2, \ldots, m-1$, $q = 1, 2, \ldots, p$, such that*

$$c = 1 - \frac{6}{m(m+1)}, \quad h = \frac{[p(m+1) - qm]^2 - 1}{4m(m+1)}. \qquad (2.4.1.2)$$

This theorem provides the necessary conditions for the existence of unitary highest weight representations. It follows from a detailed analysis of the zeros of the Kac determinant in a Verma module approach. This approach provides a uniform construction of irreducible highest weight modules as the quotient of a Verma module by its maximal submodule (see Chapter II); unitarity is not manifest, but must be checked using the Kac determinant. In the case $c \geq 1$, $h \geq 0$, the unitarity of this construction does follow from the FQS

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analysis. (See [KR] for an exposition.) For $0 < c < 1$, the necessary conditions (2.4.1.2) were shown by Goddard, Kent and Olive [GKO] to be also sufficient conditons, using a manifestly unitary *coset construction*. We postpone a description of this until Chapter III. The unitary highest weight representations with central charge $0 < c < 1$ constitute the *discrete series* of representations.

**2.4.2. Classification of irreducible positive energy representations.** The coset construction of the unitary highest weight representations of $\mathfrak{Vir}$ also constructs them as positive energy representations of $\mathrm{Diff}^+S^1$ (see Chapter III). These discrete series representations are clearly *all* the irreducible positive energy representations of $\mathrm{Diff}^+S^1$ with central charge $0 < c < 1$, up to unitary equivalence; and, in this dissertation, we shall only be concerned with them. More generally, however, a result of Goodman and Wallach [GW] guarantees that every unitary highest weight representation of the Virasoro algebra integrates to a positive energy representation of the diffeomorphism group of the circle.

## Chapter II

# Primary fields associated to the discrete series representations

We introduce the primary fields of Belavin, Polyakov and Zamolodchikov [BPZ] associated to the discrete series representations (at a fixed central charge) of the Virasoro algebra, and briefly sketch their classification in the Verma module approach of Feigin and Fuchs [FF1]–[FF3]. The basic concepts of conformal field theory — state-field correspondence, correlation functions, braiding properties, operator product expansions — are developed. In the Verma module approach, singular vectors or, equivalently, the BPZ differential equations play a central role. In particular, we use them to compute some braiding coefficients. Some technical difficulties remain unresolved in this approach, but these can be overcome using the coset construction of primary fields in Chapter III.

## 1. Definition of a primary field

**1.1. The space of densities.** For $\lambda, \mu \in \mathbb{R}$, we define the space of densities

$$V_{\lambda,\mu} = \{f(\theta)\, e^{i\mu\theta}\, (d\theta)^{\lambda} : f \in C^{\infty}(S^1)\}. \tag{1.1.1}$$

The universal covering of the diffeomorphism group of the circle acts on $V_{\lambda,\mu}$ by reparametrisations, $\theta \mapsto \phi^{-1}(\theta)$. We identify $V_{\lambda,\mu}$ with $C^{\infty}(S^1)$, with a jointly continuous action of $\overline{\mathrm{Diff}^{+}S^1}$ given by

$$(\phi, f) \mapsto f^{\phi} = e^{i\mu(\phi^{-1}(\theta)-\theta)}\, (\phi^{-1})'^{\lambda}\, f \circ \phi^{-1}. \tag{1.1.2}$$

We remark in passing that the linear map $V_{\lambda,\mu+1} \to V_{\lambda,\mu}$, $f \mapsto e^{i\theta} f$, intertwines the action of $\overline{\mathrm{Diff}^{+}S^1}$; and that the adjoint action of $\mathrm{Diff}^{+}S^1$ on $\mathrm{Vect}\, S^1$ identifies the latter with $V_{\lambda,\mu}$, $\lambda = -1$, $\mu = 0$. The vector fields on the circle, regarded as the Lie algebra of $\overline{\mathrm{Diff}^{+}S^1}$, acts jointly continuously on $V_{\lambda,\mu}$. Let $V_{\lambda,\mu}^{fin}$ be the subspace of elements with finite Fourier series; this is invariant under the Lie subalgebra of vector fields with finite Fourier series, and so can be regarded by complexification as an ordinary representation of the Virasoro algebra with vanishing central charge.

**1.2. Definition.** *Let $V_i$, $V_j$ be unitary highest weight representations of $\mathfrak{Vir}$. A primary field is a non-zero linear operator $\phi : V_i \otimes V_{\lambda,\mu}^{fin} \to V_j$ that intertwines the action of $\mathfrak{Vir}$.*

Let $V_i$, $V_j$ respectively have highest weights $h_i$, $h_j$ (at a fixed central charge $c$). It is easy to see that $\mu = h_i - h_j$. Let $h = 1 - \lambda$, the conformal dimension of $\phi$.

**1.3. Uniqueness of primary fields.** For $f \in V_{\lambda,\mu}^{fin}$, let $\phi(f) : V_i \to V_j$ be the corresponding linear operator. Let $\phi(n) = \phi(e^{in\theta})$ and define the formal expression

$$\phi(z) = \sum_{n\in\mathbb{Z}} \phi(n)\, z^{-n-(h+h_i-h_j)}, \tag{1.3.1}$$

which satisfies the covariance relations

$$[L_m, \phi(z)] = z^{m+1}\frac{d}{dz}\phi(z) + h\,(m+1)\, z^{m}\, \phi(z). \tag{1.3.2}$$

This condition essentially characterises the primary field. To see this, we note that a unitary highest weight representation of $\mathfrak{Vir}$ is generated by its one-dimensional lowest energy subspace, so that the sesquilinear form on $V_i \times V_j$ given by $(\xi, \eta) \mapsto \langle \phi(z)\,\xi, \eta\rangle$, is uniquely determined up to a scalar multiple. It follows that a primary field is characterised up to a scalar multiple by (the central charge $c$ and) the ordered triplet of numbers $(h, h_i, h_j)$.

**1.4. Conjugate primary fields.** Complex conjugation, $f \mapsto f^{*}$, defines a conjugate-linear map $* : V_{\lambda,\mu} \to V_{\lambda,-\mu}$ that intertwines the action of $\overline{\mathrm{Diff}^{+}S^1}$. If $\phi : V_i \otimes V_{\lambda,\mu}^{fin} \to V_j$ is a primary field, then there is also the primary field $\phi^{*} : V_j \otimes V_{\lambda,-\mu}^{fin} \to V_i$ conjugate to $\phi$, given by defining $\phi^{*}(f^{*})$ to be the formal adjoint of $\phi(f)$, for each $f \in V_{\lambda,\mu}^{fin}$, i.e. $\langle \phi^{*}(f^{*})\,\xi, \eta\rangle = \langle \xi, \phi(f)\,\eta\rangle$. Finite-dimensionality of the $L_0$-eigenspaces guarantees that $\phi(f)^{*}$ is defined on all of $V_j$. Let $h = 1 - \lambda$. We have, formally,

$$\phi^{*}(z) = \{\, \phi(\bar{z}^{-1})\, \bar{z}^{-2h} \,\}^{*}. \tag{1.4.1}$$

The following is a trivial example of a primary field: if $V$ is a unitary highest weight representation of $\mathfrak{Vir}$, define $\phi : V \otimes V_{\lambda,\mu}^{fin} \to V$, with $\lambda = 1$, $\mu = 0$, by $\phi(z) = I_V$.

## 2. The Verma module approach to primary fields

**2.0. Overview.** There are two complementary approaches to the unitary highest weight representation theory of the (affine Kac-Moody and) Virasoro algebra(s). The first, analogous to Borel-Weil theory in the representation theory of compact Lie groups, constructs a highest weight representation as the quotient of a Verma module by its maximal submodule, and is not manifestly unitary. The second, analogous to the Hermann Weyl theory, relies on the decomposition of tensor products of unitary highest weight representations to construct new representations in a manifestly unitary way. To each of these approaches, there is a corresponding construction of the associated primary fields. The Verma module approach of Feigin and Fuchs [FF1]–[FF3] to the study of the discrete series representations is sketched in this section.

**2.1. Discrete series representations from Verma modules.** The Virasoro algebra $\mathfrak{Vir}$ is a direct sum of linear spaces $\mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$, where $\mathfrak{g}_\pm$ is the Lie subalgebra spanned by $\{L_{\pm n} : n \geq 1\}$, and $\mathfrak{g}_0$ by $L_0$ and $C$. For $(h, c) \in \mathbb{R}^2$, the Verma module $M_{h,c}$ is defined in the following way. Let $\mathbb{C}_{h,c}$ be the one-dimensional representation of the Lie subalgebra $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_+$, given by $L_n = 0\,(n > 0)$, $L_0 = h$ and $C = c$ on $\mathbb{C}_{h,c}$. Then

$$M_{h,c} = \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{Vir}}\, \mathbb{C}_{h,c} = U(\mathfrak{Vir}) \otimes_{U(\mathfrak{p})} \mathbb{C}_{h,c} \tag{2.1.1}$$

is the representation of $\mathfrak{Vir}$ induced by $\mathbb{C}_{h,c}$, where $U(\mathfrak{l})$ denotes the universal enveloping algebra of the Lie algebra $\mathfrak{l}$. The Verma module $M_{h,c}$ satisfies the conditions (a)–(c) of § I.2.1, but not necessarily (d). However, up to scalar multiplication, there is a unique contravariant sesquilinear form $(\ .\ .\ )$ on $M_{h,c}$; without loss of generality, we require that $(\xi, \xi) > 0$ for $\xi \in \mathbb{C}_{h,c}$. We are interested only in those $(h, c)$ such that this form is positive semi-definite: the set of such $(h, c)$ is given by the FQS analysis, Theorem I.2.4.1. In this case, let $K_{h,c}$ be the kernel of the form; it is the unique maximal proper submodule of $M_{h,c}$. Then $V_{h,c}$, the quotient of $M_{h,c}$ by $K_{h,c}$, is the unitary highest weight representation of $\mathfrak{Vir}$ with highest weight $(h, c)$.

**2.1.1. Singular vectors.** A vector $\xi \in M_{h,c}$ is a *singular vector at level* $N$ if $\xi \notin \mathbb{C}_{h,c}$ and it satisfies $L_0\xi = (h + N)\xi$, $L_n\xi = 0\,(n > 0)$. Clearly, a singular vector generates a proper submodule. Let $V_{h,c}$ be a representation in the discrete series; Feigin and Fuchs [FF1]–[FF3] have the following description of the maximal proper submodule $K_{h,c}$.

**2.1.2. Theorem (Feigin-Fuchs).** *Let $M_{h,c}$ be the Verma module with highest weight $(h, c)$ in the discrete series, with*

$$c = 1 - \frac{6}{m(m+1)}, \qquad h = \frac{[p(m+1) - qm]^2 - 1}{4m(m+1)}. \tag{2.1.2.1}$$

*The maximal proper submodule $K_{h,c}$ is generated by a pair of singular vectors at levels $pq$ and $(m-p)(m+1-q)$.*

We make a choice of a vector $\zeta_{h,c} \in \mathbb{C}_{h,c}$, satisfying $(\zeta_{h,c}, \zeta_{h,c}) = 1$. Denote the pair of singular vectors of this theorem by $O_{p,q}\zeta_{h,c}$ and $O_{m-p,m+1-q}\zeta_{h,c}$ respectively.

**2.2. Classification of discrete series primary fields.** Let $M_i = M_{h_i,c}$, $i = 1, 2$, be Verma modules and $h_3 \in \mathbf{R}$. Up to scalar multiplication, there is a unique sesquilinear form $\Phi(\ .\ ,\ .\ ; z)$ on $M_1 \times M_2$ satisfying

$$\Phi(\xi,\, L_{-m}\eta;\, z) - \Phi(L_m\xi,\, \eta;\, z) = \left\{ z^{m+1}\frac{d}{dz} + h_3\,(m+1)\,z^m \right\} \Phi(\xi,\, \eta;\, z) \tag{2.2.1}$$

for $\xi \in M_1$, $\eta \in M_2$. In fact, we define $\Phi$ inductively on $M_1 \times M_2$ using (2.2.1), beginning with the vectors $\zeta_i = \zeta_{h_i,c}$, $i = 1, 2$. The form $\Phi$ defines a primary field if and only if it descends to $V_1 \times V_2 = V_{h_1,c} \times V_{h_2,c}$. If the $V_i$, $i = 1, 2$, are discrete series representations, then, by the Feigin-Fuchs description of the maximal proper submodules, this is the case if and only if

$$\Phi(O_{p_1,q_1}\zeta_1,\, \zeta_2;\, z) = \Phi(O_{m-p_1,m+1-q_1}\zeta_1,\, \zeta_2;\, z) = 0, \tag{2.2.2a}$$

$$\Phi(\zeta_1,\, O_{p_2,q_2}\zeta_2;\, z) = \Phi(\zeta_1,\, O_{m-p_2,m+1-q_2}\zeta_2;\, z) = 0. \tag{2.2.2b}$$

It is not hard to see that these are polynomial equations in $h_3$, of degree not exceeding the level of the corresponding singular vector. The singular vector $O_{p,q}\zeta_i$ is a linear combination of terms of the form $L_{-i_1} \cdots L_{-i_n}\zeta_i$ with $i_1 + \cdots + i_n = pq$. Up to scalar multiplication,

$$\Phi(\zeta_1,\, \zeta_2;\, z) = z^{-(h_1+h_3-h_2)}. \tag{2.2.3}$$

Then using (2.2.1) repeatedly, we check that

$$\Phi(O_{p_1,q_1}\zeta_1,\, \zeta_2;\, z) = z^{-(h_1+h_3-h_2)-p_1q_1}\, g_{p_1,q_1,m}(h_3, h_2), \tag{2.2.4}$$

with $P(x) = g_{p_1,q_1,m}(x, h_2)$ a polynomial of degree $\leq p_1q_1$. Its roots have been determined by Feigin and Fuchs [FF1]–[FF3].

**2.2.1. Theorem (Feigin-Fuchs).** $P(x) = g_{p_1,q_1,m}(x, h_2)$ *has the* $p_1q_1$ *roots*

$$h_{p,q} = \frac{[p(m+1) - qm]^2 - 1}{4m(m+1)}, \quad \begin{array}{l} p = p_2 - p_1 + 1,\, p_2 - p_1 + 3, \ldots, p_2 + p_1 - 1, \\ q = q_2 - q_1 + 1,\, q_2 - q_1 + 3, \ldots, q_2 + q_1 - 1. \end{array} \tag{2.2.1.1}$$

It follows that the polynomial equations $g_{p_1,q_1,m}(x,h_2) = g_{m-p_1,m+1-q_1,m}(x,h_2) = 0$. i.e. the equations (2.2.2a). have the solutions $x = h_{p,q}$.

$$
\begin{aligned}
p &= |p_2 - p_1| + 1, |p_2 - p_1| + 3, \ldots, \min(p_2 + p_1 - 1, 2m - p_2 - p_1 - 1),\\
q &= |q_2 - q_1| + 1, |q_2 - q_1| + 3, \ldots, \min(q_2 + q_1 - 1, 2(m + 1) - q_2 - q_1 - 1).
\end{aligned}
\tag{2.2.1.2}
$$

In particular. each $h_{p,q}$ is the highest weight $(h_{p,q}, c)$ of a discrete series representation.

**2.2.2. Remarks.** The equations (2.2.2b) have the same solutions as (2.2.2a). In fact. let $\Psi(\,\cdot\,,\,\cdot\,;z)$ be the sesquilinear form on $M_2 \times M_1$ satisfying

$$
\Psi(\xi, L_{-m}\eta; z) - \Psi(L_m\xi, \eta; z) = \left\{z^{m+1}\frac{d}{dz} + h_3\,(m+1)\,z^m\right\}\Psi(\xi, \eta; z). \tag{2.2.2.1}
$$

for $\xi \in M_2$, $\eta \in M_1$. Then. up to a scalar factor,

$$
\Phi(L_{-i_1}\cdots L_{-i_n}\zeta_1, \zeta_2; z) = \Psi(\zeta_2, L_{-i_1}\cdots L_{-i_n}\zeta_1; z). \tag{2.2.2.2}
$$

The claim follows from the invariance of the solutions (2.2.1.2) under the permutation $(p_1,q_1)$ — $(p_2,q_2)$. This completes the classification of the primary fields associated to the discrete series representations.

With a fixed central charge $c$. we call an ordered triplet of highest weights $(h_3,h_1,h_2)$ an *allowed vertex* if there exists a primary field $\phi : V_{h_1,c} \otimes V^{fin}_{1-h_3,h_1-h_2}$ — $V_{h_2,c}$. It is easy to check that permuting an allowed vertex produces another allowed vertex. If $h_1, h_2$ are highest weights. we let $(h_1, h_2)$ denote the subset of highest weights $h_3$ such that $(h_3,h_1,h_2)$ is an allowed vertex.

## 3. The state-field correspondence

**3.1. The $\mathfrak{Vir}$-module generated by a primary field.** Let $\phi : V_1 \otimes V^{fin}_{\lambda,\mu}$ — $V_2$ be a primary field, where $V_i = V_{h_i,c}$, $i = 1,2$, are discrete series representations and $\lambda = 1 - h_3$. The primary field $\phi(z)$ generates a $\mathfrak{Vir}$-module, given by the linear span of the elements

$$
\left\{\hat{L}_{-m_1}\cdots\hat{L}_{-m_n}\phi\right\}(z), \tag{3.1.1}
$$

defined inductively by [BPZ]

$$
\begin{aligned}
\left\{\hat{L}_m\psi\right\}(z) = &\left[\sum_{r=0}^{\infty}\binom{m+1}{r}(-z)^r L_{m-r}\right]\psi(z)\\
&- \psi(z)\left[\sum_{r=0}^{\infty}\binom{m+1}{r}(-z)^{m+1-r}L_{r-1}\right].
\end{aligned}
\tag{3.1.2}
$$

It is straightfo. .rd to check that $\hat{L}_m\phi = 0\,(m > 0)$ and $\hat{L}_0\phi = h_3\,\phi$; that

$$
\hat{L}_m\hat{L}_n\psi - \hat{L}_n\hat{L}_m\psi = (m-n)\,\hat{L}_{m+n}\psi + \frac{c}{12}(m^3 - m)\,\delta_{m,-n}\,\psi; \tag{3.1.3}
$$

and that $\{\hat{L}_{-1}\psi\}(z) = \frac{d}{dz}\psi(z)$ (check by induction). Each element $\psi(z)$ is a formal series in $z$ with linear operators $T : V_1$ — $V_2$ as coefficients; in particular, $\frac{d}{dz}\psi(z)$ makes sense. The $\mathfrak{Vir}$-module so obtained is clearly isomorphic to the quotient of the Verma module $M_{h_3,c}$ by a proper submodule, and its lowest energy subspace is spanned by the primary field $\phi(z)$. The following result is well-known but. as far as we know, not proved in the literature.

**3.2. Proposition.** *The $\mathfrak{Vir}$-module generated by a primary field $\phi$ of conformal dimension $(h_3, c)$ is isomorphic to the unitary highest weight representation $V_{h_3,c}$.*

*Proof.* Let $\psi(z) = \{\hat{O}\phi\}(z)$ be a singular vector at level $N$. Then $\hat{L}_n\psi = 0\,(n > 0)$; and $\hat{L}_0\psi = h\,\psi$, $h = h_3 + N$. Equivalently, we have a subset of the covariance relations

$$
[L_m, \psi(z)] = z^{m+1}\frac{d}{dz}\psi(z) + h\,(m+1)\,z^m\,\psi(z) \qquad (m \geq -1) \tag{3.2.1}
$$

or, taking formal adjoints term-by-term,

$$
[L_{-m}, \psi^*(z)] = z^{-m-1}[L_1, \psi^*(z)] + h\,(-m-1)\,z^{-m}\,\psi^*(z) \tag{3.2.2}
$$

for $m \geq -1$, where

$$
\psi^*(z) = \left[\psi(\frac{1}{\bar{z}})\,\bar{z}^{-2h}\right]^*. \tag{3.2.3}
$$

We claim that

$$
\psi^*(z) = \sum_{n\geq 0} z^{-n}\delta_n(z), \tag{3.2.4}
$$

a finite sum, where the $\delta_n$ are in the $\mathfrak{Vir}$-module generated by $\phi^*$, the primary field conjugate to $\phi$, such that $\hat{L}_0\delta_n = (h-n)\,\delta_n$ and $\delta_0 = (-1)^N\hat{O}\phi^*$. In fact, we note the following. Let $\chi(z)$ be an $L_0$-eigenvector in the $\mathfrak{Vir}$-module generated by $\phi$, $\hat{L}_0\chi = h_\chi\,\chi$, and let

$$
\chi^*(z) = \left[\chi(\frac{1}{\bar{z}})\,\bar{z}^{-2h_\chi}\right]^*. \tag{3.2.5}
$$

Then

$$
\begin{aligned}
\{\hat{L}_{-1}\chi\}^* &= -\{\hat{L}_{-1}\chi^*\} - 2h_\chi\,z^{-1}\,\chi^*\\
\{\hat{L}_{-2}\chi\}^* &= \{\hat{L}_{-2}\chi^*\} + 3\,z^{-1}\,\{\hat{L}_{-1}\chi^*\} + 3\,z^{-2}\,\{\hat{L}_0\chi^*\} + z^{-3}\,\{\hat{L}_1\chi^*\}\\
(m-2)\,\{\hat{L}_{-m}\chi\}^* &= \{\hat{L}_{-1}\hat{L}_{-m+1}\chi\}^* - \{\hat{L}_{-m+1}\hat{L}_{-1}\chi\}^* \quad (m \geq 3),
\end{aligned}
\tag{3.2.6}
$$

where we have extended the action of $\mathfrak{Vir}$ on elements $\zeta(z)$ to the L[illegible]r span of the elements $p(z^{-1})\,\zeta(z)$, $p$ a polynomial, in an obvious way. The claim follows from these relations. Now, for each integer $M \geq 0$, we have

$$[L_{-m}, \{\hat{L}_0^M \psi^*\}(z)] = z^{-m-1}[L_1, \{\hat{L}_0^M \psi^*\}(z)] + h\,(-m-1)\,z^{-m}\,\{\hat{L}_0^M \psi^*\}(z) \qquad (3.2.7)$$

for $m \geq -1$. It is easy to prove this by induction on $M$; the case $M = 0$ is (3.2.2). Substitution of (3.2.4) in (3.2.7) yields

$$\sum_{n\geq 0} z^{-n}\,(h-n)^M\,\{[L_{-m}.\delta_n(z)] - z^{-m-1}[L_1, \delta_n(z)] - h\,(-m-1)\,z^{-m}\,\delta_n(z)\} = 0 \quad (3.2.8)$$

for integers $m \geq -1$, $M \geq 0$. It follows that each term in the sum (3.2.8) vanishes. In particular, the zeroth term vanishes. Then, replacing $\phi$ by $\phi^*$ (the same arguments apply) and using (3.2.1) with $m = 1$, we obtain

$$[L_{-m}, \psi(z)] = z^{-m+1}\frac{d}{dz}\psi(z) + h\,(-m+1)\,z^{-m}\,\psi(z) = 0 \qquad (m \geq -1). \qquad (3.2.9)$$

Together with (3.2.1), this provides the full set of covariance relations for $\psi(z)$. If the primary field $\phi$ exists, so does the primary field $\chi : V_3 \otimes V^{fin}_{1-h_1,h_3-h_2} \to V_2$. Up to a scalar multiple,

$$\langle\{\hat{L}_{-i_1}\cdots\hat{L}_{-i_n}\phi\}(z)\zeta_1, \zeta_2\rangle = (-1)^m\,\langle\chi(z)\,L_{-i_1}\cdots L_{-i_n}\zeta_3, \zeta_2\rangle, \qquad (3.2.10)$$

where $m = \sum_{j=1}^n i_j$, and $\zeta_i \in V_i$ are lowest energy vectors. It follows from (3.2.10) and the full set of covariance relations for $\psi$ that $\langle\psi(z)\xi, \eta\rangle = 0$ when $\xi$, $\eta$ are lowest energy, and therefore arbitrary, vectors. Hence $\psi(z) = 0$, and each $\{\hat{L}_{-i_1}\cdots\hat{L}_{-i_n}\psi\}(z) = 0$. Since the maximal proper submodule of $M_{h_3,c}$ is generated by its singular vectors (Theorem 2.1.2), the Proposition is proved. □

**3.3. Remarks.** We shall write $\phi(\xi;\, z)$ for the element in the $\mathfrak{Vir}$-module generated by $\phi$ that corresponds to the vector $\xi \in V_{h_3,c}$. This requires making a choice of a lowest energy vector $\zeta_{h_3,c} \in V_{h_3,c}$, and the assignment $\phi(z) = \phi(\zeta_{h_3,c};\, z)$. In particular, we make a choice of a vacuum vector $\zeta_{0,c} = \Omega_c$, i.e. a lowest energy vector in $V_{0,c}$. With no loss of generality, we take $\zeta_{h_3,c}$ to be a unit vector. When the primary field $\phi : V_1 \otimes V^{fin}_{\lambda,\mu} \to V_2$ corresponds to $h_1 = 0$, $h_2 = h_3$, the isomorphism of Proposition 3.2 can be given by mapping an element $\psi(z)$ to $\psi(z)\Omega_c|_{z=0}$. This makes sense because $\psi(z)\Omega_c$ is a formal power series: we have

$$\phi(z)\Omega_c = \sum_{n\geq 0} z^n\,\phi(-n)\Omega_c; \qquad (3.3.1)$$

for arbitrary $\psi(z)$, it follows by induction using (3.1.2) and $L_m\Omega_c = 0\,(m \geq -1)$. Then we can make choices such that

$$\phi(\xi;\, z)\Omega_c|_{z=0} = \xi. \qquad (3.3.2)$$

We may also call $\phi(\xi;\, z)$ a secondary field (when $\xi$ is not a lowest energy vector), and a descendant of the primary field $\phi(z)$.

## 4. Correlation functions and the BPZ equations

**4.1. Conventions.** In the following, we fix a central charge $c = 1 - 6/m(m+1)$ and consider the corresponding discrete series representations $V_h = V_{h,c}$, for each of which we choose a preferred lowest energy vector $\zeta_h \in V_h$ with unit norm $||\zeta_h|| = 1$. In particular, we choose a vacuum vector $\Omega = \zeta_0$. If $\phi : V_{h_i} \otimes V^{fin}_{\lambda,\mu} \to V_{h_j}$ is a primary field of conformal dimension $h_k$, we denote by $\phi^{h_k}_{h_j\,h_i}$ its normalised form, which satisfies

$$\langle\,\phi^{h_k}_{h_j\,h_i}(\zeta_{h_k};\, z)\,\zeta_{h_i}, \zeta_{h_j}\rangle = \langle\,\phi^{h_k}_{h_j\,h_i}(z)\,\zeta_{h_i}, \zeta_{h_j}\rangle = z^{-\{h_k+h_i-h_j\}}. \qquad (4.1.1)$$

Then $\phi^{h_k\;*}_{h_j\,h_i} = \phi^{h_k}_{h_i\,h_j}$. Let $\mathcal{I} = \{i, j, \ldots\}$, be an indexing set for the discrete series representations at central charge $c$: we shall always have $0 \in \mathcal{I}$ and $h_0 = h_{1,1} = 0$. Then we may write $\zeta_i \in V_i$ for $\zeta_{h_i} \in V_{h_i}$; $\phi^k_{ji}$ for $\phi^{h_k}_{h_j\,h_i}$; $\langle i, j\rangle$ for $\langle h_i, h_j\rangle$; and $h_i = h_{p_i,q_i}$.

**4.2. The BPZ equations.** Let $\phi_i : V_{k_i} \otimes V^{fin}_{1-h_i,k_i-k_{i+1}} \to V_{k_{i+1}}$, $i = 1, \ldots, n$, be primary fields, with $k_1 = k_{n+1} = 0$. We call the formal series

$$\langle\phi_n(z_n)\cdots\phi_1(z_1)\rangle \;=\; \sum_{(m_i)} \langle\phi_n(m_n)\cdots\phi_1(m_1)\Omega, \Omega\rangle\, z_n^{-m_n-\Delta_n}\cdots z_1^{-m_1-\Delta_1}, \qquad (4.2.1)$$

where $\Delta_i = h_i + k_i - k_{i+1}$, an *n-point function* or, more ambiguously, a *correlation function*. The identities $\hat{O}_{p_i,q_i}\phi_i = \hat{O}_{m-p_i,m+1-q_i}\phi_i = 0$, and $L_m\Omega = 0\,(m = -1, 0, 1)$, imply that the $n$-point function (4.2.1) is a formal solution to a set of $2n+3$ partial differential equations, independent of the $k_i$'s, in the region $|z_n| > \cdots > |z_1|$.

In fact, if $\psi_i(z)$ is an element of the $\mathfrak{Vir}$-module generated by the primary field $\phi_i(z)$, then, using (3.1.2) and the covariance relations (1.3.2), we obtain

$$\langle\phi_n(z_n)\cdots\{\hat{L}_{-m}\psi_i\}(z_i)\cdots\phi_1(z_1)\Omega, \Omega\rangle = \mathcal{L}_{-m}\,\langle\phi_n(z_n)\cdots\psi_i(z_i)\cdots\phi_1(z_1)\Omega, \Omega\rangle \quad (4.2.2)$$

for $m \geq 0$, where

$$\mathcal{L}_{-m} = -\sum_{j\neq i}(z_j - z_i)^{-m+1}\frac{\partial}{\partial z_j} + h_j\,(-m+1)\,(z_j - z_i)^{-m} \qquad (4.2.3)$$

and we mean by $(z_j-z_i)^{-1}$ its binomial expansion in the region $|z_n| > \cdots > |z_1|$. Proceeding in this fashion, we obtain the $2n$ equations corresponding to

$$\langle \phi_n(z_n)\cdots\{\hat{O}_{p,q}\phi_i\}(z_i)\cdots\phi_1(z_1)\Omega,\ \Omega\rangle = 0 \tag{4.2.4}$$

with $(p,q) = (p_i,q_i)$ and $(m-p_i, m+1-q_i)$, and $i = 1,\ldots,n$. These are the celebrated *BPZ equations* [BPZ]. The other 3 equations are the *Möbius equations*

$$\sum_{j=1}^{n}\left\{z_j^{m+1}\frac{\partial}{\partial z_j} + h_j(m+1)z_j^{m}\right\}\langle\phi_n(z_n)\cdots\phi_1(z_1)\rangle = 0 \qquad (m=-1,0,1) \tag{4.2.5}$$

which reflect the invariance of the vacuum vector $\Omega$ under the action of the Möbius subgroup of $\mathrm{Diff}^+S^1$. In principle, we can use the regularity properties of these equations (which have to be proved) to guarantee the convergence of the $n$-point function to a multi-valued holomorphic function on the domain $|z_n| > \cdots > |z_1|$, by applying the following theorem of Yoshida and Takano [YT]. This is the method of Tsuchiya and Kanie [TK] in the loop group $LG$ case (at least for $G = SU(N)$).

**4.3. Theorem (Yoshida-Takano).** *An integrable Pfaffian system of complex partial differential equations*

$$d\bar{f} = \left\{\sum_{i=1}^{n}\frac{A_i(w)}{w_i}dw_i\right\}\bar{f} \tag{4.3.1}$$

*is said to have a regular singular point at $w=0$ if the $A_i(w)$ are holomorphic on some polydisc $D = \{w \in \mathbb{C}^n : |w_i| < r_i\}$. Then every formal power series solution*

$$\bar{f}(w) = w^{\alpha}\sum_{m\geq 0}\bar{f}_m w^m, \tag{4.3.2}$$

*where $w^{\alpha+m} = \prod w_i^{\alpha_i+m_i}$; $\alpha \in \mathbb{C}^n$, $m \in \mathbf{Z}^n$ are multi-indices, converges on $D$ to a possibly multi-valued holomorphic function.*

**4.4. Remarks.** The change of variables [TK]

$$w_n = z_n, \quad w_i = z_i/z_{i+1}\,(i \neq n) \tag{4.4.1}$$

identifies $\{z \in \mathbb{C}^n : |z_n| > \cdots > |z_1|\}$ with $\{w \in \mathbb{C}^n : |w_1| < 1; 0 < |w_n|; 0 < |w_i| < 1, 2 \leq i \leq n-1\}$. We shall let $D_n$ denote the polydisc $\{w \in \mathbb{C}^n : |w_i| < 1, 1 \leq i \leq n-1\}$.

A disadvantage of the Verma module approach to the study of correlation functions in the case of the discrete series representations is that the regularity properties of the

BPZ (plus Möbius) equations are not manifest. This is to be contrasted with the Knizhnik-Zamolodchikov (KZ) equations in the case of positive energy representations of loop groups, where they are [TK]. Our use of the BPZ equations will therefore be limited to some special cases where we are able to prove the required properties. The shortfall is compensated for using the coset construction approach of the next chapter.

**4.5. 2- and 3-point functions.** For $n$-point functions with $n \leq 3$, the Möbius equations alone are sufficient to prove convergence. It is straightforward to recast them in the form

$$w_i\frac{\partial f}{\partial w_i} = A_i(w)f \tag{4.5.1}$$

and observe that each $A_i(w)$ is holomorphic on the polydisc $D_4$. Moreover, they are readily solved.

**4.6. 4-point functions with a generating primary.** We shall mean by a *generating* primary field one that has conformal dimension $h_{1,2}$ or $h_{2,1}$. We consider the 4-point functions $\langle\phi_4(z_4)\cdots\phi_1(z_1)\rangle$ such that some $\phi_i$ is a generating primary. To be definite, we let $h_2 = h_{1,2}, h_{2,1}$ in the following. Then we have

$$\frac{3}{4h_2+2}\hat{L}_{-1}^2\phi_2 - \hat{L}_{-2}\phi_2 = 0, \tag{4.6.1}$$

yielding the BPZ equation

$$\left\{\frac{3}{4h_2+2}\frac{\partial^2}{\partial z_2^2} + \sum_{j\neq 2}(z_j-z_2)^{-1}\frac{\partial}{\partial z_j} - h_j(z_j-z_2)^{-2}\right\}f = 0, \tag{4.6.2}$$

in addition to the Möbius equations. The latter have the general solution [BPZ]

$$f = \prod_{i<j}(z_j-z_i)^{-\gamma_{ij}}g(x), \tag{4.6.3}$$

where

$$x = \frac{(z_2-z_1)(z_4-z_3)}{(z_3-z_1)(z_4-z_2)}, \tag{4.6.4}$$

is the cross-ratio, and the $\gamma_{ij}$ are real numbers satisfying

$$2h_j = \sum_{i,i<j}\gamma_{ij} + \sum_{i,i>j}\gamma_{ji} \tag{4.6.5}$$

for each $j$. In fact, we can choose $\gamma_{12}$ and $\gamma_{23}$ arbitrarily, and use (4.6.5) to solve for the remaining exponents. More precisely, we define (4.6.3) on the domain $|z_4| > \cdots > |z_1|$, with branch cuts to handle the multi-valuedness, i.e.

$$(z_j-z_i)^{\gamma_{ij}} = (w_j\cdots w_n)^{\gamma_{ij}}(1-w_i\cdots w_{j-1})^{\gamma_{ij}} \tag{4.6.6}$$

with the latter factor defined by its binomial expansion. We have $|x| < 1$. Substituting (4.6.3) in (4.6.2), we obtain

$$p_0(x)\frac{d^2g}{dx^2} + \frac{p_1(x)}{x(1-x)}\frac{dg}{dx} + \frac{p_2(x)}{x^2(1-x)^2}g = 0. \tag{4.6.7}$$

where $p_i(x)$ is a polynomial of degree $\leq i$; in fact,

$$\begin{aligned} p_0(x) &= \frac{3}{4h_2+2} \\ p_1(x) &= \left[1 - \frac{3\gamma_{12}}{2h_2-1}\right](1-x) - \left[1 - \frac{3\gamma_{23}}{2h_2+1}\right]x \\ p_2(x) &= \left[\frac{3\gamma_{12}(\gamma_{12}+1)}{4h_2+2} - \gamma_{12} - h_1\right](1-x)^2 + \left[\frac{3\gamma_{23}(\gamma_{23}+1)}{4h_2+2} - \gamma_{23} - h_3\right]x^2 - \left[\frac{3\gamma_{12}\gamma_{23}}{2h_2+1} + \gamma_{13}\right]x(1-x). \end{aligned} \tag{4.6.8}$$

In particular, (4.6.7) is Fuchsian with regular singularities at $x = 0, 1$ and $\infty$.

**4.6.1. Lemma.** *The 4-point function* $\langle \phi_4(z_4)\cdots\phi_1(z_1)\rangle$, *with some* $h_i = h_{1,2}$ *or* $h_{2,1}$, *converges to a multi-valued holomorphic function on the domain* $|z_4| > \cdots > |z_1|$.

*Proof.* We rewrite the BPZ equation (4.6.2) and the Möbius equations in the form

$$w_i\frac{\partial \bar{g}}{\partial w_i} = A_i(w)\bar{g}, \tag{4.6.1.1}$$

where $\bar{g} = (g, \frac{\partial g}{\partial w_1})$, and show that each $A_i(w)$ is holomorphic on the polydisc $D_4$. We can equivalently use $\bar{g}$ or $\bar{f} = (f, \frac{\partial f}{\partial w_1})$. For $i = 2, 3$ and $4$, this follows from

$$w_i\frac{\partial g}{\partial w_i} = \alpha_i(w)\frac{\partial g}{\partial w_1}, \tag{4.6.1.2}$$

where

$$\alpha_2(w) = -\frac{(1-w_1)(1-w_1w_2^2w_3)}{(1-w_2)(1-w_2w_3)}, \quad \alpha_3(w) = \frac{w_3(1-w_1)(1-w_1w_2)}{(1-w_3)(1-w_2w_3)}, \quad \alpha_4 = 0, \tag{4.6.1.3}$$

and its $w_1$-derivatives. For $i = 1$, it follows because (4.6.7) is Fuchsian with a regular singularity at $x = 0$. Explicitly, we have

$$\frac{\partial^2 g}{\partial w_1^2} = \beta_1(w)\frac{\partial g}{\partial w_1} - \frac{4h_2+2}{3}\left\{\beta_2(w)p_1(x)\frac{\partial g}{\partial w_1} + \beta_2(w)^2p_2(x)g\right\}, \tag{4.6.1.4}$$

with

$$\beta_1(w) = \frac{2w_2}{1-w_1w_2}, \qquad \beta_2(w) = -\frac{(1-w_2w_3)}{(1-w_1)(1-w_1w_2w_3)}. \tag{4.6.1.5}$$

Integrability of (4.6.1.1) is immediate because it is equivalent to the single-variable equation (4.6.7). The Lemma follows from the Yoshida-Takano result (Theorem 4.3). □

The proof of Lemma 4.6.1 extends to any 4-point function $\langle\phi_4(z_4)\cdots\phi_1(z_1)\rangle$ satisfying a BPZ equation that reduces to a Fuchsian equation, say of order $n$, with regular singularities at $x = 0, 1$ and $\infty$, i.e. to an equation

$$\sum_{k=0}^{n}\frac{p_k(x)}{x^k(1-x)^k}\frac{d^{n-k}g}{dx^{n-k}} = 0. \tag{4.6.1.6}$$

where $p_k(x)$ is a polynomial of degree $\leq k$.

**4.7. The general n-point function.** In principle, analogous arguments apply to the equations for arbitrary $n$-point functions. This requires manipulation of the BPZ equations, and certainly some knowledge of the expressions for singular vectors. The difficulties of this approach appear to be too great, and we defer to Chapter III a proof of the following.

**4.7.1. Theorem.** *The n-point function* $\langle\phi_n(z_n)\cdots\phi_1(z_1)\rangle$ *converges to a multi-valued holomorphic function on the domain* $|z_n| > \cdots > |z_1|$.

**4.8. Generalised correlation functions.** Let $\phi_i(z)$, $i = 1, \ldots, n$, be primary fields as in § 4.2, and let $\xi_i \in V_{h_i}$. We can also define correlation function of secondary fields,

$$\langle\phi_n(\xi_n; z_n)\cdots\phi_1(\xi_1; z_1)\Omega, \Omega\rangle. \tag{4.8.1}$$

When it is necessary to make a distinction, we shall call this a *generalised* correlation function. The properties of generalised correlation functions typically follow from those of correlation functions of primary fields. This follows from their construction using correlation functions of primary fields "with insertions" (of the "energy-momentum tensor"). We sketch the necessary arguments, following [BPZ].

Let $I$ denote the identity operator, regarded as the trivial primary field of conformal dimension 0, on each $V_h$. Define

$$T(z) = \{\hat{L}_{-2}I\}(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \tag{4.8.2}$$

the *energy-momentum tensor*. It satisfies the "anomalous" covariance relation

$$[L_m, T(z)] = z^{m+1}\frac{d}{dz}T(z) + 2(m+1)z^m T(z) + \frac{c}{12}(m^3 - m)z^{m-2}. \tag{4.8.3}$$

The basic observation is that the $n$-point function (assumed to converge by Theorem 4.7.1) with $m$-insertions

$$\langle T(x_m)\cdots T(x_1)\phi_n(z_n)\cdots\phi_1(z_1)\Omega, \Omega\rangle \tag{4.8.4}$$

is convergent for $|x_m| > \cdots > |x_1| > |z_n| > \cdots |z_1|$. This follows by commuting the $L_n\,(n \geq -1)$ to the right, and the $L_n\,(n \leq -2)$ to the left, until they annihilate the vacuum vector $\Omega$. This procedure picks up commutators which are evaluated using the covariance relations for primary fields and the energy momentum tensor. We obtain (4.8.4) as the differential operator

$$\sum_{i=1}^{m-1} \frac{1}{x_m - x_i} \frac{\partial}{\partial x_i} + \frac{2}{(x_m - x_i)^2} + \frac{c/2}{(x_m - x_i)^4} + \sum_{i=1}^{n} \frac{1}{x_m - z_i} \frac{\partial}{\partial z_i} + \frac{h_i}{(x_m - z_i)^2} \tag{4.8.5}$$

acting on

$$\langle T(x_{m-1}) \cdots T(x_1)\, \phi_n(z_n) \cdots \phi_1(z_1)\Omega,\, \Omega\rangle. \tag{4.8.6}$$

Iterating this procedure reduces (4.8.4) to differential operators of the form (4.8.5) acting on the $n$-point function. It follows that the $n$-point function with $m$-insertions is appropriately convergent. We note two points: (a) As a function in the $x$-variables, the $n$-point function with $m$-insertions analytically continues to a holomorphic function on

$$\{x \in \mathbb{C}^m : x_i \neq x_j\,(i \neq j);\, x_i \neq z_k\}; \tag{4.8.7}$$

as a function of $x_i$, it is holomorphic except at $x_j, j \neq i$, and $z_j, j = 1, \ldots, n$, where it has poles; (b) The same analysis applies when the insertions of $T(x_i)$'s are permuted and interspersed between the primary fields; the resultant holomorphic functions are all equal in the sense of analytic continuation in the $x$-variables.

Now we claim that the generalised $n$-point function (4.8.1) also converges in the domain $|z_n > \cdots > |z_1|$. Moreover, the generalised $n$-point function with $m$ insertions also converges as before, has the form of differential operators acting on the correlation function without insertions, and satisfies (a) and (b). This is proved by induction, beginning with an $n$-point function of primary fields $\phi_i(\xi_i; z_i)$, $\xi_i$ a lowest energy vector, and an arbitrary number of insertions. By taking a certain contour integral in the variable $x$, the insertion of $T(x)$ is removed while replacing a vector $\xi_i$ by $L_m\xi\,(m \in \mathbf{Z})$. The contour integral is given by

$$\oint_C \frac{dx}{2\pi i} (x - z_i)^{m+1} \langle T(x_m) \cdots T(x_1) T(x)\, \phi_n(\xi_n; z_n) \cdots \phi_1(\xi_1; z_1)\Omega,\, \Omega\rangle, \tag{4.8.8}$$

where the contour $C$ is a small anti-clockwise loop around $z_i$. The claim is that (4.8.8) is equal to

$$\langle T(x_m) \cdots T(x_1)\, \phi_n(\xi_n; z_n) \cdots \phi_i(L_m\xi_i; z_i) \cdots \phi_1(\xi_1; z_1)\Omega,\, \Omega\rangle. \tag{4.8.9}$$

Formally, we have

$$\begin{aligned}
\oint_C \frac{dx}{2\pi i} (x - z)^{m+1}\, T(x)\, \phi(\xi; z) &= \oint_{C_1} \frac{dx}{2\pi i} (x - z)^{m+1}\, T(x)\, \phi(\xi; z) \\
&\quad - \oint_{C_2} \frac{dx}{2\pi i} (x - z)^{m+1}\, \phi(\xi; z)\, T(x) \\
&= \sum_{r=0}^{\infty} \binom{m+1}{r} (-z)^r L_{m-r}\, \phi(\xi; z) \\
&\quad - \sum_{r=0}^{\infty} \binom{m+1}{r} (-z)^{m+1-r} \phi(\xi; z)\, L_{r-1} \\
&= \phi(L_m\xi; z).
\end{aligned} \tag{4.8.10}$$

Here we deform the contour $C$ into a pair of contours $C_i$, circling the origin at radii $r_i$, $r_1 = |z| + \varepsilon$, $r_2 = |z| - \varepsilon$, $\varepsilon > 0$, taken anti-clockwise and clockwise respectively. On $C_1$, since $|x| > |z|$, we can use the series expansion of $T(x)\,\phi(\xi; z)$; and on $C_2$, with $|x| < |z|$, the series expansion of $\phi(\xi; z)\, T(x)$ is valid. It is straightforward but unedifying to make the arguments rigorous.

## 5. Braiding relations of primary fields

**5.1. Definition of braiding.** Let $V_i = V_{h_i}$, $i = 1, \ldots, 4$, be discrete series representations. Then there are 4-point functions

$$\langle \phi_{0\,h_4}^{h_4}(z_4)\phi_{h_4\,h}^{h_3}(z_3)\phi_{h\,h_1}^{h_2}(z_2)\phi_{h_1\,0}^{h_1}(z_1)\Omega,\, \Omega\rangle \tag{5.1.1}$$

indexed by the highest weight (the "channel") $h \in \langle h_4, h_3\rangle \cap \langle h_2, h_1\rangle$. At the same time, there are 4-point functions

$$\langle \phi_{0\,h_4}^{h_4}(z_4)\phi_{h_4\,k}^{h_2}(z_2)\phi_{k\,h_1}^{h_3}(z_3)\phi_{h_1\,0}^{h_1}(z_1)\Omega,\, \Omega\rangle \tag{5.1.2}$$

indexed by the highest weight $k \in \langle h_4, h_2\rangle \cap \langle h_3, h_1\rangle$. Since 4-point functions satisfy the Möbius equations, they are essentially holomorphic functions in a single variable, the cross-ratio $x = (z_1 - z_2)(z_3 - z_4)/(z_1 - z_3)(z_2 - z_4)$. So let $z_1 \to 0$ and $z_4 \to \infty$ in ($z_4^{2h_4}$ times) the 4-point functions to obtain

$$\langle \phi_{h_4\,h}^{h_3}(z_3)\phi_{h\,h_1}^{h_2}(z_2)\zeta_{h_1},\, \zeta_{h_4}\rangle \tag{5.1.3}$$

and

$$\langle \phi^{h_2}_{h_4 k}(z_2)\phi^{h_3}_{k h_1}(z_3)\zeta_{h_1}, \zeta_{h_4} \rangle \tag{5.1.4}$$

respectively. We may call (5.1.3) and (5.1.4) *reduced* 4-point functions. The 4-point functions (5.1.1) and (5.1.2) satisfy the same set of BPZ and Möbius equations. but in different domains: $|x| < 1$ and $|x| > 1$ respectively. This suggests that there are matrices $(C^{h_4 h_3 h_2 h_1}_{k\,h})_{k\,h}$ such that

$$\langle \phi^{h_3}_{h_4 h}(z_3)\phi^{h_2}_{h h_1}(z_2)\zeta_{h_1}, \zeta_{h_4} \rangle = \sum_k \langle \phi^{h_2}_{h_4 k}(z_2)\phi^{h_3}_{k h_1}(z_3)\zeta_{h_1}, \zeta_{h_4} \rangle\, C^{h_4 h_3 h_2 h_1}_{k\,h} \tag{5.1.5}$$

in the sense of analytic continuation in the variable $x = z_2/z_3$. If such a *braiding relation* exists, then. using the covariance relations for primary fields. this relation must also hold when we replace the lowest energy vectors $\zeta_{h_1}, \zeta_{h_4}$ by arbitrary vectors $\xi \in V_{h_1}, \eta \in V_{h_4}$. Moreover. it must also hold with insertions of $T(z)$'s. It is therefore justifiable to write the braiding relation simply as

$$\phi^{h_3}_{h_4 h}(z_3)\phi^{h_2}_{h h_1}(z_2) = \sum_k \phi^{h_2}_{h_4 k}(z_2)\phi^{h_3}_{k h_1}(z_3)\, C^{h_4 h_3 h_2 h_1}_{k\,h}, \tag{5.1.6}$$

valid on a domain to be specified. We can check that the sets of channels $\langle h_4, h_3 \rangle \cap \langle h_2, h_1 \rangle$ and $\langle h_4, h_2 \rangle \cap \langle h_3, h_1 \rangle$ have the same number of elements. so that the *braiding matrix* $(C^{h_4 h_3 h_2 h_1}_{k\,h})_{k\,h}$ is square: as a connection matrix. it must clearly be invertible.

**5.2. "Abelian" braiding.** The simplest braiding relations occur when there is only one available channel: we call such braiding relations "abelian". An example is when a 4-point function reduces to a 3-point function. Let $x = z_2/z_3$. We have

$$\langle \phi^{h_3}_{0\,h_3}(z_3)\phi^{h_2}_{h_3 h_1}(z_2)\zeta_{h_1}, \Omega \rangle = z_3^{-2h_3}\, z_2^{-2h_2}\, z_2^{\alpha}\, (1 - x)^{-\alpha} \tag{5.2.1}$$

on $|x| < 1$; and

$$\langle \phi^{h_2}_{0\,h_2}(z_2)\phi^{h_3}_{h_2 h_1}(z_3)\zeta_{h_1}, \Omega \rangle = z_3^{-2h_3}\, z_2^{-2h_2}\, z_3^{\alpha}\, (1 - \frac{1}{x})^{-\alpha} \tag{5.2.2}$$

on $|x| > 1$, where $\alpha = h_3 + h_2 - h_1$. Then

$$\phi^{h_3}_{0\,h_3}(z_3)\phi^{h_2}_{h_3 h_1}(z_2) = \phi^{h_2}_{0\,h_2}(z_2)\phi^{h_3}_{h_2 h_1}(z_3)\, e^{\pi i\{h_3 + h_2 - h_1\}} \tag{5.2.3}$$

on $0 < \arg(x) < 2\pi$, in the sense of analytic continuation. Similarly, we have

$$\phi^{h_3}_{h_4\,h_2}(z_3)\phi^{h_2}_{h_2 0}(z_2) = \phi^{h_2}_{h_4 h_3}(z_2)\phi^{h_3}_{h_3 0}(z_3)\, e^{\pi i\{h_3 + h_2 - h_4\}} \tag{5.2.4}$$

on $0 < \arg(x) < 2\pi$. So we obtain

$$C^{0\,h_3 h_2 h}_{h_2 h_3} = C^{h\,h_3 h_2\,0}_{h_3 h_2} = e^{\pi i\{h_3 + h_2 - h\}}. \tag{5.2.5}$$

**5.3. Braiding relations of generating primaries.** The next simplest braiding relations occur of course when there are only two available channels. This is generically the case when one of the primary fields in the 4-point function is a generating primary field. Let $V_i = V_{h_i}$ be discrete series representations as in § 5.1, with $h_2 = h_{1,2}$ or $h_{2,1}$. We recall from § 4.6 that the 4-point functions (5.1.1) and (5.1.2) then satisfy a second order Fuchsian equation (4.6.7) with regular singular points at $x = 0, 1$ and $\infty$. Up to a transformation, this is just the hypergeometric equation.

Let $\gamma_{12}, \gamma_{23}$ in the equation (4.6.7) be respectively solutions to the quadratic equations

$$\begin{aligned} 3\gamma_{12}(\gamma_{12} + 1) - (4h_2 + 2)(\gamma_{12} + h_1) = 0, \\ 3\gamma_{23}(\gamma_{23} + 1) - (4h_2 + 2)(\gamma_{23} + h_3) = 0. \end{aligned} \tag{5.3.1}$$

Then (4.6.7) reduces to the hypergeometric equation

$$x(1 - x)\frac{d^2 g}{dx^2} + \{\gamma - (\alpha + \beta + 1)x\}\frac{dg}{dx} - \alpha\beta\, g = 0 \tag{5.3.2}$$

with

$$\begin{aligned} \gamma &= \frac{4h_2 + 2}{3} - 2\gamma_{12} \\ \alpha + \beta &= \frac{8h_2 + 1}{3} - 2(\gamma_{12} + \gamma_{23}) \\ \alpha\beta &= 2\gamma_{12}\gamma_{23} + \frac{4h_2 + 2}{3}\gamma_{13}. \end{aligned} \tag{5.3.3}$$

For convenience, we make definite choices for the parameters. Let

$$\begin{aligned} \gamma_{12} &= \frac{-1 - [p_1(m + 1) - q_1 m]}{2(m + 1)}, \quad \gamma_{23} = \frac{-1 - [p_3(m + 1) - q_3 m]}{2(m + 1)}, \\ \alpha &= h_{p_4, q_4 + 1} - h_{p_4, q_4} + h_2 - (\gamma_{12} + \gamma_{23}), \\ \beta &= h_{p_4, q_4 - 1} - h_{p_4, q_4} + h_2 - (\gamma_{12} + \gamma_{23}); \end{aligned} \tag{5.3.4}$$

when $h_2 = h_{1,2}$; and

$$\begin{aligned} \gamma_{12} &= \frac{1 + [p_1(m + 1) - q_1 m]}{2m}, \quad \gamma_{23} = \frac{1 + [p_3(m + 1) - q_3 m]}{2m}, \\ \alpha &= h_{p_4 + 1, q_4} - h_{p_4, q_4} + h_2 - (\gamma_{12} + \gamma_{23}), \\ \beta &= h_{p_4 - 1, q_4} - h_{p_4, q_4} + h_2 - (\gamma_{12} + \gamma_{23}); \end{aligned} \tag{5.3.5}$$

when $h_2 = h_{2,1}$. We note that $\gamma, \alpha + \beta - \gamma$ and $\alpha - \beta$ are non-integral.

The theory of the hypergeometric equation (5.3.2) is well-under[illegible]od [WW] [IKSY]; we quote the relevant results. It has the Riemann scheme

$$\begin{pmatrix} 0 & 1 & \infty \\ 0 & 0 & \alpha \\ 1-\gamma & \gamma-\alpha-\beta & \beta \end{pmatrix}. \tag{5.3.6}$$

When $\gamma$ and $\alpha-\beta$ are non-integral, the local solutions at $x=0$ and $\infty$ respectively are non-logarithmic. Let $g_p(x;\nu)$ denote the local solution at the singular point $x=p$, with characteristic exponent $\nu$,

$$\begin{aligned} g_0(x;0) &= F(\alpha,\beta,\gamma;x) \\ g_0(x;1-\gamma) &= x^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;x) \\ g_\infty(x;\alpha) &= x^{-\alpha}F(\alpha,\alpha-\gamma+1,\alpha-\beta+1;x^{-1}) \\ g_\infty(x;\beta) &= x^{-\beta}F(\beta,\beta-\gamma+1,\beta-\alpha+1;x^{-1}). \end{aligned} \tag{5.3.7}$$

where

$$F(\alpha,\beta,\gamma;x) = \sum_{m=0}^{\infty} \frac{(\alpha)_m(\beta)_m}{(\gamma)_m(1)_m} x^m \tag{5.3.8}$$

is the hypergeometric series, convergent for $|x|<1$, and $(\mu)_m = \mu(\mu+1)\cdots(\mu+m-1)$. By analytic continuation,

$$(g_0(x;0),g_0(x;1-\gamma)) = (g_\infty(x;\alpha),g_\infty(x;\beta))P(\alpha,\beta,\gamma) \tag{5.3.9}$$

on the domain $0<\arg(x)<2\pi$, where

$$P(\alpha,\beta,\gamma) = \begin{pmatrix} e^{\pi i\alpha}\frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)} & e^{\pi i(1-\gamma+\alpha)}\frac{\Gamma(2-\gamma)\Gamma(\beta-\alpha)}{\Gamma(1-\gamma+\beta)\Gamma(1-\alpha)} \\ e^{\pi i\beta}\frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)} & e^{\pi i(1-\gamma+\beta)}\frac{\Gamma(2-\gamma)\Gamma(\alpha-\beta)}{\Gamma(1-\gamma+\alpha)\Gamma(1-\beta)} \end{pmatrix} \tag{5.3.10}$$

when $\alpha$, $\beta$, $\beta-\gamma$ and $\gamma-\alpha$ are non-integral (in addition to non-integral $\gamma$ and $\alpha-\beta$). We note that: (a) Each side of (5.3.9) is analytic in $x$, $\alpha$, $\beta$ and $\gamma$; (b) The gamma function $\Gamma(z)$ is analytic except at the points $z=-n$, $n=0,1,\ldots$, where it has simple poles; it has no zeros, since $\Gamma(z)\Gamma(1-z)=\pi/\sin\pi z$ and $\sin z$ is an entire function. We deduce the following. Let $\gamma$ and $\alpha-\beta$ be non-integral. If $\alpha$, $\beta$, $\beta-\gamma$ and $\gamma-\alpha$ are also non-integral, the connection matrix $P(\alpha,\beta,\gamma)$ has no vanishing entries. When precisely one of $\alpha$, $\beta$, $\beta-\gamma$ and $\gamma-\alpha$ is an integer, there is precisely one vanishing entry.

**5.3.1. Proposition.** *Let $h_2=h_{1,2}$ or $h_{2,1}$. The braiding relation*

$$\phi^{h_3}_{h_4h}(z_3)\phi^{h_2}_{hh_1}(z_2) = \sum_k \phi^{h_2}_{h_4k}(z_2)\phi^{h_3}_{kh_1}(z_3)\,C^{h_4h_3h_2h_1}_{kh} \tag{5.3.1.1}$$

*is valid in the sense of analytic continuation on the domain $0<\arg(x)<2\pi$. The braiding coefficients are given by*

$$\begin{pmatrix} C_{k_+h_-} & C_{k_+h_+} \\ C_{k_-h_-} & C_{k_-h_+} \end{pmatrix} = e^{\pi i\gamma_{23}}\,P(\alpha,\beta,\gamma), \tag{5.3.1.2}$$

*where $C_{k_+h_-} = C^{h_4h_3h_2h_1}_{k_+h_-}$, with $k_\pm = h_{p_4,q_4\pm1}$ or $h_{p_4\pm1,q_4}$; and $h_\pm = h_{p_1,q_1\pm1}$ or $h_{p_1\pm1,q_1}$, depending on whether $h_2=h_{1,2}$ or $h_{2,1}$. Moreover, none of the braiding coefficients vanish.*

*Proof.* By comparing exponents, we must have, with $x=z_2/z_3$,

$$\begin{aligned} \langle\phi^{h_3}_{h_4h}(z_3)\phi^{h_2}_{hh_1}(z_2)\zeta_{h_1},\zeta_{h_4}\rangle &= z_3^{-\{\gamma_{13}+\gamma_{23}\}}\,z_2^{-\gamma_{12}}\,(1-x)^{-\gamma_{23}}\,g_0(x,\nu_h); \\ \langle\phi^{h_2}_{h_4k}(z_2)\phi^{h_3}_{kh_1}(z_3)\zeta_{h_1},\zeta_{h_4}\rangle &= z_3^{-\{\gamma_{13}+\gamma_{23}\}}\,z_2^{-\gamma_{12}}\,x^{-\gamma_{23}}\,(1-\frac{1}{x})^{-\gamma_{23}}\,g_\infty(x,\mu_k), \end{aligned} \tag{5.3.1.3}$$

respectively on the domains $|x|<1$ and $|x|>1$, with

$$\begin{aligned} \nu_h &= \begin{cases} 0 & \text{if } h=h_{p_1,q_1-1} \text{ (resp. } h_{p_1-1,q_1}) \\ 1-\gamma & \text{if } h=h_{p_1,q_1+1} \text{ (resp. } h_{p_1+1,q_1}); \end{cases} \\ \mu_k &= \begin{cases} \alpha & \text{if } k=h_{p_4,q_4+1} \text{ (resp. } h_{p_4+1,q_4}) \\ \beta & \text{if } k=h_{p_4,q_4-1} \text{ (resp. } h_{p_4-1,q_4}). \end{cases} \end{aligned} \tag{5.3.1.4}$$

Moreover, by analytic continuation,

$$(1-x)^{-\gamma_{23}} = e^{\pi i\gamma_{23}}\,x^{-\gamma_{23}}\,(1-\frac{1}{x})^{-\gamma_{23}} \tag{5.3.1.5}$$

on $0<\arg(x)<2\pi$. The first assertion of the Proposition follows from the connection formula (5.3.9), at least in the case when there are 2 channels. When there is a single channel, we need to check that the 4-point functions analytically continue to each other. This follows from the vanishing of an appropriate entry in the connection matrix $P(\alpha,\beta,\gamma)$. We now check this, and prove the second assertion of the Proposition. We do this for $h_2=h_{1,2}$; the other case is directly analogous.

We deduce the necessary and sufficient conditions for there to be a single channel. If

$$\langle\phi^{h_4}_{0h_4}(z_4)\phi^{h_3}_{h_4h}(z_3)\phi^{h_2}_{hh_1}(z_2)\phi^{h_1}_{h_10}(z_1)\rangle. \tag{5.3.1.6}$$

is a 4-point function with $h_2=h_{1,2}$, then $h\in\langle h_{p_1,q_1},h_{1,2}\rangle\cap\langle h_{p_3,q_3},h_{p_4,q_4}\rangle$. Recall that $h_{p,q}=h_{m-p,m+1-q}$. For each choice of the $(p_i,q_i)$, precisely one of two situations obtains: either $p_1+p_3+p_4\in 2\mathbf{Z}+1$ and $q_1+q_3+q_4\in 2\mathbf{Z}$; or $p_1+p_3+p_4$ and $q_1+q_3+q_4\in m+1+2\mathbf{Z}$. Replacing any one of the pairs $(p_i,q_i)$ by $(m-p_i,m+1-q_i)$ moves us between the two situations. So we can choose the $(p_i,q_i)$ such that the former situation holds. With this convention, there is a single channel if and only if

$$q_1+1=|q_3-q_4|+1 \quad or \quad q_1-1=\min(q_3+q_4-1,2(m+1)-q_3-q_4-1); \tag{5.3.1.7}$$

if and only if *one* of the following mutually exclusive conditions (since $1 \leq q_i \leq m$) holds:

$$\begin{aligned} q_1 + q_3 - q_4 &= 0 \\ q_1 + q_4 - q_3 &= 0 \\ q_3 + q_4 - q_1 &= 0 \\ q_1 + q_3 + q_4 &= 2m+2. \end{aligned} \tag{5.3.1.8}$$

But it is straightforward to check that

$$\begin{aligned} \alpha \in \mathbf{Z} &\Leftrightarrow q_1 + q_3 - q_4 = 0 \\ \beta \in \mathbf{Z} &\Leftrightarrow q_1 + q_3 + q_4 = 2m+2 \\ \alpha - \gamma \in \mathbf{Z} &\Leftrightarrow q_1 + q_4 - q_3 = 0 \\ \beta - \gamma \in \mathbf{Z} &\Leftrightarrow q_3 + q_4 - q_1 = 0. \end{aligned} \tag{5.3.1.9}$$

By the remarks following (5.3.10), it follows that the connection matrix $P(\alpha, \beta, \gamma)$ has a single vanishing entry when there is a single channel; and no vanishing entries otherwise. This proves the second assertion for the 2-channel case. Moreover, when there is a single channel, the vanishing entry is by inspection the required one, so that the 4-point functions are analytic continuations of each other. □

**5.4. Braiding relations of arbitrary primary fields.** In principle, arbitrary 4-point functions can be studied along the same lines as in § 5.3, allowing us to deduce the braiding relations of arbitrary primary fields. As for Theorem 4.7.1, however, technical difficulties mean that we seek an alternative method. Computing the braiding coefficients, i.e. the connection problem for special classes of $n^{th}$ order ($n \geq 3$) Fuchsian equations with 3 singular points, is left to experts. We defer the proof of the following theorem to Chapter III.

**5.4.1. Theorem.** *The braiding relation*

$$\phi^{h_3}_{h_4 h}(z_3)\phi^{h_2}_{h h_1}(z_2) = \sum_k \phi^{h_2}_{h_4 k}(z_2)\phi^{h_3}_{k h_1}(z_3)\, C^{h_4 h_3 h_2 h_1}_{k h} \tag{5.4.1.1}$$

*holds on $0 < \arg(x) < 2\pi$ in the sense of analytic continuation.*

**5.5. Braiding relations of secondary fields.** We have already noted that the braiding relations of primary fields hold with insertions of $T(x)$'s in the 4-point functions. This implies that secondary fields also satisfy braiding relations (with insertions), and that the braiding behaviour is precisely as for the corresponding ancestral primary fields. We claim that, by analytic continuation in $x = z_2/z_3$,

$$\phi^{h_3}_{h_4 h}(\xi; z_3)\phi^{h_2}_{h h_1}(\eta; z_2) = \sum_k \phi^{h_2}_{h_4 l}(\eta; z_2)\phi^{h_3}_{l h_1}(\xi; z_3)\, C^{h_4 h_3 h_2 h_1}_{k h} \tag{5.5.1}$$

on $0 < \arg(x) < 2\pi$, for all $\xi \in V_{h_3}$, $\eta \in V_{h_2}$. As always, this is understood in the sense of taking matrix elements with arbitrary vectors in $V_{h_1}$ and $V_{h_4}$; and it holds with insertions. The proof is by induction and elementary.

## 6. Operator product expansions

**6.0. Overview.** The reduced 4-point functions

$$\langle \phi^{h_3}_{h_4 h}(z_3)\phi^{h_2}_{h h_1}(z_2)\zeta_{h_1}, \zeta_{h_4} \rangle \tag{6.0.1}$$

are, up to a factor, local solutions at $x = 0$ to a Fuchsian equation with singular points at $x = 0, 1$ and $\infty$ (at least when some $h_i = h_{1,2}$ or $h_{2,1}$). Other 4-point functions provide local solutions at $x = \infty$. We have seen that the braiding relations of primary fields are just the solutions to the connection problem from $x = 0$ to $x = \infty$. Operator product expansions are the corresponding solutions to the connection problem from $x = 0$ to $x = 1$. Remarkably, the two problems are equivalent, in a sense that will become apparent; this is fusion-braiding duality.

**6.1. Lemma.** *We have*

$$\phi^{h}_{h0}(\zeta; z)\Omega = \sum_{k=0}^{\infty} \frac{z^k}{k!} L^k_{-1}\zeta. \tag{6.1.1}$$

*Proof.* Let $\phi = \phi^k_{ji}$ be a primary field, $\zeta \in V_k$ and consider $\psi(z) = \phi(\zeta; z)$. A matrix element $\langle \psi(z)\xi, \eta \rangle$ is, up to a factor $z^{-\{h_k + h_i - h_j\}}$, a Laurent polynomial in $z$. Then

$$\langle \psi(z-w)\xi, \eta \rangle = \sum_{m=0}^{\infty} \frac{(-w)^m}{m!} \frac{d^m}{dz^m} \langle \psi(z)\xi, \eta \rangle, \tag{6.1.2}$$

when $|z| > |w|$. As formal series in $z$, $w$,

$$\begin{aligned}\langle e^{-wL_{-1}}\psi(z)e^{wL_{-1}}\xi,\eta\rangle &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{w^{m+n}}{m!\,n!}(-1)^m\langle L_{-1}^m\psi(z)L_{-1}^n\xi,\eta\rangle\\ &= \sum_{r=0}^{\infty}\frac{(-w)^r}{r!}\sum_{n=0}^{r}\binom{r}{n}(-1)^n\langle L_{-1}^{n-r}\psi(z)L_{-1}^r\xi.\eta\rangle\\ &= \sum_{r=0}^{\infty}\frac{(-w)^r}{r!}\langle\{ad(L_{-1})^r\psi\}(z)\xi,\eta\rangle\\ &= \sum_{r=0}^{\infty}\frac{(-w)^r}{r!}\frac{d^r}{dz^r}\langle\psi(z)\xi,\eta\rangle,\end{aligned}\tag{6.1.3}$$

where we have used the identity

$$\{ad(L_{-1})\psi\}(z) = [L_{-1},\psi(z)] = \frac{d\psi(z)}{dz}.\tag{6.1.4}$$

It follows that the series $\langle\psi(z)e^{wL_{-1}}\xi,\eta\rangle$ converges on $|z| > |w|$ to $\langle e^{wL_{-1}}\psi(z-w)\xi,\eta\rangle$. Let $h_i = 0$; then $h_j = h_k$ and $\langle\psi(z)\xi,\eta\rangle$ is a Laurent polynomial, and

$$\langle\psi(z)\Omega,\eta\rangle = \langle e^{wL_{-1}}\psi(z-w)\Omega,\eta\rangle\tag{6.1.5}$$

on $|z| > |w|$, hence for all $(w, z)$, hence for $w = z$. It follows that

$$\phi(\zeta; z)\Omega = e^{zL_{-1}}\zeta.\tag{6.1.6}$$

The lemma could of course have been proved directly using (3.1.2). □

**6.2. Proposition.** *Let* $\xi_i \in V_i = V_{h_i}$, $i = 1,\ldots,4$, *and* $k \in \langle h_4, h_1\rangle \cap \langle h_3, h_2\rangle$. *The formal series*

$$\langle\phi^k_{h_4h_1}(\phi^{h_3}_{kh_2}(\xi_3; w)\xi_2; z)\xi_1,\xi_4\rangle\tag{6.2.1}$$

*converges on* $|w| > |z| > 0$ *to a multivalued holomorphic function; and*

$$\langle\phi^{h_3}_{h_4h}(\xi_3; z_3)\phi^{h_2}_{hh_1}(\xi_2; z_2)\xi_1,\xi_4\rangle = \sum_k F^{h_4h_3h_2h_1}_{kh}\langle\phi^k_{h_4h_1}(\phi^{h_3}_{kh_2}(\xi_3; z_3-z_2)\xi_2; z_2)\xi_1,\xi_4\rangle\tag{6.2.2}$$

*on* $|1-x| < |x| < 1$, $-\pi < \arg(x) < \pi$, $x = z_2/z_3$, *where*

$$F^{h_4h_3h_2h_1}_{kh} = C^{h_4h_1k0\ -1}_{h_1k}\,C^{h_4h_3h_1h_2}_{kh}\,C^{hh_2h_10}_{h_2h_1}.\tag{6.2.3}$$

Here $(z_3 - z_2)^\alpha$ is defined as the binomial expansion of $z_3^\alpha(1-x)^\alpha$. We write (6.2.2) as

$$\phi^{h_3}_{h_4h}(\xi; z_3)\phi^{h_2}_{hh_1}(\eta; z_2) = \sum_k F^{h_4h_3h_2h_1}_{kh}\,\phi^k_{h_4h_1}(\phi^{h_3}_{kh_2}(\xi; z_3-z_2)\eta; z_2),\tag{6.2.4}$$

the *operator product expansion* or *fusion relation*; and $(F^{h_4h_3h_2h_1}_{kh})_{kh}$ is the corresponding *fusion matrix*. We can think of (6.2.3) as a statement of "fusion-braiding duality". The following proof generalises some arguments of Goddard [Go] from "meromorphic" conformal field theories to the ones at hand with non-trivial braiding relations.

*Proof.* We shall assume that the $\xi_i$ are lowest energy vectors, $\xi_i = \zeta_i$; the proof is the same in the general case because secondary fields satisfy analogous braiding relations. Then (6.2.2) has the form

$$f(x) = \sum_k F^{h_4h_3h_2h_1}_{kh}\,x^{\alpha_k}(1-x)^{\beta_k}g_k\left(\frac{1-x}{x}\right),\tag{6.2.5}$$

where $f$ and the $g_k$ are power series converging on the unit disc; and $\alpha_k = h_4 + h_2 - h - k$, $\beta_k = k - h_3 - h_2$. To prove (6.2.2) on $|1-x| < |x| < 1$, $-\pi < \arg(x) < \pi$, it suffices to prove it on a line segment in this simply-connected domain. In the following, we take

$$z_3 = (1+\varepsilon)e^{-(\frac{3}{4}\pi+\delta)i};\quad z_2 = e^{-\frac{3}{4}\pi i},\tag{6.2.6}$$

with small $\delta$, $\varepsilon > 0$; then $x = z_2/z_3$ lies in the required domain. The series

$$\langle e^{wL_{-1}}\phi^{h_3}_{h_4h}(z_3)\phi^{h_2}_{hh_1}(z_2)\zeta_1,\zeta_4\rangle = \sum_{m=0}^{\infty}\frac{w^m}{m!}\langle L^m_{-1}\phi^{h_3}_{h_4k}(z_3)\phi^{h_2}_{kh_1}(z_2)\zeta_1,\zeta_4\rangle\tag{6.2.7}$$

and

$$\sum_{m,n=0}^{\infty}\frac{w^{m+n}}{m!\,n!}\frac{d^m}{dz_3^m}\frac{d^n}{dz_2^n}\langle\phi^{h_3}_{h_4h}(z_3)\phi^{h_2}_{hh_1}(z_2)\phi^{h_1}_{h_10}(w)\Omega,\zeta_4\rangle\tag{6.2.8}$$

are equal, at least formally. However, since the 4-point function

$$\langle\phi^{h_3}_{h_4h}(z_3)\phi^{h_2}_{hh_1}(z_2)\phi^{h_1}_{h_10}(w)\Omega,\zeta_4\rangle\tag{6.2.9}$$

converges on $|z_3| > |z_2| > |w|$ to a holomorphic function, it follows that

$$\langle\phi^{h_3}_{h_4h}(z_3)\phi^{h_2}_{hh_1}(z_2)\zeta_1,\zeta_4\rangle = \langle e^{-wL_{-1}}\phi^{h_3}_{h_4h}(z_3+w)\phi^{h_2}_{hh_1}(z_2+w)\phi^{h_1}_{h_10}(w)\Omega,\zeta_4\rangle\tag{6.2.10}$$

for sufficiently small $|w|$. Now we use the braiding relations to analytically continue the right-hand-side in the auxiliary variable $w$, obtaining

$$\begin{aligned}&\sum_k \gamma_{kh} \langle e^{-wL_{-1}} \phi^{h_1}_{h_4 k}(w) \phi^{h_3}_{k h_2}(z_3 + w) \phi^{h_2}_{h_2 0}(z_2 + w) \Omega, \zeta_4 \rangle \\ &= \sum_k \gamma_{kh} \langle e^{-wL_{-1}} \phi^{h_1}_{h_4 k}(w) \phi^{h_3}_{k h_2}(z_3 + w) e^{(z_2 + w)L_{-1}} \zeta_2, \zeta_4 \rangle,\end{aligned} \tag{6.2.11}$$

where

$$\gamma_{kh} = C^{h_4 h_3 h_1 h_2}_{k\,h} C^{h\,h_2 h_1 0}_{h_2 h_1}. \tag{6.2.12}$$

More precisely, we analytically continue in $w$ along the line segment $t\, e^{\frac{1}{4}\pi i}$, $0 \le t \le 1$. Note that $0 < \arg(w/(z_i + w)) < 2\pi$ along this path. At $t = 1$, we obtain

$$\sum_k \gamma_{kh} \langle e^{z_2 L_{-1}} \phi^{h_1}_{h_4 k}(-z_2) \phi^{h_3}_{k h_2}(z_3 - z_2) \zeta_2, \zeta_4 \rangle, \tag{6.2.13}$$

where $-z_2 = e^{\frac{1}{4}\pi i}$. We put this in the required form: introduce homogeneous orthonormal bases $\{e^{(k)}_n\}$ of $V_k$; its series expansion can be written as

$$\sum_k \gamma_{kh} \sum_n \langle e^{z_2 L_{-1}} \phi^{h_1}_{h_4 k}(-z_2) e^{(k)}_n, \zeta_4 \rangle \langle \phi^{h_3}_{k h_2}(z_3 - z_2) \zeta_2, e^{(k)}_n \rangle, \tag{6.2.14}$$

and is convergent on $|z_2| > |z_3 - z_2| > 0$. We claim that

$$\langle e^{z_2 L_{-1}} \phi^{h_1}_{h_4 k}(-z_2) e^{(k)}_n, \zeta_4 \rangle = \langle \phi^{k}_{h_4 h_1}(e^{(k)}_n; z_2) \zeta_1, \zeta_4 \rangle\, C^{h_4 h_1 k 0\; -1}_{h_1 k}. \tag{6.2.15}$$

Introduce another auxiliary variable $y$. For $|y|$ sufficiently small, the left-hand-side is

$$\begin{aligned}&\langle e^{z_2 L_{-1}} \phi^{h_1}_{h_4 k}(-z_2) e^{-yL_{-1}} \phi^{k}_{k 0}(e^{(k)}_n; y) \Omega, \zeta_4 \rangle \\ &= \langle e^{(z_2 - y)L_{-1}} \phi^{h_1}_{h_4 k}(y - z_2) \phi^{k}_{k 0}(e^{(k)}_n; y) \Omega, \zeta_4 \rangle,\end{aligned} \tag{6.2.16}$$

which has an analytic continuation in $y$ to

$$\begin{aligned}&\langle e^{(z_2 - y)L_{-1}} \phi^{k}_{h_4 h_1}(e^{(k)}_n; y) \phi^{h_1}_{h_1 0}(y - z_2) \Omega, \zeta_4 \rangle\, C^{h_4 h_1 k 0}_{h_1 k} \\ &= \langle e^{(z_2 - y)L_{-1}} \phi^{k}_{h_4 h_1}(e^{(k)}_n; y) e^{-(z_2 - y)L_{-1}} \zeta_1, \zeta_4 \rangle\, C^{h_4 h_1 k 0}_{h_1 k}.\end{aligned} \tag{6.2.17}$$

More precisely, we continue in $y$ along the line segment $t\, e^{\frac{5}{4}\pi i}$, $0 \le t \le 1$. On this path, we have $\arg(y/(y - z_2)) = \pi$; its end-point is $e^{2\pi i} z_2$. At $t = 1$, we obtain

$$\begin{aligned}&\langle \phi^{k}_{h_4 h_1}(e^{(k)}_n; e^{2\pi i} z_2) \zeta_1, \zeta_4 \rangle\, C^{h_4 h_1 k 0}_{h_1 k} \\ &= \langle \phi^{k}_{h_4 h_1}(e^{(k)}_n; z_2) \zeta_1, \zeta_4 \rangle\, C^{h_4 h_1 k 0\; -1}_{h_1 k},\end{aligned} \tag{6.2.18}$$

since

$$C^{h_4 h_1 k 0}_{h_1 k} = e^{\pi i \{h_1 + k - h_4\}}. \tag{6.2.19}$$

This proves the claim. Hence the required series expansion is given by

$$\sum_k F^{h_4 h_3 h_2 h_1}_{k\,h} \sum_n \langle \phi^{k}_{h_4 h_1}(e^{(k)}_n; z_2) \zeta_1, \zeta_4 \rangle \langle \phi^{h_3}_{k h_2}(z_3 - z_2) \zeta_2, e^{(k)}_n \rangle, \tag{6.2.20}$$

where

$$F^{h_4 h_3 h_2 h_1}_{k\,h} = C^{h_4 h_1 k 0\; -1}_{h_1 k}\, \gamma_{kh}. \tag{6.2.21}$$

This is just the left-hand-side of (6.2.2), written using homogeneous orthonormal bases for the $V_k$'s. Finally, note that, since $(F^{h_4 h_3 h_2 h_1}_{k\,h})_{k\,h}$ is an invertible matrix, each term in the sum must converge separately. □

# Chapter III
# Coset construction of primary fields

We give a new construction and existence proof of the primary fields associated to the discrete series representations by exploiting the coset construction of Goddard, Kent and Olive [GKO]. This can be regarded, especially in view of the state-field correspondence, as the natural counterpart of the result for representations. The construction makes manifest certain properties of primary fields that are hard to establish, even mysterious, in the Verma module approach. This should be unsurprising — unitarity for instance is manifest in one approach but not the other — and illustrates a general truth about the Hermann Weyl approach to representation theory, which aims to construct all the representations through the decomposition of tensor products of some "simple" ones. It requires that the simpler theories be first understood, in particular those of their properties that are inherited by sub-theories. In the case at hand, these comprise the positive energy representations of the loop group $LG$, $G = SU(2)$, and their primary fields. The corresponding conformal field theory has been studied in detail by Tsuchiya and Kanie [TK]. Moreover, the $LSU(2)$-theory at level $\ell$ can in turn be realised as a sub-theory of the $\ell$-fold tensor product of a free-field fermionic theory. Finally, we use our construction to prove Theorems 4.7.1 and 5.4.1 of Chapter II.

## 1. The loop group theory

Let $G = SU(2)$ and $\mathfrak{g}$ its Lie algebra. We briefly sketch the relevant results of the conformal field theory associated to the positive energy representations of the loop group $LG$.

**1.1. Positive energy representations of $LG$.** The loop group $LG = C^\infty(S^1, G)$, endowed with the $C^\infty$ topology and pointwise multiplication, is a topological group; it has the structure of a regular infinite-dimensional Lie group modelled on the Fréchet space $L\mathfrak{g} = C^\infty(S^1, \mathfrak{g})$ [Mi] [PS]. The group of diffeomorphisms of the circle acts on $LG$ as automorphisms, and we can form the semi-direct product $LG \rtimes \mathrm{Diff}S^1$. We identify $G$ with the subgroup of constant maps in $LG$.

A positive energy representation of $LG$ is a continuous projective unitary representation $\pi : LG \to PU(H)$ on a Hilbert space $H$ that extends to a representation of $LG \rtimes \mathrm{Rot}\,S^1$, and is positive energy as a representation of the rotation group. Then $H^{fin}$ is a representation of the affine Kac-Moody algebra $\hat{\mathfrak{g}} = \sum_{n\in\mathbb{Z}} \mathfrak{g}_\mathbb{C} \otimes \mathbb{C}e^{in\theta} \oplus \mathbb{C}\ell$,

$$[x_m, y_n] = [x, y]_{m+n} + \ell\, m\, \mathrm{tr}(xy)\, \delta_{m+n,0}, \tag{1.1.1}$$

where $x_m = x \otimes e^{im\theta}$, and $\ell$ is a central element. To this is appended an element $L_0$,

$$[L_0, x_m] = -m\, x_m, \tag{1.1.2}$$

the (negative of the) natural grading operator, corresponding to the infinitesimal generator of the rotation subgroup.

A representation $\varphi : \hat{\mathfrak{g}} \to \mathrm{End}W$ constructed from a positive energy representation of $LG$ has the following properties (cf. § I.2.1): (a) $W$ is a locally-finite graded vector space, $W = \sum_{n\in\mathbb{Z}} W(n)$, with $W(n) = 0$ for $n > 0$ and $W(0) \neq 0$; (b) $\varphi$ is a graded homomorphism; (c) The central element $\ell$ acts by scalar multiplication by some $\ell \in \mathbb{R}$, the *level*; (d) The representation is unitary, i.e. there is a contravariant inner product on $W$. Then the level $\ell$ is a strictly positive integer, and $W(0)$ a $\mathfrak{g}$-module, irreducible if $W$ is irreducible.

The irreducible representations of $\hat{\mathfrak{g}}$ that satisfy (a)–(d) are the unitary highest weight representations. Here, we note that, given the vector space decomposition $\mathfrak{g}_\mathbb{C} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, we have also $\hat{\mathfrak{g}} = \mathfrak{m}_- \oplus \mathfrak{l} \oplus \mathfrak{m}_+$, where $\mathfrak{m}_\pm = \mathfrak{n}_\pm \oplus \sum_{n\geq 1} \mathfrak{g}_\mathbb{C} \otimes \mathbb{C}e^{\pm in\theta}$ and $\mathfrak{l} = \mathfrak{h} \oplus \mathbb{C}\ell$. A unitary highest weight module $W$ is determined up to isomorphism by the highest weight $(\ell, j)$, i.e. the level $\ell$ and the spin $j$ that labels the $\mathfrak{g}$-module $W(0)$. These have been classified: at each level $\ell \geq 1$, the unitary highest weight $\hat{\mathfrak{g}}$-modules $W_{j,\ell}$ correspond to spin

$$j = 0, \frac{1}{2}, \ldots, \frac{\ell}{2}. \tag{1.1.3}$$

Let $\mathfrak{g}_\mathbb{C} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be the standard vector space decomposition, where $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) is the Lie algebra of strictly super- (resp. infra-) diagonal matrices, and $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$ the canonical Borel subalgebra. Recall that, if $U$ is an irreducible $G$-module, a primitive vector $u \in U$ (i.e. an eigenvector for $\mathfrak{b}$) is unique up to scalar multiplication. We choose unit vectors $\zeta_{j,\ell} \in W_{j,\ell}(0)$ that are primitive for $\mathfrak{b}$; then they are primitive for $\mathfrak{l} \oplus \mathfrak{m}_+$. In particular, let $\Omega_\ell = \zeta_{0,\ell}$, the vacuum vector at level $\ell$.

A positive energy representation $H$ of $LG$ is completely reducible. It is irreducible if and only if $H^{fin}$ is irreducible as a $\hat{\mathfrak{g}}$-module. In this case, $H^{fin}$ is a unitary highest weight module. Conversely, every unitary highest weight module of $\hat{\mathfrak{g}}$ integrates to a positive energy representation of $LG$.

A unitary highest weight representation $W_{j,\ell}$ extends to a representation of $\hat{\mathfrak{g}} \rtimes \mathfrak{Vir}$ by the Segal-Sugawara construction

$$L_m = -\frac{1}{\ell+2}\sum_i \left\{ \sum_{n\geq 0} x^i_{m-n}x^i_n + \sum_{n\geq 1} x^i_{-n}x^i_{m+n} \right\}, \tag{1.1.4}$$

where $\{x^i\}$ is a basis for $\mathfrak{g}$ such that $\mathrm{tr}(x^i x^j) = -\frac{1}{2}\delta_{ij}$. Then

$$\begin{aligned} [L_m, L_n] &= (m-n)\,L_{m+n} + \frac{c(\ell)}{12}\,(m^3 - m)\,\delta_{m+n,0} \\ [L_m, x_n] &= -n\,x_{m+n}, \end{aligned} \tag{1.1.5}$$

with central charge $c(\ell) = 3\ell/(\ell+2)$. In particular, we have a canonical choice for $L_0$, given by (1.1.4), with lowest eigenvalue $j(j+1)/(\ell+2)$. Moreover, an irreducible positive energy representation of $LG$ extends to a representation of $LG \rtimes \mathrm{Diff}^+S^1$.

**1.2. The associated primary fields.** The notion of primary fields associated to unitary highest weight representations of $\hat{\mathfrak{g}}$ is as for the Virasoro algebra. Let $U$ be an irreducible unitary $G$-module. Then $U_{\lambda,\mu} = V_{\lambda,\mu}\otimes U$ is a jointly continuous representation of $LG$, given by pointwise multiplication, and therefore an ordinary representation of $LG \rtimes \overline{\mathrm{Diff}^+S^1}$. The subspace $U^{fin}_{\lambda,\mu}$ of elements with finite Fourier series is an ordinary representation of $\hat{\mathfrak{g}} \rtimes \mathfrak{Vir}$, respectively at level $\ell = 0$ and central charge $c = 0$. Let $W_1$, $W_2$ be unitary highest weight representations of $\hat{\mathfrak{g}}$ at level $\ell$. A primary field is a linear map $\phi : W_1 \otimes U^{fin}_{\lambda,\mu} \to W_2$ intertwining the action of $\hat{\mathfrak{g}} \rtimes \mathfrak{Vir}$. If $W_1$, $W_2$ and $U$ correspond to spins $j_1$, $j_2$ and $j_3$, then $\mu = h_1 - h_2$ and $\lambda = 1 - h_3$, where $h_i = j_i(j_i+1)/(\ell+2)$. We say that the primary field $\phi$ has spin $j_3$.

For $u \in U$, we write $\phi(u; n) = \phi(u \otimes e^{in\theta})$ and

$$\phi(u; z) = \sum_{n\in\mathbf{Z}} \phi(u; n) z^{-n-(h_3+h_1-h_2)}. \tag{1.2.1}$$

We have the covariance relations

$$\begin{aligned} [x_m, \phi(u; z)] &= z^m \phi(xu; z) \\ [L_m, \phi(u; z)] &= z^{m+1}\frac{d}{dz}\phi(u; z) + h_3\,(m+1)\,z^m\,\phi(u; z). \end{aligned} \tag{1.2.2}$$

These are not all independent; in view of the Segal-Sugawara construction, the commutators $[L_m, \phi(u; z)]$, $m \neq 0$, are determined by the other relations. These covariance relations

essentially characterise the primary field $\phi$. It is specified up to a scalar multiple by the level $\ell$ and the ordered triplet of spins $(j_3, j_1, j_2)$. The primary fields associated to the unitary highest weight representations of $\hat{\mathfrak{g}}$ have been classified [TK]. The primary field $\phi : W_1 \otimes U^{fin}_{\lambda,\mu} \to W_2$ exists at level $\ell$ if and only if

$$\mathrm{Hom}_G(U_{j_3} \otimes U_{j_1}, U_{j_2}) = \mathbf{C}; \quad j_i \leq \frac{\ell}{2},\ i = 1,2,3; \quad \sum j_i \leq \ell, \tag{1.2.3}$$

where $U_{j_1} \cong W_1(0)$, $U_{j_2} \cong W_2(0)$ and $U_{j_3} \cong U$ as $G$-modules. This condition is invariant under the permutation of the spins $j_i$. At a fixed level $\ell$, if $U = W_{j_3}(0)$, we denote by $\phi^{j_3}_{j_2 j_1}$ the normalised form of a primary field $\phi : W_{j_1} \otimes U^{fin}_{\lambda,\mu} \to W_{j_2}$, satisfying

$$\langle\, \phi^{j_3}_{j_2 j_1}(\zeta_{j_3})\zeta_{j_1}, \zeta_{j_2} \,\rangle \;=\; 1. \tag{1.2.4}$$

Let $U$ be an irreducible unitary $G$-module, $U^*$ its dual space, and $u \mapsto u^* = \langle\, .\,, u\rangle$ the canonical conjugate-linear isomorphism of $G$-modules. Together with complex conjugation on $C^\infty(S^1)$, we obtain a conjugate-linear map $* : U_{\lambda,\mu} \to U^*_{\lambda,-\mu}$ intertwining the action of $LG \rtimes \overline{\mathrm{Diff}^+S^1}$. If $\phi : W_1 \otimes U^{fin}_{\lambda,\mu} \to W_2$ is the primary field given above, we have the conjugate primary field $\phi^* : W_2 \otimes U^{*fin}_{\lambda,-\mu} \to W_1$, given by $\langle \phi^*(g^*)\xi_2, \xi_1\rangle = \langle \xi_2, \phi(g)\xi_1\rangle$. Since $G = SU(2)$, the conjugate $G$-module $U^*$ is isomorphic to $U$; the isomorphism $U \to U^*$ is unique up to a scalar multiple. Let $U = W_{j_3}(0)$; then we can identify $U^*$ with $U$ in such a way that $\phi^{j_3\,*}_{j_2 j_1} = \phi^{j_3}_{j_1 j_2}$.

**1.3. The state-field correspondence.** A primary field $\phi$ with spin $j$ at level $\ell$ generates a $\hat{\mathfrak{g}}$-module isomorphic to the unitary highest weight module $W_{j,\ell}$ with highest weight $(\ell, j)$. The $\hat{\mathfrak{g}}$-action is given by

$$\{\hat{x}_m\psi\}(z) = \left[\sum_{r=0}^{\infty}\binom{m}{r}(-z)^r x(m-r)\right]\psi(z) - \psi(z)\left[\sum_{r=0}^{\infty}\binom{m}{r}(-z)^{m-r}x(r)\right], \tag{1.3.1}$$

and the $\mathfrak{Vir}$-action is as before. We write $\psi(z) = \phi(\xi; z)$ for that element corresponding to the vector $\xi \in W_{j,\ell}$, so that $\{\hat{x}_m\psi\}(z) = \phi(x_m\xi; z)$.

**1.4. Braiding relations and operator product expansions.** The correlation functions and braiding relations have been studied in some detail by Tsuchiya and Kanie [TK]. Fix a level $\ell \geq 1$. Let $\phi_i = \phi^{j_i}_{l_{i+1} l_i}$, $i = 1, \ldots, n$, be primary fields, with $l_1 = l_{n+1} = 0$. An n-point function is a formal series

$$\langle \phi_n(\,.\,; z_n)\cdots\phi_1(\,.\,; z_1)\Omega, \Omega\rangle, \tag{1.4.1}$$

regarded as a form on $W_{j_n}(0)\otimes\cdots\otimes W_{j_1}(0)$. It is a formal solution to an integrable Pfaffian system of partial differential equations, the Knizhnik-Zamolodchikov equations, at a regular singular point; and converge on $|z_n| > \cdots > |z_1|$.

The local solutions of the KZ equations are spanned by the $n$-point functions, and we have the braiding relations

$$\phi^{j_3}_{j_4 j}(\xi;\, z_3)\phi^{j_2}_{j\, j_1}(\eta;\, z_2) \;=\; \sum_l \phi^{j_2}_{j_4 l}(\eta;\, z_2)\phi^{j_3}_{l\, j_1}(\xi;\, z_3)\, C^{j_4 j_3 j_2 j_1}_{l\, j}, \tag{1.4.2}$$

valid on $0 < \arg(x) < 2\pi$, $x = z_2/z_3$, in the sense of analytic continuation, and in the sense of taking matrix elements. By explicit calculation, the braiding matrix $(C^{j_4 \frac{1}{2}\, j_2 j_1}_{l\, j})_{l\, j}$, has no vanishing entries.

The operator product expansion

$$\phi^{j_3}_{j_4 j}(\xi;\, z_3)\phi^{j_2}_{j\, j_1}(\eta;\, z_2) \;=\; \sum_l F^{j_4 j_3 j_2 j_1}_{l\, j}\, \phi^{l}_{j_4 j_1}(\phi^{j_3}_{l\, j_2}(\xi;\, z_3 - z_2)\eta;\, z_2) \tag{1.4.3}$$

is valid on $|1 - x| < |x| < 1$, $x = z_2/z_3$, in the sense of taking matrix elements, with

$$F^{j_4 j_3 j_2 j_1}_{l\, j} \;=\; C^{j_4 j_1 l 0}_{j_1\, l}{}^{-1} C^{j_4 j_3 j_1 j_2}_{l\, j} C^{j\, j_2 j_1 0}_{j_2 j_1}. \tag{1.4.4}$$

This is proved in the same way as Proposition 6.2.

## 2. Coset construction of discrete series representations

**2.1. Coset construction of discrete series representations.** The construction by Goddard, Kent and Olive [GKO] of the discrete series representations proceeds as follows.

(i) The tensor product $W_1 \otimes W_2$ of unitary highest weight $\hat{\mathfrak{g}}$-modules, respectively at levels $\ell_1$ and $\ell_2$, is naturally a $\hat{\mathfrak{g}}$-module satisfying the conditions (a)–(d) of § 1.1, at level $\ell_1 + \ell_2$. (ii) By the Segal-Sugawara construction, each $W_i$ is also a $\mathfrak{Vir}$-module satisfying the conditions (a)–(d) of § I.2.1, at central charge $c_i = 3\ell_i/(\ell_i + 2)$. Therefore, the tensor product $W_1 \otimes W_2$ is naturally a $\mathfrak{Vir}$-module satisfying the same conditions, at central charge $c_1 + c_2$; let $L_m$, $m \in \mathbf{Z}$, be the corresponding Virasoro algebra elements. (iii) The Segal-Sugawara construction for the level $\ell_1 + \ell_2$ representation of $\hat{\mathfrak{g}}$ makes the tensor product space a representation of $\mathfrak{Vir}$ at central charge $3(\ell_1 + \ell_2)/(\ell_1 + \ell_2 + 2)$; let $K_m$, $m \in \mathbf{Z}$, be the corresponding Virasoro algebra elements. (iv) The deficit

$$\frac{3\ell_1}{\ell_1 + 2} + \frac{3\ell_2}{\ell_2 + 2} - \frac{3(\ell_1 + \ell_2)}{\ell_1 + \ell_2 + 2} \tag{2.1.1}$$

is the central charge of the "coset" $\mathfrak{Vir}$-action, corresponding to the Virasoro algebra elements $T_m = L_m - K_m$, $m \in \mathbf{Z}$. These commute with the $\hat{\mathfrak{g}}$-action, and therefore also with the $\mathfrak{Vir}$-action given by the $K_m$'s. The inner product on the tensor product space is contravariant with respect to the natural $\hat{\mathfrak{g}}$ and $\mathfrak{Vir}$ actions, and therefore also the coset $\mathfrak{Vir}$ action. (v) The tensor product space $W_1 \otimes W_2$ therefore supports commuting actions of $\hat{\mathfrak{g}}$ and $\mathfrak{Vir}$, at level $\ell_1 + \ell_2$ and central charge (2.1.1) respectively. As a $\hat{\mathfrak{g}}$-module, $W_1 \otimes W_2$ is completely reducible to a direct sum of (irreducible) unitary highest weight modules at level $\ell_1 + \ell_2$. There are only a finite number of these, and we have

$$W_1 \otimes W_2 \;=\; \sum_j W_j \otimes X_j, \tag{2.1.2}$$

where the $W_j$ are distinct unitary highest weight $\hat{\mathfrak{g}}$-modules at level $\ell_1 + \ell_2$, and the $X_j$ are multiplicity spaces. Then each $X_j$ is a $\mathfrak{Vir}$-module at central charge (2.1.1), satisfying (a)–(d) of § I.2.1, and therefore completely reducible to a (possibly infinite) direct sum of unitary highest weight $\mathfrak{Vir}$-modules. (vi) The central charge (2.1.1) is strictly less than one if and only if one of the $\ell_i$ is equal to one. So let $\ell_1 = \ell$ and $\ell_2 = 1$. Then each $X_j$ is a finite direct sum of discrete series representations.

**2.2. Theorem (Goddard-Kent-Olive).** *We have*

$$W_{l,\ell} \otimes W_{\varepsilon,1} \;\cong\; \sum_j W_{j,\ell+1} \otimes V_{h_{2l+1,2j+1},\, c(\ell)}, \tag{2.2.1}$$

*where* $c(\ell) = 1 - 6/(\ell + 2)(\ell + 3)$, *and the sum is over* $j = 0, \frac{1}{2}, \ldots, \frac{\ell+1}{2}$, *with* $j + l + \varepsilon \in \mathbb{Z}$.

It is easy to check that every discrete series representation can be constructed in this way.

## 3. Coset construction of discrete series primary fields

In the following, let $l$ (resp. $l_1, l_2, \ldots$) denote the highest weight $(l, \ell)$ corresponding to spin $l$ at level $\ell$. Let $W_l$ be the unitary highest weight $\hat{\mathfrak{g}}$-module with highest weight $(l, \ell)$; let $\phi^{l_3}_{l_2\, l_1}$ be a normalised primary field at level $\ell$; and let $h_l = l(l + 1)/(\ell + 2)$. Similarly, we use the indices $\varepsilon, \varepsilon_1, \varepsilon_2, \ldots$, at level 1; and $j, j_1, j_2, \ldots$, at level $\ell + 1$. We also let $h_{p,q} = h_{2l+1,2j+1}$, corresponding to the central charge $c(\ell) = 1 - 6/(\ell + 2)(\ell + 3)$; and let $V_{h_{p,q}}$ be the discrete series representation with highest weight $(h_{p,q}, c(\ell))$. When there is no confusion, we also let $h_i = h_{p_i,q_i}$.

**3.1. Coset construction for primary fields.** Using the state-field correspondence, we can replace $\hat{\mathfrak{g}}$-modules $W_{l_3}$ and $W_{\varepsilon_3}$ by the isomorphic $\hat{\mathfrak{g}}$-modules generated by primary fields $\phi^{l_3}_{l_2\,l_1}(\,.\,;z)$ and $\phi^{\varepsilon_3}_{\varepsilon_2\,\varepsilon_1}(\,.\,;z)$. Corresponding to $\xi\otimes\eta\in W_{l_3}\otimes W_{\varepsilon_3}$, we have

$$Y(\xi\otimes\eta;z) \;=\; \phi^{l_3}_{l_2\,l_1}(\xi;z)\otimes\phi^{\varepsilon_3}_{\varepsilon_2\,\varepsilon_1}(\eta;z). \tag{3.1.1}$$

This is a formal series whose coefficients are linear maps from $W_{l_1}\otimes W_{\varepsilon_1}$ to $W_{l_2}\otimes W_{\varepsilon_2}$. The linear span of these elements is a $\hat{\mathfrak{g}}$-module isomorphic to $W_{l_3}\otimes W_{\varepsilon_3}$. To be sure, if $L_0\xi=\{h_{l_3}+n(\xi)\}\,\xi$, then

$$\phi^{l_3}_{l_2\,l_1}(\xi;z) \;=\; \sum_{n\in\mathbb{Z}}\phi^{l_3}_{l_2\,l_1}(\xi;n)\,z^{-n-\{h_{l_3}+n(\xi)+h_{l_1}-h_{l_2}\}}, \tag{3.1.2}$$

where the integral moding of each coefficient is its degree as a graded linear map $W_{l_1}\to W_{l_2}$. And if also $L_0\eta=\{h_{\varepsilon_3}+n(\eta)\}\,\eta$, then

$$Y(\xi\otimes\eta;z) \;=\; \sum_{n\in\mathbb{Z}} z^{-n-n(\xi)-n(\eta)-\Delta(l)-\Delta(\varepsilon)}\sum_{m\in\mathbb{Z}}\phi^{l_3}_{l_2\,l_1}(\xi;m)\otimes\phi^{\varepsilon_3}_{\varepsilon_2\,\varepsilon_1}(\eta;n-m), \tag{3.1.3}$$

where $\Delta(l)=h_{l_3}+h_{l_1}-h_{l_2}$, $\Delta(\varepsilon)=h_{\varepsilon_3}+h_{\varepsilon_1}-h_{\varepsilon_2}$. Each of the tensor product spaces $W_{l_i}\otimes W_{\varepsilon_i}$ decomposes into irreducibles for the action of $\hat{\mathfrak{g}}\times\mathfrak{Vir}$,

$$W_{l_i}\otimes W_{\varepsilon_i} \;=\; \sum_{j_i} W_{j_i}\otimes V_{h_{\varepsilon_i,\varepsilon_i}} \tag{3.1.4}$$

according to the GKO theorem. Let $P_{j_i}$ be the projection onto the spin-$j_i$ summand. Then

$$Y(\,.\,;z) \;=\; \sum_{j_1,j_2,j_3} Y^{j_3}_{j_2\,j_1}(\,.\,;z), \tag{3.1.5}$$

where

$$\langle Y^{j_3}_{j_2\,j_1}(\xi_3;z)\xi_2,\xi_1\rangle \;=\; \langle Y(P_{j_3}\xi_3;z)\,P_{j_1}\xi_1,\,P_{j_2}\xi_2\rangle \tag{3.1.6}$$

for $\xi_i\in W_{l_i}\otimes W_{\varepsilon_i}$. When necessary, we write explicitly

$$\begin{aligned} Y(\,.\,;z) &= Y\begin{bmatrix} l_3 & \varepsilon_3 \\ l_2\;l_1 & \varepsilon_2\;\varepsilon_1\end{bmatrix}(\,.\,;z) \\ Y^{j_3}_{j_2\,j_1}(\,.\,;z) &= Y^{j_3}_{j_2\,j_1}\begin{bmatrix} l_3 & \varepsilon_3 \\ l_2\;l_1 & \varepsilon_2\;\varepsilon_1\end{bmatrix}(\,.\,;z). \end{aligned} \tag{3.1.7}$$

For $u\in W_{j_3}(0)$, we define in addition

$$\psi(u;z) \;=\; \psi^{j_3}_{j_2\,j_1}(u;z) \;=\; \psi^{j_3}_{j_2\,j_1}\begin{bmatrix} l_3 & \varepsilon_3 \\ l_2\;l_1 & \varepsilon_2\;\varepsilon_1\end{bmatrix}(u;z) \;=\; Y^{j_3}_{j_2\,j_1}(u\otimes\zeta_{h_3};z), \tag{3.1.8}$$

and

$$\psi^{j_3}(u;z) \;=\; \sum_{j_2,j_1}\psi^{j_3}_{j_2\,j_1}(u;z), \tag{3.1.9}$$

where $u\otimes\zeta_{h_3}\in W_{j_3}(0)\otimes V_{h_3}(0)\subset W_{l_3}\otimes W_{\varepsilon_3}$. Then $\psi(\,.\,;z)$ consists of linear maps from $W_{j_1}\otimes V_{h_1}$ into $W_{j_2}\otimes V_{h_2}$. With the notation of § 2.1,

$$\begin{aligned} &\{\hat{x}_0\psi\}(\,.\,;z)=\psi(x\,.\,;z); && \hat{K}_0\psi=h_{j_3}\psi; && \hat{T}_0\psi=h_3\psi; \\ &\hat{x}_m\psi=0\;(m>0); && \hat{K}_m\psi=0\;(m>0); && \hat{T}_m\psi=0\;(m>0); \end{aligned} \tag{3.1.10}$$

and

$$\{\hat{L}_{-1}\psi\}(\,.\,;z) \;=\; \frac{d}{dz}\psi(\,.\,;z). \tag{3.1.11}$$

As in the proof of Proposition II.3.2, these relations imply that

$$[x_m,\psi(\,.\,;z)] \;=\; z^m\,\psi(x\,.\,;z) \tag{3.1.12a}$$

$$[K_m,\psi(\,.\,;z)] \;=\; z^m\,[K_0,\psi(\,.\,;z)]+h_{j_3}\,m\,z^m\,\psi(\,.\,;z) \tag{3.1.12b}$$

$$[T_m,\psi(\,.\,;z)] \;=\; z^m\,[T_0,\psi(\,.\,;z)]+h_3\,m\,z^m\,\psi(\,.\,;z) \tag{3.1.12c}$$

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and

$$[L_m,\psi(\,.\,;z)] \;=\; z^{m+1}\frac{d}{dz}\psi(\,.\,;z)+\{h_{j_3}+h_3\}\,(m+1)\,z^m\,\psi(\,.\,;z) \tag{3.1.13d}$$

for all $m\in\mathbf{Z}$.

**3.2. Lemma.** *If $\psi$ does not vanish identically, then the primary fields $\phi^{j_3}_{j_2\,j_1}$ and $\phi^{h_3}_{h_2h_1}$ exist and*

$$\psi(\,.\,;z) \;=\; \phi^{j_3}_{j_2\,j_1}(\,.\,;z)\otimes\phi^{h_3}_{h_2h_1}(z) \tag{3.2.1}$$

*up to a non-zero scalar multiple.*

*Proof.* Claim: the commutation relations (3.1.12) determine $\psi$ uniquely up to a scalar multiple. Let $\zeta_i\in V_{h_i}(0)$ and $u_i\in W_{j_i}(0)$. The $L_0$-relation gives

$$\langle\psi(u_3;z)\,u_1\otimes\zeta_1,\,u_2\otimes\zeta_2\rangle \;=\; \langle u_3,u_1;u_2\rangle\,z^{-\{h_{j_3}+h_{j_1}-h_{j_2}\}}\,z^{-\{h_3+h_1-h_2\}}, \tag{3.2.2}$$

where $\langle\,.\,,\,.\,;\,.\,\rangle$ is a $G$-invariant form. Since we can use (3.1.12) to evaluate $\psi(\,.\,;z)$ on arbitrary vectors, this proves the claim. In particular, $\psi$ vanishes identically if and only if it vanishes between the lowest energy subspaces. Suppose $\psi\neq0$. Let

$$\alpha(\,.\,;z) \;=\; z^{h_{j_3}+h_{j_1}-h_{j_2}}\,\psi(\,.\,;z), \tag{3.2.3}$$

and $\xi_i \in V_{h_i}$, $i = 1, 2$, be arbitrary vectors. Since $L_0 = K_0 + T_0$,

$$\langle [T_m, \alpha(u_3; z)]\, u_1 \otimes \xi_1,\ u_2 \otimes \xi_2 \rangle \ = \ \left\{ z^{m+1} \frac{d}{dz} + h_3\,(m+1)\, z^m \right\} \langle \alpha(u_3; z)\, u_1 \otimes \xi_1,\ u_2 \otimes \xi_2 \rangle . \tag{3.2.4}$$

It follows that the primary field $\phi^{h_3}_{h_2 h_1}(z)$ exists. Similarly, let

$$\beta(\,.\,; z) \ = \ z^{h_3 + h_1 - h_2}\, \psi(\,.\,; z), \tag{3.2.5}$$

and $\eta_i \in W_{j_i}$, $i = 1, 2$, be arbitrary vectors. Then

$$\langle [K_m, \beta(u_3; z)]\, \eta_1 \otimes \zeta_1,\ \eta_2 \otimes \zeta_2 \rangle \ = \ \left\{ z^{m+1} \frac{d}{dz} + h_3\,(m+1)\, z^m \right\} \langle \beta(u_3; z)\, \eta_1 \otimes \zeta_1,\ \eta_2 \otimes \zeta_2 \rangle . \tag{3.2.6}$$

Together with (3.1.12a), this shows that the primary field $\phi^{j_3}_{j_2\, j_1}(\,.\,; z)$ exists. Finally, note that the tensor product of primary fields

$$\phi^{j_3}_{j_2\, j_1}(\,.\,; z) \otimes \phi^{h_3}_{h_2 h_1}(z) \tag{3.2.7}$$

also satisfies (3.1.12), and therefore coincides with $\psi(\,.\,; z)$ up to a scalar multiple. □

We note that if $\psi(u; z)$ vanishes identically for all $u \in W_{j_3}(0)$, then $Y^{j_3}_{j_2\, j_1}(\xi; z)$ vanishes identically for all $\xi \in W_{j_3} \otimes V_{h_3}$. From the lemma, it is sufficient to consider those $Y^{j_3}_{j_2\, j_1}(\,.\,; z)$ for which $(j_1, j_2, j_3)$ is an allowed vertex. For each pair of allowed vertices $(l_1, l_2, l_3)$ and $(j_1, j_2, j_3)$, it is convenient to define branching coefficients

$$Y^{j_3}_{j_2\, j_1}\left[ {l_3 \atop l_2\ l_1}\ {\epsilon_3 \atop \epsilon_2\ \epsilon_1} \right] \tag{3.2.8}$$

by

$$\psi^{j_3}_{j_2\, j_1}\left[ {l_3 \atop l_2\ l_1}\ {\epsilon_3 \atop \epsilon_2\ \epsilon_1} \right](\,.\,; z) \ = \ Y^{j_3}_{j_2\, j_1}\left[ {l_3 \atop l_2\ l_1}\ {\epsilon_3 \atop \epsilon_2\ \epsilon_1} \right] \phi^{j_3}_{j_2\, j_1}(\,.\,; z) \otimes \phi^{h_3}_{h_2 h_1}(z). \tag{3.2.9}$$

If $\psi = 0$, the coefficient is defined to be zero. We shall prove the following.

**3.3. Theorem.** $\psi^{j_3}_{j_2 j_1}(\,.\,; z)$ *does not vanish identically if*

$$\mathrm{Hom}_G(U_{j_3} \otimes U_{j_1}, U_{j_2}) = \mathbf{C}, \quad j_i \leq \frac{\ell + 1}{2}\ (i = 1, 2, 3), \quad \sum_i j_i \leq \ell + 1. \tag{3.3.1}$$

We note that these are precisely the necessary and sufficient conditions for $(j_1, j_2, j_3)$ to be an allowed vertex, so we can certainly replace "if" by "if and only if". The proof will be given in stages, and incorporates an existence proof for the discrete series primary fields. Convergence of correlation functions and braiding relations will also become clear. In particular, we shall obtain a proof of Theorems 4.7.1 and 5.4.1 of Chapter II.

**3.4. Convergence of n-point functions of discrete series primary fields.** Let $\xi_i \otimes \eta_i \in W_{j_i} \otimes V_{h_i}$, $i = 1, 4$; and let $u \otimes u_3 \in W_j(0) \otimes W_{j_3}(0)$. The 4-point function

$$\begin{aligned} &\langle\, \psi^{j}\left[ {l \atop l_4\ l_2}\ {\epsilon \atop \epsilon_4\ \epsilon_2} \right](u; w)\, \psi^{j_3}\left[ {l_3 \atop l_2\ l_1}\ {\epsilon_3 \atop \epsilon_2\ \epsilon_1} \right](u_3; z)\ \ \xi_1 \otimes \eta_1,\, \xi_4 \otimes \eta_4 \,\rangle \ = \\ &\qquad \sum_{j_2} \langle\, \psi^{j}_{j_4\, j_2}\left[ {l \atop l_4\ l_2}\ {\epsilon \atop \epsilon_4\ \epsilon_2} \right](u; w)\, \psi^{j_3}_{j_2\, j_1}\left[ {l_3 \atop l_2\ l_1}\ {\epsilon_3 \atop \epsilon_2\ \epsilon_1} \right](u_3; z)\, \xi_1 \otimes \eta_1,\, \xi_4 \otimes \eta_4 \,\rangle \end{aligned} \tag{3.4.1}$$

which, by the previous considerations, is equal to

$$\begin{aligned} &\sum_{j_2} Y^{j}_{j_4\, j_2}\left[ {l \atop l_4\ l_2}\ {\epsilon \atop \epsilon_4\ \epsilon_2} \right] Y^{j_3}_{j_2\, j_1}\left[ {l_3 \atop l_2\ l_1}\ {\epsilon_3 \atop \epsilon_2\ \epsilon_1} \right] \\ &\qquad \langle\, \phi^{j}_{j_4\, j_2}(u; w)\phi^{j_3}_{j_2\, j_1}(u_3; z)\, \xi_1,\, \xi_4 \,\rangle\ \langle\, \phi^{h}_{h_4 h_2}(w)\phi^{h_3}_{h_2 h_1}(z)\, \eta_1,\, \eta_4 \,\rangle \end{aligned} \tag{3.4.2}$$

converges on $|w| > |z| > 0$. Moreover, this convergence holds separately for each

$$\langle\, \phi^{j}_{j_4 j_2}(u; w)\phi^{j_3}_{j_2 j_1}(u_3; z)\, \xi_1,\, \xi_4 \,\rangle \tag{3.4.3}$$

in the sum. From the operator product expansion

$$\phi^{j}_{j_4\, j_2}(u; w)\phi^{j_3}_{j_2\, j_1}(u_3; z) \ = \ \sum_{j'} F^{j_4 j\, j_3 j_1}_{j'\ j_2}\ \phi^{j'}_{j_4\, j_1}(\phi^{j}_{j' j_3}(u; w - z)u_3; z), \tag{3.4.4}$$

it follows by projecting onto the irreducible $G$-submodules of $W_j(0) \otimes W_{j_3}(0)$ that

$$\sum_{j_2} Y^{j}_{j_4\, j_2}\left[ {l \atop l_4\ l_2}\ {\epsilon \atop \epsilon_4\ \epsilon_2} \right] Y^{j_3}_{j_2\, j_1}\left[ {l_3 \atop l_2\ l_1}\ {\epsilon_3 \atop \epsilon_2\ \epsilon_1} \right] F^{j_4 j\, j_3 j_1}_{j'\ j_2}\ \langle\, \phi^{h}_{h_4 h_2}(w)\phi^{h_3}_{h_2 h_1}(z)\, \eta_1,\, \eta_4 \,\rangle \tag{3.4.5}$$

converges on $|w| > |z|$, for each $j'$. Since fusion matrices are invertible, this implies that

$$Y^{j}_{j_4\, j_2}\left[ {l \atop l_4\ l_2}\ {\epsilon \atop \epsilon_4\ \epsilon_2} \right] Y^{j_3}_{j_2\, j_1}\left[ {l_3 \atop l_2\ l_1}\ {\epsilon_3 \atop \epsilon_2\ \epsilon_1} \right] \langle\, \phi^{h}_{h_4 h_2}(w)\phi^{h_3}_{h_2 h_1}(z)\eta_1,\, \eta_4 \,\rangle \tag{3.4.6}$$

converges on $|w| > |z|$, for each $j_2$. It may, of course, simply vanish identically. The arguments easily generalise to $n$-point functions to show that, provided that the corresponding branching coefficients are non-zero, an $n$-point function of discrete series primary fields converges on $|z_n| > \cdots > |z_1|$.

**3.5. Braiding relations of discrete series primary fields.** The braiding relations

$$\psi^{j}\left[\begin{smallmatrix} l & \varepsilon \\ l_4\; l_2 & \varepsilon_4\; \varepsilon_2 \end{smallmatrix}\right](u;w)\,\psi^{j_3}\left[\begin{smallmatrix} l_3 & \varepsilon_3 \\ l_2\; l_1 & \varepsilon_2\; \varepsilon_1 \end{smallmatrix}\right](u_3;z) =$$

$$\sum_{l'} \psi^{j_3}\left[\begin{smallmatrix} l_3 & \varepsilon_3 \\ l_4\; l' & \varepsilon_4\; \varepsilon' \end{smallmatrix}\right](u_3;z)\,\psi^{j}\left[\begin{smallmatrix} l & \varepsilon \\ l'\; l_1 & \varepsilon'\; \varepsilon_1 \end{smallmatrix}\right](u;w)\, C_{l'l_2}^{l_4 l l_3 l_1}\, C_{\varepsilon'\varepsilon_2}^{\varepsilon_4\varepsilon\varepsilon_3\varepsilon_1} \tag{3.5.1}$$

can be written on the left-hand-side as

$$\sum_{j_2} Y^{j}_{j_4\,j_2}\left[\begin{smallmatrix} l & \varepsilon \\ l_4\; l_2 & \varepsilon_4\; \varepsilon_2 \end{smallmatrix}\right] Y^{j_3}_{j_2\,j_1}\left[\begin{smallmatrix} l_3 & \varepsilon_3 \\ l_2\; l_1 & \varepsilon_2\; \varepsilon_1 \end{smallmatrix}\right] \phi^{j}_{j_4\,j_2}(u;w)\phi^{j_3}_{j_2\,j_1}(u_3;z) \otimes \phi^{h}_{h_4 h_2}(w)\phi^{h_3}_{h_2 h_1}(z) \tag{3.5.2}$$

and on the right-hand-side as

$$\sum_{l'} C_{l'l_2}^{l_4 l l_3 l_1}\, C_{\varepsilon'\varepsilon_2}^{\varepsilon_4\varepsilon\varepsilon_3\varepsilon_1} \sum_{j'} Y^{j_3}_{j_4\,j'}\left[\begin{smallmatrix} l_3 & \varepsilon_3 \\ l_4\; l' & \varepsilon_4\; \varepsilon' \end{smallmatrix}\right] Y^{j}_{j'\,j_1}\left[\begin{smallmatrix} l & \varepsilon \\ l'\; l_1 & \varepsilon'\; \varepsilon_1 \end{smallmatrix}\right]$$

$$\phi^{j_3}_{j_4\,j'}(u_3;z)\phi^{j}_{j'\,j_1}(u;w) \otimes \phi^{h_3}_{h_4 h'}(z)\phi^{h}_{h' h_1}(w). \tag{3.5.3}$$

Equating the two, we obtain the braiding relations

$$\sum_{j_2} Y^{j}_{j_4\,j_2}\left[\begin{smallmatrix} l & \varepsilon \\ l_4\; l_2 & \varepsilon_4\; \varepsilon_2 \end{smallmatrix}\right] Y^{j_3}_{j_2\,j_1}\left[\begin{smallmatrix} l_3 & \varepsilon_3 \\ l_2\; l_1 & \varepsilon_2\; \varepsilon_1 \end{smallmatrix}\right] C_{j'j_2}^{j_4 j j_3 j_1}\, \phi^{h}_{h_4 h_2}(w)\phi^{h_3}_{h_2 h_1}(z) =$$

$$\sum_{l'} C_{l'l_2}^{l_4 l l_3 l_1}\, C_{\varepsilon'\varepsilon_2}^{\varepsilon_4\varepsilon\varepsilon_3\varepsilon_1}\, Y^{j_3}_{j_4\,j'}\left[\begin{smallmatrix} l_3 & \varepsilon_3 \\ l_4\; l' & \varepsilon_4\; \varepsilon' \end{smallmatrix}\right] Y^{j}_{j'\,j_1}\left[\begin{smallmatrix} l & \varepsilon \\ l'\; l_1 & \varepsilon'\; \varepsilon_1 \end{smallmatrix}\right] \phi^{h_3}_{h_4 h'}(z)\phi^{h}_{h' h_1}(w). \tag{3.5.4}$$

Here again we have projected onto the spin-$j'$ submodules of $W_j(0) \otimes W_{j_3}(0)$ to single out particular terms.

**3.6. Lemma.** *Let $\sigma$ be a permutation of $\{1,2,3\}$; then*

$$Y^{j_3}_{j_2\,j_1}\left[\begin{smallmatrix} l_3 & \varepsilon_3 \\ l_2\; l_1 & \varepsilon_2\; \varepsilon_1 \end{smallmatrix}\right] = 0 \quad \Leftrightarrow \quad Y^{j_{\sigma(3)}}_{j_{\sigma(2)}\,j_{\sigma(1)}}\left[\begin{smallmatrix} l_{\sigma(3)} & \varepsilon_{\sigma(3)} \\ l_{\sigma(2)}\; l_{\sigma(1)} & \varepsilon_{\sigma(2)}\; \varepsilon_{\sigma(1)} \end{smallmatrix}\right] = 0. \tag{3.6.1}$$

*Proof.* It is easy to see that Theorem 3.3 holds when $l_1 = \varepsilon_1 = j_1 = 0$, i.e.

$$Y^{j_3}_{j_3\,0}\left[\begin{smallmatrix} l_3 & \varepsilon_3 \\ l_3\; 0 & \varepsilon_3\; 0 \end{smallmatrix}\right] \neq 0. \tag{3.6.2}$$

This follows from the observation that

$$\phi^{l_3}_{l_3\,0}(\xi;z)\Omega_\ell \otimes \phi^{\varepsilon_3}_{\varepsilon_3\,0}(\eta;z)\Omega_1\,|_{z=0} \;=\; \xi \otimes \eta \tag{3.6.3}$$

for $\xi \otimes \eta \in W_{l_3} \otimes W_{\varepsilon_3}$, and that $\Omega_\ell \otimes \Omega_1 = \Omega_{\ell+1} \otimes \Omega_{c(\ell)}$. In the braiding relation (3.5.4), let $l_1 = j_1 = 0$:

$$Y^{j}_{j_4\,j_3}\left[\begin{smallmatrix} l & \varepsilon \\ l_4\; l_3 & \varepsilon_4\; \varepsilon_3 \end{smallmatrix}\right] Y^{j_3}_{j_3\,0}\left[\begin{smallmatrix} l_3 & \varepsilon_3 \\ l_3\; 0 & \varepsilon_3\; 0 \end{smallmatrix}\right] C_{j\,j_3}^{j_4 j j_3 0}\, \phi^{h}_{h_4 h_3}(w)\phi^{h_3}_{h_3\,0}(z) =$$

$$C_{l\,l_3}^{l_4 l l_3 0} C_{\varepsilon\varepsilon_3}^{\varepsilon_4\varepsilon\varepsilon_3 0}\, Y^{j_3}_{j_4\,j}\left[\begin{smallmatrix} l_3 & \varepsilon_3 \\ l_4\; l & \varepsilon_4\; \varepsilon \end{smallmatrix}\right] Y^{j}_{j\,0}\left[\begin{smallmatrix} l & \varepsilon \\ l\; 0 & \varepsilon\; 0 \end{smallmatrix}\right] \phi^{h_3}_{h_4 h}(z)\phi^{h}_{h\,0}(w). \tag{3.6.4}$$

It follows immediately that

$$Y^{j}_{j_4\,j_3}\left[\begin{smallmatrix} l & \varepsilon \\ l_4\; l_3 & \varepsilon_4\; \varepsilon_3 \end{smallmatrix}\right] = 0 \quad \Leftrightarrow \quad Y^{j_3}_{j_4\,j}\left[\begin{smallmatrix} l_3 & \varepsilon_3 \\ l_4\; l & \varepsilon_4\; \varepsilon \end{smallmatrix}\right] = 0. \tag{3.6.5}$$

Moreover, by taking formal adjoints, we must also have that

$$Y^{j}_{j_4\,j_3}\left[\begin{smallmatrix} l & \varepsilon \\ l_4\; l_3 & \varepsilon_4\; \varepsilon_3 \end{smallmatrix}\right] = 0 \quad \Leftrightarrow \quad Y^{j}_{j_3\,j_4}\left[\begin{smallmatrix} l & \varepsilon \\ l_3\; l_4 & \varepsilon_3\; \varepsilon_4 \end{smallmatrix}\right] = 0. \tag{3.6.6}$$

This proves the lemma. □

**3.7. Special cases.** We prove Theorem 3.3 in the case: $l_3 + j_3 \leq \frac{1}{2}$. When $l_3 = j_3 = 0$, the result is obvious because

$$\phi^{0}_{l_1 l_1}(z) \otimes \phi^{0}_{\varepsilon_1\varepsilon_1}(z) = I_{l_1} \otimes I_{\varepsilon_1} \tag{3.7.1}$$

is just the identity. Consider the tensor product of primary fields

$$\phi^{l_3}_{l_2\,l_1}(\,\cdot\,;z) \otimes \phi^{\varepsilon_3}_{\varepsilon_2\,\varepsilon_1}(\,\cdot\,;z) \;=\; \sum_{j_3=|l_3\pm\varepsilon|}\;\sum_{j_1,j_2} \psi^{j_3}_{j_2\,j_1}(\,\cdot\,;z) \tag{3.7.2}$$

with $l_3, \varepsilon_3 = 0$ or $\frac{1}{2}$. The operator product expansion of the primary field $\phi^{l_3}_{l_2\,l_1}(\,\cdot\,;z)$ with its conjugate is given by

$$\phi^{l_3}_{l_1\,l_2}(v;z_3)\phi^{l_3}_{l_2\,l_1}(u;z_2) \;=\; \sum_{l} F_{l\,l_2}^{l_1 l_3 l_3 l_1}\, \phi^{l}_{l_1\,l_1}(\phi^{l_3}_{l\,l_3}(v;z_3-z_2)u;z_2). \tag{3.7.3}$$

Since, for $l_3 = 0$ or $\frac{1}{2}$,

$$F_{l\,l_2}^{l_1 l_3 l_3 l_1} \;=\; C_{l_1 l}^{l_1 l_1 l 0\;-1} C_{l\,l_2}^{l_1 l_3 l_1 l_3} C_{l_3 l_1}^{l_2 l_3 l_1 0} \;\neq\; 0, \tag{3.7.4}$$

the leading term in (3.7.3) as $z_3 - z_2 \to 0$ is

$$F_{0\,l_2}^{l_1 l_3 l_3 l_1}\,\langle\, \phi^{l_3}_{0\,l_3}(v;0)u,\, \Omega\,\rangle\,(z_3-z_2)^{-2h_l}\, I_{l_1}. \tag{3.7.5}$$

The same considerations apply to the operator product expansion of $\phi^{\varepsilon_3}_{\varepsilon_2\,\varepsilon_1}(\,\cdot\,;z)$ with its conjugate. It follows that (3.7.2) cannot vanish identically on any vector in $W_{l_1} \otimes W_{\varepsilon_1}$, nor can the linear combination

$$\sum_{j_1,j_2} \psi^{j_3}_{j_2\,j_1}(\,\cdot\,;z) \tag{3.7.6}$$

for each $j_3$. Hence for each fixed $j_3$, $j_1$, there exists some $j_2$ such that $\psi^{j_3}_{j_2\,j_1}(\,\cdot\,;z) \neq 0$.

Let $(l_3, j_3) = (\frac{1}{2}, 0)$. By (3.3.1), $j_2 = j_1$ only, so the Theorem necessarily holds in this case. Now let $(l_3, j_3) = (0, \frac{1}{2})$. By (3.3.1), with a fixed $j_1$, $j_2$ can take either a single value or a pair of values in the sum, and it is sufficient to consider the latter case. Then $\frac{1}{2} \leq j_1 \leq \frac{\ell}{2}$ and $j_2 = j_\pm = |j_1 \pm \frac{1}{2}|$. We shall assume that one of the pair $\psi^{\frac{1}{2}}_{j_\pm\, j_1}(\,\cdot\,; z)$ vanishes identically and obtain a contradiction. So let $\psi^{\frac{1}{2}}_{j_-\, j_1} = 0$ in (3.7.2); we obtain

$$I_{l_1} \otimes \phi^{\varepsilon_3}_{\varepsilon_2\, \varepsilon_1}(\,\cdot\,; z) \;=\; \psi^{\frac{1}{2}}_{j_+\, j_1}(\,\cdot\,; z) \;=\; \lambda\, \phi^{\frac{1}{2}}_{j_+\, j_1}(\,\cdot\,; z) \otimes \phi^{h_{1,2}}_{h_+\, h_1}(z) \tag{3.7.7}$$

on $W_{j_1} \otimes V_{h_1}$, for some non-zero scalar $\lambda$, where $h_+ = h_{p_2, q_+}$. Left-multiply this by its conjugate to obtain

$$I_{l_1} \otimes \phi^{\varepsilon_3}_{\varepsilon_1\, \varepsilon_2}(\,\cdot\,; z)\phi^{\varepsilon_3}_{\varepsilon_2\, \varepsilon_1}(\,\cdot\,; w) = |\lambda|^2\, \phi^{\frac{1}{2}}_{j_1\, j_+}(\,\cdot\,; z)\phi^{\frac{1}{2}}_{j_+\, j_1}(\,\cdot\,; w) \otimes \phi^{h_{1,2}}_{h_1\, h_+}(z)\phi^{h_{1,2}}_{h_+\, h_1}(w) \tag{3.7.8}$$

on $W_{j_1} \otimes V_{h_1}$. Now analytically continue to $|w| > |z|$. On the left-hand-side, we obtain

$$I_{l_1} \;\otimes\; \phi^{\varepsilon_3}_{\varepsilon_1\, \varepsilon_2}(\,\cdot\,; w)\phi^{\varepsilon_3}_{\varepsilon_2\, \varepsilon_1}(\,\cdot\,; z)\; C^{\varepsilon_1 \varepsilon_3 \varepsilon_3 \varepsilon_1}_{\varepsilon_2 \varepsilon_2} \tag{3.7.9}$$

which, by (3.7.8), is

$$|\lambda|^2\, \phi^{\frac{1}{2}}_{j_1\, j_+}(\,\cdot\,; w)\phi^{\frac{1}{2}}_{j_+\, j_1}(\,\cdot\,; z) \;\otimes\; \phi^{h_{1,2}}_{h_1\, h_+}(w)\phi^{h_{1,2}}_{h_+\, h_1}(z)\; C^{\varepsilon_1 \varepsilon_3 \varepsilon_3 \varepsilon_1}_{\varepsilon_2 \varepsilon_2}. \tag{3.7.10}$$

On the right-hand-side, we obtain

$$|\lambda|^2 \left\{ \sum_j \phi^{\frac{1}{2}}_{j_1\, j}(\,\cdot\,; w)\phi^{\frac{1}{2}}_{j\, j_1}(\,\cdot\,; z)\; C^{j_1 \frac{1}{2} \frac{1}{2} j_1}_{j\, j_+} \right\} \otimes \{\cdots\}, \tag{3.7.11}$$

where the factor $\{\cdots\}$ is irrelevant in the following. Note that we do not need to know a braiding relation for the discrete series primary fields to obtain this factor; the required analytic continuation exists anyway because it exists for the left-hand-side and for the other factor. The sum is over $j = j_\pm$, and

$$C^{j_1 \frac{1}{2} \frac{1}{2} j_1}_{j\, j_+} \neq 0. \tag{3.7.12}$$

We claim that (3.7.10) and (3.7.11) are not equal. This would establish the required contradiction. Left-multiply each expression by

$$\phi^{\frac{1}{2}}_{j_-\, j_1}(\,\cdot\,; x) \otimes I_{h_1} \tag{3.7.13}$$

and compare their respective behaviours as functions of $x$. Since

$$\phi^{\frac{1}{2}}_{j_-\, j_1}(u; x)\phi^{\frac{1}{2}}_{j_1\, j_\pm}(v; w) \;=\; \sum_j F^{j_- \frac{1}{2} \frac{1}{2} j_\pm}_{j\, j_1}\; \phi^{j}_{j_-\, j_\pm}(\phi^{\frac{1}{2}}_{j\, \frac{1}{2}}(u; x-w)\, v; w), \tag{3.7.14}$$

where the sum is over $j = 0, 1$ for $j_-$; and $j = 1$ for $j_+$, and since

$$F^{j_- \frac{1}{2} \frac{1}{2} j_-}_{0\, j_1} \neq 0, \tag{3.7.15}$$

we obtain different leading terms as $x - w \to 0$. This proves the claim, and establishes the Theorem when $(l_3, j_3) = (0, \frac{1}{2})$.

**3.8. The generating primary fields.** In the following, we assume that the Theorem holds for a fixed $(l_3, j_3)$, i.e.

$$Y^{j_3}_{j_2\, j_1}\begin{bmatrix} l_3 & \varepsilon_3 \\ l_2\;\, l_1 & \varepsilon_2\;\, \varepsilon_1 \end{bmatrix} \neq 0 \tag{3.8.1}$$

for all possible $j_2, j_1, l_2, l_1$. From § 3.7, this holds also for $(l_3, j_3) = (0, \frac{1}{2})$ and $(\frac{1}{2}, 0)$. Moreover, by (3.6.2),

$$Y^{j_1}_{j_1\, 0}\begin{bmatrix} l_1 & \varepsilon_1 \\ l_1\;\, 0 & \varepsilon_1\;\, 0 \end{bmatrix} \neq 0 \tag{3.8.2}$$

for all $l_1$, $j_1$. From § 3.4, if $h = h_{1,2}$ or $h_{2,1}$, the 4-point function

$$(\,\phi^{h}_{h_4 h_2}(w)\phi^{h_3}_{h_2 h_1}(z)\phi^{h_1}_{h_1 0}(x)\Omega,\, \zeta_{h_4}\,) \tag{3.8.3}$$

converges on $|w| > |z| > 0$. From § 3.5,

$$\begin{aligned} &Y^{0}_{j_4\, j_4}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ l_4\;\, l_2 & \varepsilon_4\;\, \varepsilon_2 \end{bmatrix} Y^{j_3}_{j_4\, j_1}\begin{bmatrix} l_3 & \varepsilon_3 \\ l_2\;\, l_1 & \varepsilon_2\;\, \varepsilon_1 \end{bmatrix} \phi^{h_{2,1}}_{h_4 h_2}(w)\phi^{h_3}_{h_2 h_1}(z) \;=\; \\ &\quad \sum_{l'} C^{l_4 \frac{1}{2} l_3 l_1}_{l' l_2}\, C^{\varepsilon_4 \frac{1}{2} \varepsilon_3 \varepsilon_1}_{\varepsilon' \varepsilon_2}\, Y^{j_3}_{j_4\, j_1}\begin{bmatrix} l_3 & \varepsilon_3 \\ l_4\;\, l' & \varepsilon_4\;\, \varepsilon' \end{bmatrix} Y^{0}_{j_1\, j_1}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ l'\;\, l_1 & \varepsilon'\;\, \varepsilon_1 \end{bmatrix} \phi^{h_3}_{h_4 h'}(z)\phi^{h_{2,1}}_{h' h_1}(w). \end{aligned} \tag{3.8.4}$$

It follows that

$$\phi^{h_{2,1}}_{h_4 h_2}(w)\phi^{h_3}_{h_2 h_1}(z) \;=\; \sum_{h'} \phi^{h_3}_{h_4 h'}(z)\phi^{h_{2,1}}_{h' h_1}(w)\, C^{h_4 h_{2,1} h_3 h_1}_{h' h_2} \tag{3.8.5}$$

and, for each value of $h_2$, $C^{h_4 h_{2,1} h_3 h_1}_{h' h_2}$ vanishes if and only if it vanishes for all values of $h'$. It therefore cannot vanish, since the 4-point functions corresponding to different values of $h_2$ or $h'$ are manifestly linearly independent. Similarly, we have

$$\begin{aligned} &\sum_{j_2} Y^{\frac{1}{2}}_{j_4\, j_2}\begin{bmatrix} 0 & \frac{1}{2} \\ l_4\;\, l_4 & \varepsilon_4\;\, \varepsilon_2 \end{bmatrix} Y^{j_3}_{j_2\, j_1}\begin{bmatrix} l_3 & \varepsilon_3 \\ l_4\;\, l_1 & \varepsilon_2\;\, \varepsilon_1 \end{bmatrix} C^{j_4 \frac{1}{2} j_3 j_1}_{j' j_2}\; \phi^{h_{1,2}}_{h_4 h_2}(w)\phi^{h_3}_{h_2 h_1}(z) \;=\; \\ &\qquad C^{\varepsilon_4 \frac{1}{2} \varepsilon_3 \varepsilon_1}_{\varepsilon' \varepsilon_2}\, Y^{j_3}_{j_4\, j'}\begin{bmatrix} l_3 & \varepsilon_3 \\ l_4\;\, l_1 & \varepsilon_4\;\, \varepsilon' \end{bmatrix} Y^{\frac{1}{2}}_{j'\, j_1}\begin{bmatrix} 0 & \frac{1}{2} \\ l_1\;\, l_1 & \varepsilon'\;\, \varepsilon_1 \end{bmatrix} \phi^{h_3}_{h_4 h'}(z)\phi^{h_{1,2}}_{h' h_1}(w). \end{aligned} \tag{3.8.6}$$

It follows that

$$\phi^{h_{1,2}}_{h_4 h_2}(w)\phi^{h_3}_{h_2 h_1}(z) \;=\; \sum_{h'} \phi^{h_3}_{h_4 h'}(z)\phi^{h_{1,2}}_{h' h_1}(w)\, C^{h_4 h_{1,2} h_3 h_1}_{h' h_2}, \tag{3.8.7}$$

and the braiding coefficients likewise cannot vanish. By Lemma 3.6, we also have

$$\phi^{h_3}_{h_2 h_1}(w)\phi^{h_1}_{h_1 0}(z) \;=\; \phi^{h_3}_{h_2 h_3}(z)\phi^{h_3}_{h_3 0}(w)\,C^{h_2 h_3 h_1 0}_{h_3 h_1}. \tag{3.8.8}$$

It is now straightforward to see that we have sufficient data to use the proof of Proposition II.6.2 to obtain the operator product expansion

$$\phi^{h}_{h_4 h_2}(w)\phi^{h_3}_{h_2 h_1}(z) \;=\; \sum_{h'} F^{h_4 h\, h_3 h_1}_{h' h_2}\, \phi^{h'}_{h_4 h_1}(\phi^{h}_{h' h_3}(w-z)\zeta_{h_3};\, z) \tag{3.8.9}$$

for $h = h_{2,1}$ or $h_{1,2}$, with

$$F^{h_4 h\, h_3 h_1}_{h' h_2} \;\neq\; 0. \tag{3.8.10}$$

**3.9. Proof of Theorem 3.3.** The proof of Theorem 3.3 now proceeds by induction on $l_3 + j_3$. In the following, we assume that the Theorem holds for a fixed $(l_3, j_3)$, i.e.

$$Y^{j_3}_{j_2 j_1}\left[\begin{matrix} l_3 & \varepsilon_3 \\ l_2\; l_1 & \varepsilon_2\; \varepsilon_1 \end{matrix}\right] \;\neq\; 0 \tag{3.9.1}$$

for all possible $j_2, j_1, l_2, l_1$. We show that it holds also for $(l_3 \pm \frac{1}{2}, j_3)$ and $(l_3, j_3 \pm \frac{1}{2})$ whenever these cases are defined.

We shall be considering the operator product expansion of

$$\psi^{j}\left[\begin{matrix} l & \varepsilon \\ l_4\; l_2 & \varepsilon_4\; \varepsilon_2 \end{matrix}\right](\,.\,;\, w)\,\psi^{j_3}\left[\begin{matrix} l_3 & \varepsilon_3 \\ l_2\; l_1 & \varepsilon_2\; \varepsilon_1 \end{matrix}\right](\,.\,;\, z) \tag{3.9.2}$$

when $(l, j) = (\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$. Since

$$\begin{aligned} \phi^{l}_{l_4\, l_2}(\xi;\, w)\phi^{l_3}_{l_2\, l_1}(\eta;\, z) \;&=\; \sum_{l'} F^{l_4 l\, l_3 l_1}_{l' l_2}\, \phi^{l'}_{l_4\, l_1}(\phi^{l}_{l'\, l_3}(\xi;\, w-z)\,\eta;\, z) \\ \phi^{\varepsilon}_{\varepsilon_4\, \varepsilon_2}(\nu;\, w)\phi^{\varepsilon_3}_{\varepsilon_2\, \varepsilon_1}(v;\, z) \;&=\; F^{\varepsilon_4 \varepsilon \varepsilon_3 \varepsilon_1}_{\varepsilon' \varepsilon_2}\phi^{\varepsilon'}_{\varepsilon_4 \varepsilon_1}(\phi^{\varepsilon}_{\varepsilon'\, \varepsilon_3}(\nu;\, w-z)\, v;\, z), \end{aligned} \tag{3.9.3}$$

the terms of the operator product expansion of (3.9.2) are indexed by a spin $l'$ and an integer $n \geq 0$, and given by

$$F^{l_4 l\, l_3 l_1}_{l' l_2}\, F^{\varepsilon_4 \varepsilon \varepsilon_3 \varepsilon_1}_{\varepsilon' \varepsilon_2}\,(w-z)^{\{h_{l'} + h_{\varepsilon'} + n\} - \{h_l + h_\varepsilon + h_{j_3} + h_3\}}\; Y\left[\begin{matrix} l' & \varepsilon' \\ l_4\; l_1 & \varepsilon_4\; \varepsilon_1 \end{matrix}\right](\xi_{l',n};\, z), \tag{3.9.4}$$

for some $\xi_{l',n} \in W_{l'} \otimes W_{\varepsilon'}$, an $L_0$-eigenvector with eigenvalue $h_{l'} + h_{\varepsilon'} + n$. Moreover, the vector $\xi_{l',n}$ does not depend on $l_i$, $\varepsilon_i$, for $i = 1, 2, 4$. The dependence on these variables is only through the fusion coefficients. In the cases of interest, i.e. $(l, j) = (\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$, there are at most two values that $l'$ can take; and when there are two, the difference of the corresponding values of $h_{l'}$ is non-integral.

The first case: $(l, j) = (\frac{1}{2}, 0)$. Then (3.9.2) is given by

$$\sum_{j_2, j_1} Y^{0}_{j_2\, j_2}\left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ l_4\; l_2 & \varepsilon_4\; \varepsilon_2 \end{matrix}\right] Y^{j_3}_{j_2\, j_1}\left[\begin{matrix} l_3 & \varepsilon_3 \\ l_2\; l_1 & \varepsilon_2\; \varepsilon_1 \end{matrix}\right] \phi^{j_3}_{j_2\, j_1}(\,.\,;\, z) \otimes \phi^{h_{2,1}}_{h_4 h_2}(w)\phi^{h_3}_{h_2 h_1}(z), \tag{3.9.5}$$

where $h_i = h_{p_i, q_i}$. We now use, from § 3.8, the operator product expansion

$$\phi^{h_{2,1}}_{h_4 h_2}(w)\phi^{h_3}_{h_2 h_1}(z) \;=\; \sum_{h'} F^{h_4 h_{2,1} h_3 h_1}_{h' h_2}\, \phi^{h'}_{h_4 h_1}(\phi^{h_{2,1}}_{h' h_3}(w-z)\zeta_{h_3};\, z), \tag{3.9.6}$$

where the sum is over $h' = h_{p', q_3} \in \langle h_{2,1}, h_3\rangle \cap \langle h_4, h_1\rangle$; and also the fact that the fusion coefficients do not vanish. The coefficients of corresponding powers of $w - z$ in (3.9.6) and (3.9.4) can therefore be equated. In particular, for each $u \in W_{j_3}(0)$,

$$\sum_{j_2, j_1} Y^{0}_{j_2\, j_2}\left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ l_4\; l_2 & \varepsilon_4\; \varepsilon_2 \end{matrix}\right] Y^{j_3}_{j_2\, j_1}\left[\begin{matrix} l_3 & \varepsilon_3 \\ l_2\; l_1 & \varepsilon_2\; \varepsilon_1 \end{matrix}\right] F^{h_4 h_{2,1} h_3 h_1}_{h' h_2}\, \phi^{j_3}_{j_2\, j_1}(u;\, z) \otimes \phi^{h'}_{h_4 h_1}(z) \tag{3.9.7}$$

is equal to

$$F^{l_4 \frac{1}{2} l_3 l_1}_{l' l_2}\, F^{\varepsilon_4 \frac{1}{2} \varepsilon_3 \varepsilon_1}_{\varepsilon' \varepsilon_2}\, Y\left[\begin{matrix} l' & \varepsilon' \\ l_4\; l_1 & \varepsilon_4\; \varepsilon_1 \end{matrix}\right](\xi_u;\, z), \tag{3.9.8}$$

for some $\xi_u \in W_{l'} \otimes W_{\varepsilon'}$. It follows that the $\xi_u$ are the vectors in the lowest energy subspace of the summand $W_{j_3} \otimes V_{h_{p', q_3}}$ in the decomposition of $W_{l'} \otimes W_{\varepsilon'}$. Hence

$$\begin{aligned} &F^{l_4 \frac{1}{2} l_3 l_1}_{l' l_2}\, F^{\varepsilon_4 \frac{1}{2} \varepsilon_3 \varepsilon_1}_{\varepsilon' \varepsilon_2}\, Y^{j_3}_{j_2\, j_1}\left[\begin{matrix} l' & \varepsilon' \\ l_4\; l_1 & \varepsilon_4\; \varepsilon_1 \end{matrix}\right] \;= \\ &\qquad Y^{0}_{j_2\, j_2}\left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ l_4\; l_2 & \varepsilon_4\; \varepsilon_2 \end{matrix}\right] Y^{j_3}_{j_2\, j_1}\left[\begin{matrix} l_3 & \varepsilon_3 \\ l_2\; l_1 & \varepsilon_2\; \varepsilon_1 \end{matrix}\right] F^{h_4 h_{2,1} h_3 h_1}_{h' h_2}, \end{aligned} \tag{3.9.9}$$

up to a non-zero scalar factor that depends at most on $j_3$, $l_3$ and $l'$. This proves the Theorem for $(l', j_3)$, and therefore for $(l_3 \pm \frac{1}{2}, j_3)$ whenever defined.

The second case: $(l, j) = (0, \frac{1}{2})$. Then (3.9.2) is given by

$$\sum_{j_4, j_2, j_1} Y^{\frac{1}{2}}_{j_4\, j_2}\left[\begin{matrix} 0 & \frac{1}{2} \\ l_2\; l_2 & \varepsilon_4\; \varepsilon_2 \end{matrix}\right] Y^{j_3}_{j_2\, j_1}\left[\begin{matrix} l_3 & \varepsilon_3 \\ l_2\; l_1 & \varepsilon_2\; \varepsilon_1 \end{matrix}\right] \phi^{\frac{1}{2}}_{j_4\, j_2}(\,.\,;\, w)\phi^{j_3}_{j_2\, j_1}(\,.\,;\, z) \otimes \phi^{h_{1,2}}_{h_4 h_2}(w)\phi^{h_3}_{h_2 h_1}(z). \tag{3.9.10}$$

We now use the operator product expansion

$$\phi^{\frac{1}{2}}_{j_4\, j_2}(u;\, w)\phi^{j_3}_{j_2\, j_1}(v;\, z) \;=\; \sum_{j'} F^{j_4 \frac{1}{2} j_3 j_1}_{j' j_2}\, \phi^{j'}_{j_4\, j_1}(\phi^{\frac{1}{2}}_{j' j_3}(u;\, w-z)v;\, z); \tag{3.9.11}$$

and also, from § 3.8,

$$\phi^{h_{1,2}}_{h_4 h_2}(w)\phi^{h_3}_{h_2 h_1}(z) \;=\; \sum_{h''} F^{h_4 h_{1,2} h_3 h_1}_{h'' h_2}\, \phi^{h''}_{h_4 h_1}(\phi^{h_{1,2}}_{h'' h_3}(w-z)\zeta_{h_3};\, z) \tag{3.9.12}$$

where $h'' = h_{p_3,q''} \in (h_{1,2}, h_3) \cap (h_1, h_4)$; and the fact that the fusion coefficients again do not vanish. We play the same game of matching up coefficients of corresponding powers of $w - z$ in (3.9.10) and (3.9.4). The situation is marginally more complicated than in the previous case, because there are now two independent sums, over $j'$ and over $j''$ in $h_{p_3,2j''+1}$. They range over the same values. Claim: the terms with $j' \neq j''$ vanish. This follows because

$$\begin{aligned} n &= \{h_l + h_\epsilon + h_{j_3} + h_3 - h_{l'} - h_{\epsilon'}\} - \{h_{1,2} + h_3 - h'' + h_{j=\frac{1}{2}} + h_{j_3} - h_{j'}\} \\ &= (j'' - l_3)^2 - (\varepsilon_3 - \frac{1}{2})^2 + \frac{j'(j'+1) - j''(j''+1)}{\ell + 3} \end{aligned} \tag{3.9.13}$$

is an integer if and only if $j'' = j'$. Moreover, they must vanish separately, since they correspond to different values of $h'' + h_{j'}$. Hence

$$\sum_{j_2} Y^{\frac{1}{2}}_{j_4\, j_2}\begin{bmatrix} 0 & \frac{1}{2} \\ l_2\ l_2 & \epsilon_4\ \epsilon_2 \end{bmatrix} Y^{j_3}_{j_2\, j_1}\begin{bmatrix} l_3 & \epsilon_3 \\ l_2\ l_1 & \epsilon_2\ \epsilon_1 \end{bmatrix} F^{j_4 \frac{1}{2} j_3 j_1}_{j' j_2} F^{h_4 h_{1,2} h_3 h_1}_{h'' h_2} \tag{3.9.14}$$

vanishes when $j'' \neq j'$. Since fusion matrices are invertible, it also follows that this cannot vanish when $j'' = j'$. Hence, for each $u \in W_{j'}$,

$$\sum_{j_1, j_4} \sum_{j_2} Y^{\frac{1}{2}}_{j_4\, j_2}\begin{bmatrix} 0 & \frac{1}{2} \\ l_2\ l_2 & \epsilon_4\ \epsilon_2 \end{bmatrix} Y^{j_3}_{j_2\, j_1}\begin{bmatrix} l_3 & \epsilon_3 \\ l_2\ l_1 & \epsilon_2\ \epsilon_1 \end{bmatrix} F^{j_4 \frac{1}{2} j_3 j_1}_{j' j_2} F^{h_4 h_{1,2} h_3 h_1}_{h' h_2} \phi^{j'}_{j_4\, j_1}(u; z) \otimes \phi^{h'}_{h_4 h_1}(z), \tag{3.9.15}$$

where $h' = h_{p_3,q'}$, is equal to

$$F^{l_2 \frac{1}{2} l_3 l_1}_{l_3 l_2} F^{\epsilon_4 \frac{1}{2} \epsilon_3 \epsilon_1}_{\epsilon' \epsilon_2} Y^{j'}_{j_4\, j_1}\begin{bmatrix} l_3 & \epsilon' \\ l_2\ l_1 & \epsilon_4\ \epsilon_1 \end{bmatrix} (\xi_u; z) \tag{3.9.16}$$

for some $\xi_u \in W_{l_3} \otimes W_{\epsilon'}$. It follows that the $\xi_u$ are the vectors in the lowest energy subspace of the summand $W_{j'} \otimes V_{h_{p_3,q'}}$ in the decomposition of $W_{l_3} \otimes W_{\epsilon'}$. Hence

$$\begin{aligned} &F^{l_2 \frac{1}{2} l_3 l_1}_{l_3 l_2} F^{\epsilon_4 \frac{1}{2} \epsilon_3 \epsilon_1}_{\epsilon' \epsilon_2} Y^{j'}_{j_4\, j_1}\begin{bmatrix} l_3 & \epsilon' \\ l_2\ l_1 & \epsilon_4\ \epsilon_1 \end{bmatrix} = \\ &\qquad \sum_{j_2} Y^{\frac{1}{2}}_{j_4\, j_2}\begin{bmatrix} 0 & \frac{1}{2} \\ l_2\ l_2 & \epsilon_4\ \epsilon_2 \end{bmatrix} Y^{j_3}_{j_2\, j_1}\begin{bmatrix} l_3 & \epsilon_3 \\ l_2\ l_1 & \epsilon_2\ \epsilon_1 \end{bmatrix} F^{j_4 \frac{1}{2} j_3 j_1}_{j' j_2} F^{h_4 h_{1,2} h_3 h_1}_{h' h_2} \end{aligned} \tag{3.9.17}$$

up to a non-zero scalar factor that depends at most on $j_3, l_3$ and $j'$. This proves the Theorem for $(l_3, j')$, and therefore for $(l_3, j \pm \frac{1}{2})$ when this is defined.

**3.10. Remarks.** We have proved the existence of the discrete series primary fields by an explicit construction analogous to the Goddard-Kent-Olive construction for discrete series representations. We have also seen in § 3.4 and § 3.5 that the convergence of $n$-point functions and braiding relations follow directly from the corresponding properties in the loop group theory. In particular, we have proved Theorems 4.7.1 and 5.4.1 of Chapter II.

IV+V

# Chapter IV

# Localised fields and braiding relations

We apply the construction of discrete series primary fields in Chapter III to establish Sobolev inequalities for these operators. They extend a primary field $\phi : H_1^{fin} \otimes V_{\lambda,\mu}^{fin} \to H_2^{fin}$ to a jointly continuous linear map $H_1^\infty \otimes V_{\lambda,\mu} \to H_2^\infty$. The smeared primary field $\phi(f)$ is a densely-defined, closeable operator. At least when $\phi$ has conformal dimension $h_{1,2}$ or $h_{2,2}$, it has bounded closure and satisfies a stronger $L^2$-inequality. We describe the construction of localised fields by smearing with bump functions, and obtain the braiding relations they satisfy when they have disjoint support.

## 1. Sobolev and $L^2$ inequalities for discrete series primary fields

**1.1. Inequalities for loop group primary fields.** The discrete series theory at central charge $c(\ell)$ and the $LSU(2)$ theory at level $\ell+1$ occur as subtheories of the $(\ell+1)$-fold tensor product of the $LSU(2)$ theory at level 1. This in turn occurs as a subtheory of the $LU(2)$ theory at level 1, which is realised as a free fermion theory. In this latter theory, Cliff$(H)$, the CAR algebra of $H = L^2(S^1, \mathbb{C}^2)$ acts on fermionic Fock space $\mathcal{F}_P = \Lambda PH \hat{\otimes} \Lambda(P^\perp H)^*$. This is the GNS representation corresponding to the pure state $\varphi_P$, where $P$ is the Hardy space projection onto the non-positive modes of $H$; and $LU(2)$ acts in a continuous projective unitary way on $\mathcal{F}_P$ by Bogoliubov automorphisms. In the $LU(2)$ theory at level 1, the smeared primary fields are the fermion fields, which are therefore bounded and satisfy an $L^2$-inequality. Because all the primary fields of the $LSU(2)$ and discrete series theories can be "constructed" from the fermion fields by taking tensor products and compressing, we can deduce the required Sobolev and $L^2$ inequalities. We refer to [JW] [Wa1]-[Wa5] for the free fermion theories built on $H = L^2(S^1, \mathbb{C}^N)$, and for the $LSU(N)$ theories. It is sufficient for our purposes to use the following result from the $LSU(2)$ theory: if $\phi$ is a primary field of spin-$\frac{1}{2}$ at level 1, then the smeared field satisfies the $L^2$-inequality $\|\phi(f)\xi\| \leq K\,\|f\|\,\|\xi\|$.

**1.2. The GKO construction revisited.** The GKO construction (Theorem III.2.2) easily extends to a construction of the discrete series as positive energy representations of Diff$^+S^1$. Let $G = SU(2)$, and $H = H_{l,\ell} \otimes H_{\varepsilon,1}$. As a tensor product of positive energy representations of $LG$ at level $\ell$ and level 1. $H$ is a positive energy representation at level $\ell+1$. By complete reducibility,

$$H = \bigoplus_j H_j \otimes K_j, \tag{1.2.1}$$

where the $H_j$ are distinct irreducible positive energy representations of $LG$ at level $\ell+1$, and the $K_j$ are multiplicity spaces. Consideration of the Lie algebra action on the finite energy vectors $H^{fin}$ and comparison with (III.2.2.1) gives

$$H = \bigoplus_j H_{j,\ell+1} \otimes H_{h_{p,q},c(\ell)}, \tag{1.2.2}$$

at least as $LG$-modules. However, an irreducible positive energy representation of $LG$ at level $\ell$ supports a positive energy representation of $\mathrm{Diff}^+S^1$ with central charge $3\ell/(\ell+2)$ that occurs by exponentiating the Segal-Sugawara operators (we can prove this by induction on the level $\ell$ using the arguments to follow). Then $H$ is a positive energy representation of $\mathrm{Diff}^+S^1$ at central charge $3\ell/(\ell+2)+1$. Consideration of the corresponding Lie algebra action shows that this action preserves each summand $H_{j,\ell+1}\otimes H_{h_{p,q},c(\ell)}$, and factors as a tensor product of positive energy representations on $H_{j,\ell+1}$ and $H_{h_{p,q},c(\ell)}$. These follow from the relation $L_m = K_m + T_m$ from § III.2.1, and the fact that the image of the exponential map on $\mathrm{Diff}^+S^1$ generates the group. So we can write $\pi = \oplus_j \pi_j \otimes \sigma_j$ for the positive energy representation $\pi$ of $\mathrm{Diff}^+S^1$ on $H$. Continuity of $\pi_j$, $\sigma_j$ follows since $\pi$, and hence $\pi_j \otimes \sigma_j$, is continuous.

**1.3. Proposition.** *Let $\phi$ be a discrete series primary field. There are $s, t \geq 0$ such that*

$$\|\phi(f)\xi\| \leq K\,\|f\|_s\|\xi\|_t. \tag{1.3.1}$$

*If $\phi$ has conformal dimension $h_{1,2}$ or $h_{2,2}$, then we have the stronger $L^2$-inequality*

$$\|\phi(f)\xi\| \leq K\,\|f\|\,\|\xi\|, \tag{1.3.2}$$

*where $\|f\|^2 = \sum_n |f_n|^2$.*

*Proof.* The proof is as for loop groups [Wa5]. An $LG$ primary field $\phi$ at level 1 is either spin-0 and just the identity, or spin-$\frac{1}{2}$ and satisfies the $L^2$-inequality

$$\|\phi(f)\xi\| \leq K\,\|f\|\,\|\xi\|. \tag{1.3.3}$$

From § III.3.1, the $\ell+1$-fold tensor product of $LG$ primary fields at level 1,

$$\psi(\,.\,\otimes\cdots\otimes\,.\,;z) = \psi_1(\,.\,;z)\otimes\cdots\otimes\psi_\ell(\,.\,;z) \tag{1.3.4}$$

is a sum of terms of the form

$$\phi_0(\,.\,;z)\otimes\phi_1(z)\otimes\cdots\otimes\phi_\ell(z), \tag{1.3.5}$$

where $\phi_0(\,.\,;z)$ is an $LG$ primary field at level $\ell+1$, and $\phi_i(z)$ $(i=1,\ldots,\ell)$ a discrete series primary field at central charge $c(i)$. It is easy to check using Theorem 3.3 that every $LG$ primary field at level $\ell+1$ occurs as some $\phi_0$ by choosing the $\psi_i$'s appropriately; and the discrete series primary fields at central charge $c(i)$ that can occur as some $\phi_i$ are precisely those with conformal dimension $h_{p,q}$, $q=p$, $p\pm1$. Fix an arbitrary integer $M$ and let

$$\psi(m) = \sum_{(m_i)} \psi_1(m_1)\otimes\cdots\otimes\psi_\ell(m_\ell), \tag{1.3.6}$$

where the sum is such that $m_1+\cdots+m_\ell = m+M$.

When only one of the $\psi_i$'s is spin-$\frac{1}{2}$, we clearly have

$$\|\psi(f)\xi\| \leq K\,\|f\|\,\|\xi\|. \tag{1.3.7}$$

In this case, the $\phi_0$'s that occur are precisely those with spin-$\frac{1}{2}$; and the discrete series primary fields $\phi_i$ that occur are precisely those with conformal dimension $h_{1,2}$ and $h_{2,2}$. We claim that, for each $i$,

$$\|\phi_i(f)\xi\| \leq K\,\|f\|\,\|\xi\|. \tag{1.3.8}$$

More generally, suppose that $k$ of the $\psi_i$'s are spin-$\frac{1}{2}$. If $\eta$ is an $L_0$-eigenvector in the $\ell+1$-fold tensor product space, $L_0\eta = h\eta$, then

$$\psi_1(m_1)\otimes\cdots\otimes\psi_\ell(m_\ell)\,\eta = 0 \tag{1.3.9}$$

if some $m_i > h$. Since the operator norms $\|\psi_i(m_i)\|$ are uniformly bounded,

$$\begin{aligned}\|\psi(m)\eta\| &\leq K(1+h+|m+M-(k-1)h|)^k\,\|\eta\| \\ &\leq K(1+|M|)^k(1+k)^k(1+|m|)^k\,\|\eta\|_k\end{aligned} \tag{1.3.10}$$

and therefore

$$\|\psi(f)\xi\| \leq K(1+|M|)^k(1+k)^k\,\|f\|_k\,\|\xi\|_k \tag{1.3.11}$$

for all $\xi$. We claim in this case that

$$\|\phi_i(f)\xi\| \leq K\,\|f\|_k\,\|\xi\|_k. \tag{1.3.12}$$

We shall be content to prove (1.3.8) and (1.3.12) when $i = \ell$; the proofs are identical for each $i$. Let $\xi = \xi_0 \otimes \cdots \otimes \xi_\ell$ in the relevant submodule, where the $\xi_i$ are lowest energy vectors if $i \neq \ell$. Then, for an appropriate choice of the shift $M$,

$$\begin{aligned} ||v(f)\xi||^2 &\geq ||\sum_{(m_i)} f_{m_0+\cdots+m_\ell}\, \phi_0(m_0)\xi_0 \otimes \cdots \otimes \phi_\ell(m_\ell)\xi_\ell||^2 \\ &\geq ||\phi_0(0)\xi_0||^2 \cdots ||\phi_{\ell-1}(0)\xi_{\ell-1}||^2\, ||\phi_\ell(f)\xi_\ell||^2 \end{aligned} \tag{1.3.13}$$

since the $\phi_i(m_i)\xi_i$, $i \neq \ell$, with different $m_i$ are mutually orthogonal. Each of the factors $||\phi_i(0)\xi_i||$, $i \neq \ell$, is certainly non-zero, so the claims follow from (1.3.7) and (1.3.11) respectively. This proves the second assertion of the Proposition; and also the first assertion when $\phi$ has conformal dimension $h_{p,q}$, $q = p, p \pm 1$. It also proves the corresponding results for the $LG$ primary fields at level $\ell + 1$.

If $\phi$ is an $LG$ primary field at level $\ell$, then we certainly have $||\phi(f)\xi|| \leq K\, ||f||_\ell\, ||\xi||_\ell$. Let $\chi(z) = \phi(\eta; z) = \sum_{n\in\mathbb{Z}} \chi(n)\, z^{-n-\Delta}$, where $\eta$ is an $L_0$-eigenvector. We claim that

$$||\chi(n)\xi|| \leq K\,(1 + |n|)^s\, ||\xi||_t \tag{1.3.14}$$

for some $s, t \geq 0$. The proof is by induction. Let (1.3.14) hold for $\chi(z)$ and consider

$$\{\hat{x}_m \lambda\}(z) = \sum_n \{\hat{x}_m \lambda\}(n)\, z^{-n+m-\Delta}. \tag{1.3.15}$$

Let $\xi$ be an $L_0$-eigenvector, $L_0\xi = h\,\xi$. Then

$$\begin{aligned} ||\{\hat{x}_m \lambda\}(n)\,\xi|| &= ||\sum_{r=0}^{\infty} \binom{m}{r} \{(-1)^r x_{m-r}\chi(n+r-m)\xi + (-1)^{1+m-r}\chi(n-r)x_r\xi\}|| \\ &\leq \sum_{r=0}^{[h]+m-n} \left|\binom{m}{r}\right|\, ||x_{m-r}\chi(n+r-m)\xi|| + \sum_{r=0}^{[h]} \left|\binom{m}{r}\right|\, ||\chi(n-r)x_r\xi|| \\ &\leq K\,|x|\,||\xi||_{t+\frac{1}{2}} \left\{ \sum_{r=0}^{[h]+m-n} \left|\binom{m}{r}\right| (1+|m-r|)^{\frac{1}{2}}.(1+|n+r-m|)^{s+\frac{1}{2}} \right. \\ &\qquad \left. + \sum_{r=0}^{[h]} \left|\binom{m}{r}\right| (1+|n-r|)^s (1+r)^{t+\frac{1}{2}} \right\}, \end{aligned} \tag{1.3.16}$$

where we have used

$$||x_m\xi||_s \leq K\,|x|\,(1+|m|)^{|s|+\frac{1}{2}}\, ||\xi||_{s+\frac{1}{2}}. \tag{1.3.17}$$

It is sufficent to consider $m \leq 0$. The terms in braces have only polynomial growth in $|n|$ and $h$. The first term is

$$\leq C_m\,(1+h)^{s+\frac{1}{2}}\,(1+h+|n|)^{\frac{1}{2}+|m|}, \tag{1.3.18}$$

and the second term is

$$\leq C_m\,(1+h+|n|)^s\,(1+h)^{t+\frac{1}{2}+|m|}, \tag{1.3.19}$$

for some constant $C_m$ depending on $m$. Hence

$$||\{\hat{x}(m)\chi\}(n)\xi|| \leq K_m\,|x|\,(1+|n|)^{s+|m|+\frac{1}{2}}\, ||\xi||_{t+s+|m|+1}, \tag{1.3.20}$$

which proves the claim.

Now let $\chi_i(z) = \phi(\eta_i; z) = \sum_{n\in\mathbb{Z}} \chi_i(n)\, z^{-n-\Delta_i}$, where the $\eta_i$'s are $L_0$-eigenvectors, and consider their tensor product

$$\chi(z) = \chi_1(z) \otimes \chi_2(z) = \sum_{n\in\mathbb{Z}} z^{-n-\Delta} \sum_{m\in\mathbb{Z}} \chi_1(n-m) \otimes \chi_2(m) \tag{1.3.21}$$

We claim that

$$||\chi(f)\,\xi|| \leq K\,||f||_s\,||\xi||_t \tag{1.3.22}$$

for some $s, t \geq 0$. We have

$$||\chi_i(n)\,\xi_i|| \leq K\,(1+|n|)^{s_i}\, ||\xi_i||_{t_i} \tag{1.3.23}$$

for some $s_i$, $t_i \geq 0$, so that

$$||\chi_i(n)\,\xi_i||_s \leq K\,(1+|n|)^{|s|+s_i}\, ||\xi_i||_{s+t_i}. \tag{1.3.24}$$

Let $\xi$ be an $L_0$-eigenvector in the tensor product space with eigenvalue $h$. Then

$$\begin{aligned} ||\chi(n)\,\eta|| &= ||\sum_{m\in\mathbb{Z}} \chi_1(m) \otimes \chi_2(n-m)\,\eta|| \\ &\leq K \sum_{m=n-[h]}^{[h]} (1+|m|)^{s_1}(1+|n-m|)^{s_2}\, ||\eta||_{t_1+t_2}. \end{aligned} \tag{1.3.25}$$

It follows that

$$||\chi(f)\,\xi|| \leq K\,||f||_{s_1+s_2+1}\, ||\xi||_{s_1+s_2+t_1+t_2+1}, \tag{1.3.26}$$

which proves the claim.

By (1.3.22) and the construction of discrete series primary fields in § III.3, if

$$\psi(z) \;=\; \phi^{j_3}_{j_2\,j_1}(u;\,z) \otimes \phi^{h_3}_{h_2 h_1}(z), \tag{1.3.27}$$

then

$$||\psi(f)\,\xi|| \;\leq\; K\,||f||_s\,||\xi||_t \tag{1.3.28}$$

for some $s, t \geq 0$; and the same must hold for $\phi^{h_3}_{h_2 h_1}(f)$, cf. (1.3.13) above. □

**1.4. Corollary.** *The primary field* $\phi : H_1^{fin} \otimes V^{fin}_{\lambda,\mu} \to H_2^{fin}$ *extends to a continuous linear operator* $H_1^\infty \otimes V_{\lambda,\mu} \to H_2^\infty$. *For* $f \in V_{\lambda,\mu}$, *the smeared primary field* $\phi(f)$ *is a densely-defined, closeable operator. If* $\phi$ *has conformal dimension* $h_{1,2}$ *or* $h_{2,2}$, *then* $\phi(f)$ *is defined for square-integrable* $f$ *and has bounded closure.*

*Proof.* This is immediate. noting that $\phi(f) \subset \phi^*(f^*)^*$. □

**1.5. Intertwining property of primary fields.** Let $\phi : H_1^\infty \otimes V_{\lambda,\mu} \to H_2^\infty$ be a discrete series primary field. and let $\xi \in H_1^\infty$. Recall that $\mathrm{Diff}^+S^1$ and $\mathrm{Vect}S^1$ leave the smooth vectors invariant. Let $\phi_t$ be the one-parameter subgroup of $\overline{\mathrm{Diff}^+S^1}$ generated by $g \in \mathrm{Vect}S^1$. For $f \in V_{\lambda,\mu}$, let

$$f_t \;=\; e^{i\mu(\phi_{-t}(\theta)-\theta)}\,\phi'^{\lambda}_{-t}\,f \circ \phi_{-t}. \tag{1.5.1}$$

Then

$$\frac{d}{dt}\,e^{itL(g)}\,\phi(f_{-t})\,e^{-itL(g)}\,\xi \;=\; e^{itL(g)}\,\left\{[iL(g),\phi(f_{-t})] + \phi(\frac{df_{-t}}{dt})\right\}\,e^{-itL(g)}\,\xi. \tag{1.5.2}$$

Claim: the term in braces vanishes on $H_1^\infty$. By replacing $f$ by $f_{-t}$, it is sufficient to show this at $t = 0$. Since the maps $\phi : H_1^\infty \otimes V_{\lambda,\mu} \to H_2^\infty$ and $H_i^\infty \otimes \mathrm{Vect}S^1 \to H_i^\infty, \xi \otimes g \mapsto L(g)\xi$, are jointly continuous, it is also sufficient to show this on the finite energy vectors $H_1^{fin}$, and for $f, g$ with finite Fourier series. But this is immediate, since

$$\phi(\frac{df_{-t}}{dt}|_{t=0}) \;=\; \phi(gf' + \lambda g' f + i\mu g f). \tag{1.5.3}$$

Hence

$$e^{itL(g)}\,\phi(f)\,\xi = \phi(f_t)\,e^{itL(g)}\,\xi. \tag{1.5.4}$$

We think of this relation in the following way. Each $H_i$ can be thought of (proof?) as a continuous unitary representation of a central extension of $\mathrm{Diff}^+S^1$, determined by the highest weight $(h_i, c)$. Let $\mathbf{Z} \oplus \mathbf{R} \to E \to \mathrm{Diff}^+S^1$ be the universal central extension of $\mathrm{Diff}^+S^1$ [Seg]. Then (1.5.4) is the statement that $\phi : H_1^\infty \otimes V_{\lambda,\mu} \to H_2^\infty$ intertwines $E$, which acts on each $H_i$ by factoring through the central extension, and on $V_{\lambda,\mu}$ by factoring through $\overline{\mathrm{Diff}^+S^1}$.

## 2. Braiding relations of localised fields

**2.1. Localised fields.** We shall always mean by an interval $I \subset S^1$ a connected subset such that both $I$ and $I^c = S^1 \backslash I$ have non-empty interiors. Let $S^1 = \mathbf{R}/2\pi\mathbf{Z}$ and identify $C^\infty(S^1)$ with $C^\infty_{2\pi}(\mathbf{R})$, the $2\pi$-periodic smooth functions.

Let $I \subset S^1$ be an interval and $C^\infty_I(S^1)$ the smooth functions with support in $I$. Let $\tilde{I} \subset \mathbf{R}$ be an interval such that $\exp(i\tilde{I}) = I$; it is defined modulo translation by $2\pi$. Then we can identify $C^\infty_I(S^1)$ with $C^\infty_{\tilde{I}}(\mathbf{R})$, the smooth functions with support in $\tilde{I}$. With this identification, if $\alpha \in \mathbf{R}$ and $f \in C^\infty_I(S^1)$, we also have $g \in C^\infty_I(S^1)$, defined by

$$g(\theta) \;=\; e^{i\alpha\theta}\,f(\theta) \tag{2.1.1}$$

for $\theta \in \tilde{I}$. Replacing $\tilde{I}$ by its $2\pi$-translate multiplies $g$ by a phase $e^{2\pi i\alpha}$.

Let $\phi : H_1^\infty \otimes V_{\lambda,\mu} \to H_2^\infty$ be a primary field and $f \in C^\infty_I(S^1)$. Define

$$\tilde{\phi}(f) \;=\; \phi(e^{-i\mu\theta}f), \tag{2.1.2}$$

a *localised field* with *support* in the interval $I$ (but we really mean $\tilde{I}$). The reason for smearing primary fields in this way will be apparent when we consider the braiding relations satisfied by localised fields with disjoint support. If we regard $g = e^{-i\mu\theta}f$ as an element of $V_{\lambda,\mu}$, then $\psi \in \overline{\mathrm{Diff}^+S^1}$ maps $g$ to

$$h \;=\; e^{-i\mu(\psi^{-1}-\theta)}\,(\psi^{-1})'^{\lambda}\,g \circ \psi^{-1}, \tag{2.1.3}$$

which is given by

$$h(\theta) \;=\; e^{-i\mu\theta}\,\{(\psi^{-1})'^{\lambda}\,f \circ \psi^{-1}\}\,(\theta) \tag{2.1.4}$$

for $\theta \in \psi(\tilde{I})$. Therefore, we can think of $f$ as transforming under the action of $\mathrm{Vect}S^1$ as an element of $V_{\lambda,0}$, i.e. independent of $\mu$. Its transform under $\psi$ is not really defined unless $\psi$ itself has support in $I$, in which case $f$ transforms as an element of $V_{\lambda,0}$.

We should think of the support of a localised field not as an interval $I \subset S^1$ but as the subset $\tilde{I} \subset \mathbf{R}$. Let the standard localised fields $\tilde{\phi}(f)$ be primary fields $\phi$ smeared with the smooth function $e^{-i\mu\theta} f$, with $f \in C^\infty_J(\mathbf{R})$, $J = \tilde{I}$ or $\tilde{I}^c$. Conjugating by an element $\psi \in \overline{\mathrm{Diff}^+S^1}$ maps its support $\tilde{I}$ to $\psi(\tilde{I})$. If $\psi(\tilde{I}) \subset 2\pi n + \tilde{I}$ or $2\pi n + \tilde{I}^c$, for some integer $n$, then it makes sense to make a negative translation of $2\pi n$ to recover a standard localised field; this introduces a phase factor.

In the rest of this chapter. and often in the subsequent ones, we let $I \subset S^1$ be the open upper half-circle; $\tilde{I} = \{\theta :\ 0 < \theta < \pi\}$; and $\tilde{I}^c = \{\theta :\ \pi \le \theta \le 2\pi\}$. It is easy to see that. with this convention.

$$\begin{aligned} e^{\pi i L_0}\, \tilde{\phi}(f)\, e^{-\pi i L_0} &= \tilde{\phi}(f \circ r_\pi); \\ e^{\pi i L_0}\, \tilde{\phi}(g)\, e^{-\pi i L_0} &= e^{-2\pi i \mu}\, \tilde{\phi}(g \circ r_\pi), \end{aligned} \tag{2.1.5}$$

where $r_t(\theta) = \theta + t$. We shall see later that the additional phase factor is quite important.

**2.2. Proposition.** *Let $a_{kj} = \phi^{h_a}_{h_k h_j}$ and $b_{ji} = \phi^{h_b}_{h_j h_i}$ be discrete series primary fields; let $f \in C^\infty_I(S^1)$ and $g \in C^\infty_{I^c}(S^1)$. Then*

$$\tilde{a}_{kj}(f)\, \tilde{b}_{ji}(g) \;=\; \sum_l \tilde{b}_{kl}(g)\, \tilde{a}_{li}(f)\, C^{h_k h_a h_b h_i}_{h_l h_j} \tag{2.2.1}$$

*on $H_i^\infty$, where the coefficients are given by the braiding relation*

$$a_{kj}(z)\, b_{ji}(w) \;=\; \sum_l b_{kl}(w)\, a_{li}(z)\, C^{h_k h_a h_b h_i}_{h_l h_j} \tag{2.2.2}$$

*on $0 < \arg(w/z) < 2\pi$.*

*Proof.* By continuity, it is sufficient to prove the equality (2.2.1) in the sense of taking matrix elements with finite energy vectors. Moreover, since primary fields are intertwiners for $\mathfrak{Vir}$, it is sufficient to consider lowest energy vectors. If $I_n \nearrow I$ is an increasing sequence of open intervals with $\overline{I_n} \subset I \backslash \partial I$ and $\cup_n I_n = I$, then we can find $f_n \in C^\infty_{I_n}(S^1)$ such that $f_n \to f$ in $C^\infty(S^1)$. Hence it also suffices to take $f$ with slightly shrunken support, say in $I_\varepsilon \subset S^1$, corresponding to the interval $(\varepsilon, \pi - \varepsilon) \subset \mathbf{R}$, for some $\varepsilon > 0$. We have

$$\langle a_{kj}(f)\, b_{ji}(g) \zeta_i,\, \zeta_k \rangle \;=\; \sum_{n=0}^{\infty} h_{-n}\, \langle a_{kj}(n)\, b_{ji}(-n) \zeta_i,\, \zeta_k \rangle, \tag{2.2.3}$$

where $h_{-n} = f_n\, g_{-n}$, so that

$$h(\theta) \;=\; \sum_{n \in \mathbf{Z}} h_n\, e^{in\theta} \;=\; \frac{1}{2\pi} \int_0^{2\pi} f(t-\theta)\, g(t)\, dt. \tag{2.2.4}$$

We write $h = f * g$. Then $h \in C^\infty_{\{1\}}(S^1)$, the smooth functions vanishing to all orders at $e^{i0} \in S^1$, since $f$ and $g$ have disjoint support. Moreover, by the choice of $f$, the function $h$ has support in the interval $J_\varepsilon \subset S^1$ corresponding to $(\varepsilon, 2\pi - \varepsilon) \subset \mathbf{R}$. From the previous results on 4-point functions, we know that the power series

$$\sum_{n=0}^{\infty} \langle a_{kj}(n)\, b_{ji}(-n) \zeta_i,\, \zeta_k \rangle\, x^n \;=\; z^{h_a + h_j - h_k}\, w^{h_b + h_i - h_j}\, \langle a_{kj}(z)\, b_{ji}(w) \zeta_i,\, \zeta_k \rangle \tag{2.2.5}$$

converges on $|x| < 1$, where $x = w/z$, and has an analytic continuation to a holomorphic function $F(x)$ on $0 < \arg(x) < 2\pi$, given by the series

$$\begin{aligned} &\sum_l C^{h_k h_a h_b h_i}_{h_l h_j}\, z^{h_a + h_j - h_k}\, w^{h_b + h_i - h_j}\, \langle b_{kl}(w)\, a_{li}(z) \zeta_i,\, \zeta_k \rangle \;= \\ &\qquad\qquad \sum_l C^{h_k h_a h_b h_i}_{h_l h_j} \sum_{n=0}^{\infty} \langle b_{kl}(n)\, a_{li}(-n) \zeta_i,\, \zeta_k \rangle\, x^{-n-\{h_j + h_l - h_i - h_k\}} \end{aligned} \tag{2.2.6}$$

when $|x| > 1$. It is clear that the function $(r, \theta) \mapsto F(re^{i\theta})$ is continuous and bounded on the rectangle $\delta \le \theta \le 2\pi - \delta$; $R_1 \le r \le R_2$, for all $\delta > 0$, $R_2 > R_1 > 0$. It follows that the function $(r, \theta) \mapsto h(\theta)\, F(re^{i\theta})$ is continuous and bounded on $0 \le \theta \le 2\pi$, $R_1 \le r \le R_2$. By Lebesgue dominated convergence,

$$\lim_{r \nearrow 1} \int_0^{2\pi} h(\theta)\, F(re^{i\theta})\, \frac{d\theta}{2\pi} \;=\; \int_0^{2\pi} h(\theta)\, F(e^{i\theta})\, \frac{d\theta}{2\pi} \;=\; \lim_{r \searrow 1} \int_0^{2\pi} h(\theta)\, F(re^{i\theta})\, \frac{d\theta}{2\pi}. \tag{2.2.7}$$

The left-hand-side is

$$\lim_{r \nearrow 1} \left\{ \sum_{n=0}^{\infty} \langle a_{ki}(n)\, b_{il}(-n) \zeta_i,\, \zeta_k \rangle\, r^n \int_0^{2\pi} h(\theta)\, e^{in\theta}\, \frac{d\theta}{2\pi} \right\} \tag{2.2.8}$$

and recovers (2.2.3); the right-hand-side is

$$\sum_l C^{h_k h_a h_b h_i}_{h_l h_j} \lim_{r \searrow 1} \left\{ \sum_{n=0}^{\infty} \langle b_{kl}(n)\, a_{li}(-n) \zeta_i,\, \zeta_k \rangle\, r^{-n-\alpha} \int_0^{2\pi} h(\theta)\, e^{-i(n+\alpha)\theta}\, \frac{d\theta}{2\pi} \right\}, \tag{2.2.9}$$

where $\alpha = h_j + h_l - h_i - h_k$. With $0 < \theta < 2\pi$,

$$e^{-i\alpha\theta}\, h(\theta) \;=\; \int_0^{2\pi} e^{i\alpha(\phi-\theta)}\, f(\phi - \theta)\, e^{-i\alpha\phi}\, g(\phi)\, \frac{d\phi}{2\pi}, \tag{2.2.10}$$

which, with our conventions, is the statement that

$$\{e^{-i\alpha\theta} h\} \;=\; \{e^{i\alpha\theta} f\} * \{e^{-i\alpha\theta} g\}. \tag{2.2.11}$$

Since

$$\sum_{n=0}^{\infty}\{e^{-i\alpha\theta}h\}_n\,\langle b_{kl}(n)\,a_{li}(-n)\,\zeta_i,\zeta_k\rangle = \langle b_{kl}(e^{-i\alpha\theta}g)\,a_{li}(e^{i\alpha\theta}f)\zeta_i,\zeta_k\rangle, \qquad (2.2.12)$$

it follows that

$$\langle a_{kj}(f)\,b_{ji}(g)\,\zeta_i,\zeta_k\rangle = \sum_l C^{h_k h_\bullet h_\bullet h_i}_{h_l h_j}\,\langle b_{kl}(e^{-i\alpha\theta}g)\,a_{li}(e^{i\alpha\theta}f)\,\zeta_i,\zeta_k\rangle. \qquad (2.2.13)$$

This completes the proof. □

The corresponding braiding relations when the upper half-circle $I$ is replaced by an arbitrary interval $J$ is obtained simply by conjugating (2.2.1) with diffeomorphism group elements. Moreover, any two intervals are diffeomorphic by a Möbius transformation. Having chosen an interval $J$. we have then to decide on which interval of the real line to use to define the localised field. just as we had to above. The braiding coefficients change accordingly. In particular, replacing $I$ by $I^c$ in (2.2.1) above multiplies the braiding coefficient $C^{h_k h_\bullet h_\bullet h_i}_{h_l h_j}$ by a factor $e^{2\pi i\{h_k+h_i-h_l-h_j\}}$.

# Chapter V
# Von Neumann algebras of local diffeomorphism groups

We give a brief exposition of some results of Wassermann's [Wa4] on the von Neumann algebras generated by local diffeomorphism groups acting on the discrete series representations. Together with other results, they imply the construction of quantum field theories satisfying the axioms of Doplicher-Haag-Roberts theory. The method is by descent from tensor products of the $LSU(2)$ theories to the discrete series theories, which are realised as sub-theories by the GKO construction. A key tool is the Tomita-Takesaki-Connes theory of modular operators and Takesaki devissage.

## 1. Local diffeomorphism groups

**1.1 Definitions.** If $I \subset S^1$ is an interval, we define $\mathrm{Diff}_I S^1$ to be the subgroup of diffeomorphisms with support in $I$, i.e. the diffeomorphisms that are just the identity on the complement of $I$; we call these *local diffeomorphisms* with support in $I$. We also define $\mathrm{Vect}_I S^1$ to be the subspace of vector fields on the circle that vanish on $I^c$. If $A \subset S^1$ is a finite subset, we define $\mathrm{Diff}^A S^1$ to be the subgroup of diffeomorphisms that fix $A$ to all orders, i.e. the diffeomorphisms $\phi$ satisfying $\phi(\theta) = \theta$; $\phi'(\theta) = 1$; and $\phi^{(n)}(\theta) = 0$, $n \geq 2$, for all $\theta$ such that $e^{i\theta} \in A$. In particular, if $A = \partial\overline{I}$, we can identify $\mathrm{Diff}^A S^1$ with $\mathrm{Diff}_I S^1 \times \mathrm{Diff}_{I^c} S^1$.

**1.2. Proposition.** *Let $I_1, \ldots, I_n$ be a covering of $S^1$ by open intervals. Then the local subgroups $\mathrm{Diff}_{I_1} S^1, \ldots, \mathrm{Diff}_{I_n} S^1$ generate $\mathrm{Diff}^+ S^1$.*

*Proof.* Let $D$ be the subgroup generated by $\mathrm{Diff}_{I_1} S^1, \ldots, \mathrm{Diff}_{I_n} S^1$. Let $J \subset S^1$ be an interval and $\phi \in \mathrm{Diff}_J S^1$. For $\psi \in \mathrm{Diff}^+ S^1$, we have $\psi\phi\psi^{-1} \in \mathrm{Diff}_{\psi(J)} S^1$. By conjugating $\phi$ by elements of $D$, we can shrink the support until it lies in some $I_i$: to see this, observe that if $U \subset V \subset I$ are open intervals such that $\overline{V} \subset I$, then $U$ and $V$ are diffeomorphic by an element of $\mathrm{Diff}_I S^1$. It follows that $\phi \in D$ and $\mathrm{Diff}_J S^1 \subset D$ for every interval $J$. In particular, $D$ is independent of the choice of cover $\{I_i\}$. One way to conclude the proof would be to note that $D$ is a normal subgroup of $\mathrm{Diff}^+ S^1$, which is known to be a simple group (a theorem of Epstein, Herman and Thurston; see [Mi]).

We give an elementary argument that does not invoke this result. For each $\phi \in \mathrm{Diff}^+S^1$, we can always find disjoint closed intervals $I_1$, $I_2$ and a $\psi \in \mathrm{Diff}^+S^1$ such that $\psi = \phi$ on $I_1$, and $\psi = \mathrm{Id}$ on $I_2$. To see this, take $\phi$ to be an element of $C^\infty(\mathbf{R}, \mathbf{R})$ satisfying $\phi' > 0$; $\phi(x + 2\pi) = \phi(x) + 2\pi$; and $0 \le \phi(0) < 2\pi$. Let $I_1 = [0, \varepsilon]$, with $\varepsilon > 0$ sufficiently small that $\phi(\varepsilon) < 2\pi$; and let $I_2 = [a, b]$ with $\phi(\varepsilon) < a < b < 2\pi$. Then there is certainly a $\psi \in C^\infty(\mathbf{R}, \mathbf{R})$ with $\psi = \phi$ on $I_1$; $\psi = \mathrm{Id}$ on $I_2$; and $\psi' > 0$, $\psi(x + 2\pi) = \psi(x) + 2\pi$. Hence $\{I_1^c, I_2^c\}$ is an open cover of $S^1$, with $\psi \in \mathrm{Diff}_{I_2^c}S^1$ and $\psi^{-1}\phi \in \mathrm{Diff}_{I_1^c}S^1$. So $\phi \in D$ and we have $D = \mathrm{Diff}^+S^1$. □

## 2. Technical preliminaries

**2.1. Proposition.** *Let $\pi : \mathrm{Diff}^+S^1 \to PU(H)$ be a positive energy representation and $I \subset S^1$ an open interval. Then*

$$\pi(\mathrm{Diff}_I S^1)'' \subset \pi(\mathrm{Diff}_{I^c} S^1)'. \tag{2.1.1}$$

*Proof.* We show that $\pi(\mathrm{Diff}_I S^1)$ and $\pi(\mathrm{Diff}_{I^c} S^1)$ commute, i.e. their representative elements in $U(H)$ commute (a "locality" property of the cocycle). They obviously commute up to a phase since $\mathrm{Diff}_I S^1$ and $\mathrm{Diff}_{I^c} S^1$ commute. Let $f \in \mathrm{Vect}_I S^1$ and $g \in \mathrm{Vect}_{I^c} S^1$. From § I.1.14, we have

$$e^{it\,L(g)}\, e^{is\,L(f)}\, e^{-it\,L(g)} \;=\; e^{its\,c(g,f)}\, e^{is\,L(Ad(\phi_g(t))\,f)}, \tag{2.1.2}$$

where $\phi_g(t)$ is the one-parameter subgroup generated by $g$, and

$$c(g, f) \;=\; \frac{c}{24\pi}\int_0^{2\pi} \{f''' + f'\}\, g\, d\theta. \tag{2.1.3}$$

Since $f$ and $g$ have disjoint supports, $c(g, f) = 0$ and $Ad(\phi_g(t))f = f$. It follows that $\pi(\phi)\pi(\psi) = \pi(\psi)\pi(\phi)$ when $\phi \in \overline{G_I}$ and $\psi \in \overline{G_{I^c}}$, where $\overline{G_J}$ is the closure of $G_J$, the subgroup generated by $\exp(\mathrm{Vect}_J S^1) \subset \mathrm{Diff}_J S^1$. The Proposition follows if $\overline{G_J} = \mathrm{Diff}_J S^1$; cf. $\exp(\mathrm{Vect}S^1)$ generates $\mathrm{Diff}^+S^1$.

Let $J \subset S^1$ be an open interval and $J_n \nearrow J$ a sequence of open intervals increasing to $J$, with each $\overline{J_n} \subset J$. For each $\phi \in \mathrm{Diff}_J S^1$, we can find $\phi_n \in \mathrm{Diff}_{J_n} S^1$ such that $\phi_n \to \phi$ in $\mathrm{Diff}^+S^1$, since $\phi = \mathrm{Id}$ on $\partial \overline{I}$ to all orders. Hence it suffices to show that $\mathrm{Diff}_K S^1 \subset \overline{G_J}$ for intervals $K \subset \overline{K} \subset J$. Using results of McDuff [Mc], we can show that $\mathrm{Diff}_K S^1 \subset G_J$. Clearly, $\mathrm{Diff}_K S^1 \subset H_J = \mathrm{Diff}_c J$, the diffeomorphisms in $\mathrm{Diff}^+J$ that have compact support. $H_J$ is connected, since an element of $H_J$ is isotopic to the identity by an isotopy with the same or smaller support. Let $N_J$ be the normal subgroup of $\mathrm{Diff}^+J$ generated by $\exp(\mathrm{Vect}_c J)$, where $\mathrm{Vect}_c J$ are the vector fields with compact support. Since $\mathrm{Vect}_c J \subset \mathrm{Vect}_J S^1$, we have $N_J \subset G_J$. By Theorems 1.1 and 1.2 of [Mc],

$$(H_J \supset)\, N_J \supset [H_J, H_J] \;=\; H_J. \tag{2.1.4}$$

Hence $\mathrm{Diff}_K S^1 \subset H_J = N_J \subset G_J$. □

**2.2. Theorem (Reeh-Schlieder).** *Let $(H, \pi) = (H_{h,c}, \pi_{h,c})$ be a discrete series representation, $\Omega = \Omega_c \in H_{0,c}$ the vacuum vector, $\phi$ the primary field $\phi^h_{h0}$, and $I \subset S^1$ an interval. The set of vectors*

$$K \;=\; \{\pi(g)\phi(f)\Omega : g \in \mathrm{Diff}_I S^1,\ f \in C_I^\infty(S^1)\} \tag{2.2.1}$$

*is total in $H^n$, for each integer $n \ge 0$.*

*Proof.* The proof is identical to the one for local loop groups considered by Wassermann. We clearly have $K \subset H^\infty = \cap_{n \ge 0} H^n$. Let $J \subset I$ be a sub-interval and $\varepsilon > 0$ such that $r_\theta(J) \subset I$ for rotations $r_\theta$ with $|\theta| < \varepsilon$. Let $\langle\, .\, ,\, .\, \rangle_n$ be the inner product on the Hilbert space $H^n$, $\langle \xi, \eta\rangle_n = \langle (1 + L_0)^n\xi, (1 + L_0)^n\eta\rangle$. Let $\eta \in H^n$ satisfy $\langle K, \eta\rangle_n = 0$. Then

$$\langle \pi(r_{t_k}\, g_k\, r_{-t_k}) \cdots \pi(r_{t_1}\, g_1\, r_{-t_1})\, \phi(f \circ r_{-t_0})\Omega,\, \eta\rangle_n \;=\; 0 \tag{2.2.2}$$

for $g_i \in \mathrm{Diff}_J S^1$, $f \in C_J^\infty(S^1)$ and $|t_i| < \varepsilon$. Setting $d = L_0 - h$, this can be written as

$$\langle \{e^{it_k d}\pi(g_k)\, e^{-it_k d}\} \cdots \{e^{it_1 d}\pi(g_1)\, e^{-it_1 d}\}\, e^{it_0 d}\phi(f)\,\Omega,\, \eta\rangle_n \;=\; 0. \tag{2.2.3}$$

Hence there exists $\delta_i > 0$ such that

$$F(s_k, \dots, s_0) \;=\; \langle e^{is_k d}\pi(g_k) \cdots e^{is_1 d}\pi(g_1)\, e^{is_0 d}\phi(f)\Omega,\, \eta\rangle_n \tag{2.2.4}$$

is zero when $|s_i| < \delta_i$, $i = 0, \dots, k$. Claim: $F \equiv 0$. Fix a $j$ and freeze the $s_i$, $i \ne j$, at some values with $|s_i| < \delta_i$. Let

$$G(e^{is_j}) \;=\; F(s_k, \dots, s_0) \;=\; \sum_{m=0}^{\infty} e^{ims_j}\,\langle x\,P_m\,\nu,\, \eta\rangle_n, \tag{2.2.5}$$

where

$$\begin{aligned} x \;&=\; e^{is_k d}\,\pi(g_k) \cdots e^{is_{j+1} d}\,\pi(g_{j+1}) \\ \nu \;&=\; \pi(g_j)\, e^{is_{j-1} d}\,\pi(g_{j-1}) \cdots e^{is_1 d}\,\pi(g_1)\, e^{is_0 d}\,\phi(f)\Omega \end{aligned} \tag{2.2.6}$$

and $P_m$ is the projection onto the $L_0$-eigenspace with eigenvalue $h+m$. Recall that $\mathrm{Diff}^+S^1$ acts continuously on $H^n$; let $T_n(x)$ be the adjoint of $x \in B(H^n)$ with respect to the inner product on $H^n$. We have

$$\begin{aligned}\sum_m |\langle x\,P_m\,\nu,\,\eta\rangle_n| &= \sum_m |\langle P_m\,\nu,\,P_m\,T_n(x)\,\eta\rangle_n| \\ &\leq \{\sum_m \|P_m\,\nu\|_n^2\}^{\frac{1}{2}}\{\sum_m \|P_m\,T_n(x)\,\eta\|_n^2\}^{\frac{1}{2}} \qquad (2.2.7)\\ &= \|\nu\|_n\,\|T_n(x)\,\eta\|_n.\end{aligned}$$

It follows that the power series

$$G(z) \;=\; \sum_{m\geq 0} z^m\,\langle x\,P_m\,\nu,\,\eta\rangle_n \qquad (2.2.8)$$

is absolutely convergent when $|z| \leq 1$. But $G(e^{i\theta}) = 0$ when $|\theta| < \delta_j$. Therefore $z \mapsto G(z)$ is a holomorphic function on the open unit disc, continuous on the closed unit disc, and vanishes on an interval of the unit circle. By the Schwarz reflection principle and the identity theorem, $G(z) \equiv 0$. Thus $F(s_k, \ldots, s_0) = 0$ when $|s_i| < \delta_i$, $i = 0, \ldots, k$, with $\delta_j = \infty$. Repeating the arguments with different $j$'s, we find that $F \equiv 0$. Therefore,

$$\langle \pi(r_{t_k}\,g_k\,r_{-t_k})\cdots\pi(r_{t_1}\,g_1\,r_{-t_1})\,\phi(f\circ r_{-t_0})\,\Omega,\,\eta\rangle_n \;=\; 0 \qquad (2.2.9)$$

for all $t_i$, and $g_i \in \mathrm{Diff}_J S^1$, $f \in C_J^\infty(S^1)$. Using Proposition 1.2, we conclude that

$$\langle \pi(g)\,\xi,\,\eta\rangle_n \;=\; 0 \qquad (2.2.10)$$

for all $g \in \mathrm{Diff}^+S^1$ and some $\xi \in H^\infty$.

Now $\pi(\mathrm{Diff}^+S^1)\,\xi$ spans a subspace of $H^n$ invariant under $\mathrm{Diff}^+S^1$ which, by irreducibility of $\pi$, is dense in $H$. Claim: a rotationally-invariant subspace of $H^n$ that is dense in $H$ is also dense in $H^n$. Let $K^n$ be such a subspace; without loss of generality, we can assume that it is closed in $H^n$. Observe that $H^n$ is unitary as a representation of $\mathrm{Rot}S^1$. Then $K^n$ is the direct sum of its $L_0$-eigenspaces and each $P_m\,K^n$ is a subspace of $K^n$. If $\eta \in H^n$ and $\langle K^n, \eta\rangle_n = 0$, then $\langle P_m\,K^n, \eta\rangle_n = 0$, and thus $\langle P_m\,K^n, \eta\rangle = 0$, for each $m$. Hence $\langle K^n, \eta\rangle = 0$. Since $K^n$ is dense in $H$, we have $\eta = 0$. It follows that $K^n = H^n$. □

The proof applies verbatim if the single smeared primary field $\phi(f)$, $f \in C_I^\infty(S^1)$, in the Theorem is replaced by a chain of smeared primary fields

$$\phi_k(f_k)\cdots\phi_1(f_1), \quad f_i \in C_I^\infty(S^1), \qquad (2.2.11)$$

mapping the vacuum sector into $H^\infty$. From the results of Chapter IV, such a chain can always be constructed using only the primary fields with conformal dimension $h_{1,2}$ and $h_{2,2}$; then (2.2.11) is a bounded operator.

## 3. Von Neumann algebras generated by local diffeomorphism groups

We describe properties of the von Neumann algebras $\pi(\mathrm{Diff}_I S^1)''$ generated by local diffeomorphism groups acting on discrete series representations $(H, \pi)$. These have been deduced by Jones and Wassermann [JW] [Wa1] [Wa4], along with the corresponding results for local loop groups acting on positive energy representations. The method is by descent from larger but understood theories, ultimately tensor products of the free fermion theories, to sub-theories. Here we describe the descent from tensor products of the $LSU(2)$ theories to the discrete series theories via the GKO construction.

**3.1. Takesaki devissage.** We briefly recall how the modular theories of the von Neumann algebras $M_i$ are related to that of the tensor product $M_1\overline{\otimes}M_2$; and also Takesaki's theorem relating the modular theory of a von Neumann algebra $M$ to that of a subalgebra $N$ that is invariant under the modular automorphism group.

Let $M_i \subset B(H_i)$ be a von Neumann algebra with a cyclic, separating vector $\Omega_i$; and let $J_i$, $\Delta_i^{it}$ be the corresponding modular operators. Then $(M_1\,\overline{\otimes}\,M_2)' = M_1'\,\overline{\otimes}\,M_2'$; the vector $\Omega_1 \otimes \Omega_2$ is cyclic and separating for $M_1\,\overline{\otimes}\,M_2$; and the corresponding modular operators are given by $J_1 \otimes J_2$ and $\Delta_1^{it} \otimes \Delta_2^{it}$ (see [KaR2]).

We sketch Takesaki's theorem [Ta] (see also [Su]). Let $M \subset B(H)$ be a von Neumann algebra with a cyclic, separating vector $\Omega$. Let $J_M$, $\Delta_M^{it}$ be the corresponding modular operators, and $\sigma_t = Ad(\Delta_M^{it})$ the modular automorphism group. Let $N \subset M$ be a von Neumann subalgebra. The following are equivalent:

(i) There is a conditional expectation $E : M \to N$ that preserves the vector state $x \mapsto \langle x\Omega, \Omega\rangle$, i.e. $\langle E(x)\Omega, \Omega\rangle = \langle x\Omega, \Omega\rangle$ for all $x \in M$;

(ii) $\sigma_t(N) \subset N$ for all $t \in \mathbb{R}$.

Suppose that (i) and (ii) hold. Then $\overline{N\Omega} = \overline{M\Omega}$ if and only if $N = M$. Let $P$ be the projection onto $\overline{N\Omega}$. Then $PMP = NP$; the conditional expectation $E$ is unique and given by $PxP = E(x)P$, $x \in M$. The representation of $N$ on $\overline{N\Omega}$ is the GNS representation corresponding to the vector state $x \mapsto \langle x\Omega, \Omega\rangle$. In particular, the restriction of $N$ to $\overline{N\Omega}$ is a von Neumann algebra isomorphic to $N$. Let $J_N$, $\Delta_N^{it}$ be the corresponding modular operators, and $\alpha_t = Ad(\Delta_N^{it})$ the modular automorphism group. Then $J_M$, $\Delta_M^{it}$ respectively restrict to $J_N$, $\Delta_N^{it}$ on $\overline{N\Omega}$; and $\sigma_t$ restricts to the modular automorphism group $\alpha_t$ on $N$, i.e. $\sigma_t(x)P = \alpha_t(xP)$, $x \in N$.

**3.2. The loop group theory.** We sketch some results from the $LSU(2)$ theory, together with other relevant facts. Let $G = SU(2)$. We fix a level $\ell$ and let $(H_j, \pi_j)$ denote the positive energy representation of $LG$ with spin $j$. Let $I \subset S^1$ be an interval. We are concerned with the von Neumann algebras $\pi_j(L_IG)''$, where $L_IG$ is the local loop group consisting of elements $g \in LG$ with support in $I$, i.e. $g(z) = e$, the unit in $G$, for all $z \in I^c$. For each $j$, we have the inclusion

$$\pi_j(L_IG)'' \subset \pi_j(L_{I^c}G)'. \tag{3.2.1}$$

**3.2.1. Geometric modular operators and Haag duality in the vacuum sector.** The vacuum vector $\Omega \in H_0$ is cyclic and separating for the local loop group $\pi_0(L_IG)''$. The corresponding modular operators $J$ and $\Delta^{it}$ are *geometric* in the following sense. When $I$ is the upper interval, they are given respectively by the flip $c : z \mapsto \bar{z}$ and the one-parameter group of Möbius transformations $\psi_t$ preserving $\partial I = \{-1, 1\}$, i.e.

$$\Delta^{it} = e^{itL(f)}, \tag{3.2.1.1}$$

where $f \in \mathrm{Vect}S^1$ generates $\psi_t$; and

$$\begin{aligned} J\, e^{itX(g)}\, J &= e^{itX(g\circ c)}; \\ J\, e^{itL(h)}\, J &= e^{-itL(h\circ c)}, \end{aligned} \tag{3.2.1.2}$$

for $g \in L\mathfrak{g}$ and $h \in \mathrm{Vect}S^1$. Since $\exp(L\mathfrak{g})$ and $\exp(\mathrm{Vect}S^1)$ respectively generate $LG$ and $\mathrm{Diff}^+S^1$, we have in particular that

$$\pi_0(L_{I^c}G)' = \pi_0(L_IG)'' \quad (= J\,\pi_0(L_{I^c}G)''\,J). \tag{3.2.1.3}$$

This is known as *Haag duality* in the vacuum sector.

**3.2.2. Quasiregularity and ergodicity.** The representation $t \mapsto \Delta^{it}$ of $\mathbf{R}$ on $H_0$ is a direct sum of the trivial representation $\mathbb{C}\Omega$ and a quasiregular representation.

A continuous unitary representation of $\mathbf{R}$ is *quasiregular* if it is a sub-representation of a direct sum of regular representations $L^2(\mathbf{R}) \otimes \ell^2$, where $\mathbf{R}$ acts trivially on $\ell^2$. Since $L^2(\mathbf{R})$ has no vectors that are fixed by $\mathbf{R}$, the same holds for a quasiregular representation. The direct sum and the tensor product of quasiregular representations are also quasiregular. The latter follows because, if $H$ is an arbitrary continuous unitary representation of $\mathbf{R}$, then the tensor product with the regular representation $L^2(\mathbf{R}) \otimes H$ is unitarily equivalent to

the representation $L^2(\mathbf{R}) \otimes H_{triv}$, where $H_{triv}$ is the same space as $H$, but with the trivial action of $\mathbf{R}$.

In particular, the only vector $\xi \in H_0$ satisfying $\Delta^{it}\xi = \xi$ for all $t$ is the vacuum vector $\Omega$. Hence the modular automorphism group $Ad(\Delta^{it})$ acts ergodically on $\pi_0(L_IG)''$. This implies that $\pi_0(L_IG)''$ is a Type III factor.

If $M$ is a von Neumann algebra with a cyclic, separating vector, and corresponding modular automorphism group $\sigma_t$, we say that the action of $\sigma_t$ on $M$ is *ergodic* if

$$M^\sigma \overset{\text{def}}{=} \{x \in M : \sigma_t(x) = x \text{ for all } t\} = \mathbb{C}. \tag{3.2.2.1}$$

If $\sigma_t$ acts ergodically, then $M$ must be a Type III factor. The proof proceeds as follows: since $M^\sigma \supset Z(M) = M \cap M'$, ergodicity of $\sigma_t$ implies $Z(M) = \mathbb{C}$, i.e. $M$ is a factor. If $M$ is a Type I or Type II factor, then the modular automorphism group $\sigma_t$ is always inner, and cannot therefore be ergodic, so $M$ is Type III.

**3.2.3. Local equivalence.** The restrictions $\pi_j|_{L_IG}$ are unitarily equivalent representations of the local loop group $L_IG$. In particular, the von Neumann algebras $\pi_j(L_IG)''$ are unitarily equivalent. Moreover, $\pi_j(\mathrm{Diff}_IS^1) \subset \pi_j(L_IG)''$ and the intertwiners for $L_IG$ also intertwine $\mathrm{Diff}_IS^1$. More precisely, if $U_j : H_0 \to H_j$ is a unitary intertwiner for $L_IG$, we mean that

$$\begin{aligned} U_j\, e^{itX(g)} &= e^{itX(g)}\, U_j; \\ U_j\, e^{itL(h)} &= e^{itL(h)}\, U_j \end{aligned} \tag{3.2.3.1}$$

for $g \in L_I\mathfrak{g}$ and $h \in \mathrm{Vect}_IS^1$. This implies that the von Neumann algebras generated by local loop groups $L_IG$ acting on a positive energy representation are all unitarily equivalent. Using Haag duality, we see also that each is unitarily equivalent to its commutant.

**3.2.4. Local factorisation and hyperfiniteness.** If $J_1$, $J_2$ are intervals with disjoint closures, then $L_{J_1\cup J_2}G = L_{J_1}G \times L_{J_2}G$. *Local factorisation* is the statement that the representations $\pi_j|_{L_{J_1\cup J_2}G}$ and $\pi_j|_{L_{J_1}G} \otimes \pi_j|_{L_{J_2}G}$ are unitarily equivalent. This implies hyperfiniteness of $\pi_0(L_IG)''$ by the following argument. If $I_n \nearrow I$ is a sequence of open intervals increasing to $I$, then

$$\pi_0(L_IG)'' = \bigvee_n \pi_0(L_{I_n}G)''. \tag{3.2.4.1}$$

If $I_n \subsetneq I_{n+1}$, then $J_1 = I_n$ and $J_2 = \overline{I_{n+1}}^{\,c}$ are open intervals with disjoint closures, and $\pi_0|_{L_{J_1\cup J_2}G}$ and $\pi_0|_{L_{J_1}G} \otimes \pi_0|_{L_{J_2}G}$ are unitarily equivalent. Since

$$\pi_0(L_{J_1}G)'' \otimes \mathrm{Id} \subset B(H_0) \otimes \mathrm{Id} \subset \{\mathrm{Id} \otimes \pi_0(L_{J_2}G)\}', \tag{3.2.4.2}$$

there is a Type I hyperfinite factor $M_n$, such that

$$\pi_0(L_{I_n}G)'' \subset M_n \subset \pi_0(L_{I_{n+1}}G)''. \tag{3.2.4.3}$$

It follows that $\pi_0(L_IG)'' = \vee_n M_n$, which implies hyperfiniteness.

**3.3. The discrete series theory.** Consider $\pi = \pi_{l,\ell} \otimes \pi_{\varepsilon,1}$, the tensor product of projective unitary representations of $LG \rtimes \mathrm{Diff}^+S^1$, and consider the GKO decomposition

$$H = H_{l,\ell} \otimes H_{\varepsilon,1} = \bigoplus_j H_{j,\ell+1} \otimes H_{h,c}, \tag{3.3.1}$$

where $c = c(\ell)$ and $h = h_{p,q} = h_{2l+1,2j+1}$. Then

$$\begin{aligned} \pi(g) &= \bigoplus_j \pi_{j,\ell+1}(g) \otimes \mathrm{Id} \\ \pi(\phi) &= \bigoplus_j \pi_{j,\ell+1}(\phi) \otimes \pi_{h,c}(\phi) \end{aligned} \tag{3.3.2}$$

for $g \in LG$ and $\phi \in \mathrm{Diff}^+S^1$, cf. § IV.1.2. Let $I \subset S^1$ be an interval. From § 3.2 above, we know that the irreducible positive energy representations of $LG$ at a fixed level restrict to unitarily equivalent representations of $L_IG \rtimes \mathrm{Diff}_IS^1$. Then we can define

$$\sigma = \bigoplus_j \pi_{j,\ell+1}|_{L_IG \rtimes \mathrm{Diff}_IS^1} \otimes \mathrm{Id}, \tag{3.3.3}$$

a projective unitary representation of $L_IG \rtimes \mathrm{Diff}_IS^1$; and

$$\tau = \bigoplus_j \mathrm{Id} \otimes \pi_{h,c}|_{\mathrm{Diff}_IS^1}, \tag{3.3.4}$$

a projective unitary representation of $\mathrm{Diff}_IS^1$, defined by

$$\tau(\psi) = \sigma(\psi)^*\pi(\psi), \quad \psi \in \mathrm{Diff}_IS^1. \tag{3.3.5}$$

Let

$$M = \pi_{l,\ell}(L_IG)'' \overline{\otimes} \pi_{\varepsilon,1}(L_IG)''. \tag{3.3.6}$$

For all $j$, $\ell$, we have $\pi_{j,\ell}(\mathrm{Diff}_IS^1)'' \subset \pi_{j,\ell}(L_IG)''$, so that

$$\begin{aligned} \pi(\mathrm{Diff}_IS^1)'' &\subset M; \\ \sigma(\mathrm{Diff}_IS^1)'' &\subset \pi(L_IG)'' \subset M; \\ \tau(\mathrm{Diff}_IS^1)'' &\subset M, \end{aligned} \tag{3.3.7}$$

where the latter inclusion follows from the former two and (3.3.5).

**3.3.1. Geometric modular operators, quasiregularity and Haag duality.** Consider the case $l = \varepsilon = 0$. Let $I \subset S^1$ be the upper half of the circle. At each level $\ell$, the vacuum vector $\Omega_\ell \in H_{0,\ell}$ is cyclic and separating for the von Neumann algebra $\pi_{0,\ell}(L_IG)''$; the corresponding modular operators $J_\ell$, $\Delta_\ell^{it}$ are geometric; and the representation $t \mapsto \Delta_\ell^{it}$ of $\mathbb{R}$ on $H_{0,\ell}$ is a direct sum of $\mathbb{C}\Omega_\ell$ and a quasiregular representation. It follows that $\Omega = \Omega_\ell \otimes \Omega_1$ is a cyclic and separating vector for $M = \pi_{0,\ell}(L_IG)'' \overline{\otimes} \pi_{0,1}(L_IG)''$; the corresponding modular operators are $J = J_\ell \otimes J_1$, $\Delta^{it} = \Delta_\ell^{it} \otimes \Delta_1^{it}$, and therefore also geometric; and the representation $t \mapsto \Delta^{it}$ of $\mathbb{R}$ on $H_{0,\ell} \otimes H_{0,1}$ is also a direct sum of $\mathbb{C}\Omega$ and a quasiregular representation.

Let $N_1 = \tau(\mathrm{Diff}_IS^1)''$ and $N_2 = \pi(L_IG)''$. Since $\Delta^{it}$ is a diffeomorphism group element, the modular automorphism group $Ad(\Delta^{it})$ leaves each $N_i$ invariant, and we can apply Takesaki's theorem. Let $P_i$ be the projection onto $\overline{N_i\Omega}$. On $\overline{N_i\Omega}$, $J$ and $\Delta^{it}$ restrict to the modular operators for $N_iP_i$. Since $\Omega = \Omega_{\ell+1} \otimes \Omega_c$, we have $\overline{N_1\Omega} = \Omega_{\ell+1} \otimes H_{0,c}$ and $\overline{N_2\Omega} = H_{0,\ell+1} \otimes \Omega_c$ (using the Reeh-Schlieder theorem).

Clearly, $\Delta^{it}$ restricts to

$$\Delta_{\ell+1}^{it} \otimes \Delta_c^{it} = e^{itL(f)} \otimes e^{itL(f)} \tag{3.3.1.1}$$

on $H_{0,\ell+1} \otimes H_{0,c}$, where $f \in \mathrm{Vect}\,S^1$ generates the one-parameter subgroup of Möbius transformations that preserve $\partial\overline{I} = \{-1, 1\}$. So $\Delta_c^{it}$ is the modular operator for $\pi_{0,c}(\mathrm{Diff}_IS^1)''$. The representation $t \mapsto \Delta_c^{it}$ of $\mathbb{R}$ on $H_{0,c}$ is a direct sum of $\mathbb{C}\Omega_c$ and a quasiregular representation, since it is a sub-representation of the representation $t \mapsto \Delta^{it}$ on $H_{0,\ell} \otimes H_{0,1}$ that contains $\mathbb{C}\Omega$. It follows that the modular automorphism group on $\tau(\mathrm{Diff}_IS^1)''$ corresponding to the vector state $x \mapsto \langle x\Omega, \Omega\rangle$, acts ergodically. Hence $\tau(\mathrm{Diff}_IS^1)''$, or $\pi_{0,c}(\mathrm{Diff}_IS^1)''$, is a Type III factor.

Let $J$ restrict to $J_{\ell+1} \otimes \mathrm{Id}$ and $\mathrm{Id} \otimes J_c$ on $H_{0,\ell+1} \otimes \Omega_c$ and $\Omega_{\ell+1} \otimes H_{0,c}$ respectively. It is easy to see that

$$J = J_{\ell+1} \otimes J_c \tag{3.3.1.2}$$

on the summand $H_{0,\ell+1} \otimes H_{0,c} \subset H$. We simply check this on the dense subspace

$$N_1N_2\Omega = \pi_{0,\ell+1}(L_IG)''\Omega_{\ell+1} \otimes \pi_{0,c}(\mathrm{Diff}_IS^1)''\Omega_c. \tag{3.3.1.3}$$

It follows that, on $H_{0,\ell+1} \otimes H_{0,c}$,

$$\begin{aligned} J_{\ell+1}\,\pi_{0,\ell+1}(\phi)\,J_{\ell+1} \otimes J_c\,\pi_{0,c}(\phi)\,J_c &= J\,\pi(\phi)\,J \\ &= \pi(c\phi c) \\ &= \pi_{0,\ell+1}(c\phi c) \otimes \pi_{0,c}(c\phi c) \end{aligned} \tag{3.3.1.4}$$

for $\phi \in \mathrm{Diff}^+S^1$, and up to phases, whence

$$J_c \pi_{0,c}(\phi) J_c = \pi_{0,c}(c\phi c) \tag{3.3.1.5}$$

up to a phase. Hence

$$\pi_{0,c}(\mathrm{Diff}_I S^1)' = J_c \pi_{0,c}(\mathrm{Diff}_I S^1)'' J_c = \pi_{0,c}(\mathrm{Diff}_{I^c} S^1)'', \tag{3.3.1.6}$$

which is Haag duality in the vacuum sector $H_{0,c}$.

**3.3.2. Local equivalence.** Recall that the positive energy representations $H_{j,\ell}$ restrict to unitarily equivalent representations of $L_I G \rtimes \mathrm{Diff}_I S^1$ for all $(j, \ell)$. It follows that there is a unitary map $U : H_{0,\ell} \otimes H_{0,1} \to H_{l,\ell} \otimes H_{\epsilon,1}$ intertwining $(L_I G \rtimes \mathrm{Diff}_I S^1) \times (L_I G \rtimes \mathrm{Diff}_I S^1)$. Claim: $U$ also intertwines the representations $\tau$ of $\mathrm{Diff}_I S^1$. (We should of course index the representations $\pi$, $\sigma$ and $\tau$, cf. § 3.3 above, by the spins $l$, $\epsilon$, but we can without confusion omit them in the following.)

The map $U$ obviously intertwines the representations $\pi$ of $L_I G \rtimes \mathrm{Diff}_I S^1$. Hence, for $\phi \in \mathrm{Diff}_I S^1$, we have

$$U^* \sigma(\phi) U U^* \tau(\phi) U = \sigma(\phi) \tau(\phi), \tag{3.3.2.1}$$

so that

$$\sigma(\phi)^* \{U^* \sigma(\phi) U\} = \tau(\phi) \{U^* \tau(\phi) U\}^*. \tag{3.3.2.2}$$

The left-hand-side lies in $\sigma(L_I G)''$, and the right-hand-side in the commutant. Since $\sigma(L_I G)''$ is a factor,

$$U^* \tau(\phi) U = \tau(\phi) \tag{3.3.2.3}$$

up to a phase. In fact, since $\sigma(\mathrm{Diff}_I S^1) \subset \sigma(L_I G)''$ and $U$ intertwines $\sigma(L_I G)''$, it must also intertwine $\sigma(\mathrm{Diff}_I S^1)$; so the left-hand-side of (3.3.2.2) can only be the identity. Hence

$$U^* e^{itL(h)} U = e^{itL(h)} \tag{3.3.2.4}$$

for $h \in \mathrm{Vect}_I S^1$. Hence the von Neumann algebras $\tau(\mathrm{Diff}_I S^1)''$ are all unitarily equivalent; in particular, from § 3.3.1, they are Type III factors.

The commutant of a Type III factor is also a Type III factor; for Type III factors, any two non-zero projections are equivalent. It follows that the sub-representations of $\tau$ are unitarily equivalent to $\tau$; and therefore that the von Neumann algebras $\pi_{h,c}(\mathrm{Diff}_I S^1)''$ are unitarily equivalent for all $h$. This holds for an arbitrary interval $I \subset S^1$. By conjugating by suitable diffeomorphism group elements, we see that the von Neumann algebras $\pi_{h,c}(\mathrm{Diff}_I S^1)''$ are unitarily equivalent for all $h$ and intervals $I$. Using Haag duality, we deduce also that they are unitarily equivalent to their commutants $\pi_{h,c}(\mathrm{Diff}_I S^1)'$.

**3.3.3. Local factorisation and hyperfiniteness.** Let $J_1, J_2 \subset S^1$ be intervals with disjoint closures. If $(H, \pi)$ is a positive energy representation of $LG$, recall that we have a unitary equivalence $\pi|_{L_{J_1 \cup J_2} G} \cong \pi|_{L_{J_1} G} \otimes \pi_j|_{L_{J_2} G}$, i.e. a unitary map $T : H \to H \otimes H$ that intertwines $L_{J_1 \cup J_2} G = L_{J_1} G \times L_{J_2} G$. Since $\pi(\mathrm{Diff}_{J_i} S^1) \subset \pi(L_{J_i} G)''$, $T$ is also an intertwiner for $\mathrm{Diff}_{J_1} S^1 \times \mathrm{Diff}_{J_2} S^1$.

Let $H = H_{l,\ell} \otimes H_{\epsilon,1}$ and $\pi$, $\sigma$, $\tau$ be as in § 3.3 above, where we let $I \supset J_1, J_2$. Then there is a unitary map $T : H \to H \otimes H$ intertwining $(L_{J_1} G \times L_{J_2} G) \times (L_{J_1} G \times L_{J_2} G)$, and the corresponding local diffeomorphisms. In particular, $T$ provides a unitary equivalence $\sigma|_{L_{J_1 \cup J_2} G} \cong \sigma|_{L_{J_1} G} \otimes \sigma|_{L_{J_2} G}$, which therefore also intertwines the corresponding local diffeomorphisms. Since $\tau(\phi) = \pi(\phi)\sigma(\phi)^*$ for all $\phi \in \mathrm{Diff}^+S^1$, $T$ is also a unitary equivalence $\tau|_{\mathrm{Diff}_{J_1 \cup J_2} S^1} \cong \tau|_{\mathrm{Diff}_{J_1} S^1} \otimes \tau|_{\mathrm{Diff}_{J_2} S^1}$. This is the required local factorisation property. Clearly, we can replace $\tau$ by the unitarily equivalent subrepresentations $\pi_{h,c}|_{\mathrm{Diff}_I S^1}$ to obtain

$$\pi_{h,c}|_{\mathrm{Diff}_{J_1 \cup J_2} S^1} \cong \pi_{h,c}|_{\mathrm{Diff}_{J_1} S^1} \otimes \pi_{h,c}|_{\mathrm{Diff}_{J_2} S^1}. \tag{3.3.3.1}$$

As for the von Neumann algebras generated by local loop groups, local factorisation implies hyperfiniteness. We have only to note that if $\phi \in \mathrm{Diff}_I S^1$, then we can find $\phi_n \in \mathrm{Diff}_{I_n} S^1$ such that $\phi_n \to \phi$ in $\mathrm{Diff}^+S^1$. The other arguments are verbatim as for loop groups.

**3.3.4. Duality.** Let $(H_i, \pi_i)$ be the discrete series representations of $\mathrm{Diff}^+S^1$ at a fixed central charge $c$. Let $I \subset S^1$ be an interval. The restrictions $\pi_i|_{\mathrm{Diff}_I S^1}$ are unitarily equivalent, as are the restrictions $\pi_i|_{\mathrm{Diff}_{I^c} S^1}$. Let

$$H = \bigoplus_i H_i; \quad \varrho = \bigoplus_i \pi_i|_{\mathrm{Diff}_I S^1}; \quad \rho = \bigoplus_i \pi_i|_{\mathrm{Diff}_{I^c} S^1}, \tag{3.3.4.1}$$

and $P_i : H \to H_i$ the corresponding projections. Then $H$ is a continuous projective unitary representation of $\mathrm{Diff}_I S^1 \times \mathrm{Diff}_{I^c} S^1$. For each $i$, let $\vartheta_i : \pi_0(\mathrm{Diff}_I S^1)'' \to \pi_i(\mathrm{Diff}_I S^1)''$ be the spatial isomorphism obtained from the unitary equivalence $\pi_0|_{\mathrm{Diff}_I S^1} \cong \pi_i|_{\mathrm{Diff}_I S^1}$.

The smeared primary fields $\phi(f)$ that are bounded and satisfy the $L^2$-inequality provide, for $f$ with support in $I^c$, a natural collection of bounded intertwiners for $\mathrm{Diff}_I S^1$. From Proposition IV.1.3, such intertwiners can be obtained using the primary fields with conformal dimension $h_{1,2}$ and $h_{2,2}$, and it is sufficient for our purposes to restrict attention to these. This subset $C$ of primary fields is *generating* in the sense that, for each pair $H_j$, $H_k$, there is a chain $\phi(f) = \phi_1(f_1) \cdots \phi_n(f_n) \neq 0$, with $\phi_i \in C$, mapping $H_j$ into $H_k$.

Let $\mathcal{A}_{I^c} \subset B(H)$ be the $*$-algebra generated by the smeared primary fields $\phi(f)$, with $\phi \in C$ and $f \in L^2_{I^c}(S^1)$. Each $P_i \mathcal{A}_{I^c} P_j$ is a non-zero subspace of $\mathcal{A}_{I^c}$ normalised by

$\mathrm{Diff}_{I^c}S^1$. By the Reeh-Schlieder theorem,

$$\{x\,\rho(g)\,\Omega : x \in P_i\,\mathcal{A}_{I^c}P_0,\ g \in \mathrm{Diff}_{I^c}S^1\} \qquad (3.3.4.2)$$

is total in $H_i$. Let $T \in \mathcal{A}_{I^c}'$. If $j \neq i$, then $P_j\,T\,P_i\,x = 0$ for all $x \in P_i\,\mathcal{A}_{I^c}P_0$, so that $P_j\,T\,P_i = 0$. Hence $\mathcal{A}_{I^c}' \subset \oplus_i B(H_i)$. In fact, we have the *duality* relation

$$\mathcal{A}_{I^c}' \;=\; \varrho(\mathrm{Diff}_I S^1)'' \;=\; \bigoplus_i \pi_i(\mathrm{Diff}_I S^1)''. \qquad (3.3.4.3)$$

The inclusion $\supset$ is obvious. Let $x = \oplus_i\,x_i \in \mathcal{A}_{I^c}'$; we show that $x \in \oplus_i\,\pi_i(\mathrm{Diff}_I S^1)''$. Let $T \in P_k\,\mathcal{A}_{I^c}P_0$. Then $x_k\,T = T\,x_0$. But $\vartheta_k(y)\,T = T\,y$ for all $y \in \pi_0(\mathrm{Diff}_I S^1)''$. By choosing $T$ to be unitary, it is clear that, in order to show $x \in \varrho(\mathrm{Diff}_I S^1)''$, it is sufficient to show that $x_0 \in \pi_0(\mathrm{Diff}_I S^1)''$. That is, we are required to show that

$$\mathcal{A}_{I^c}'P_0 \;\subset\; \pi_0(\mathrm{Diff}_I S^1)''. \qquad (3.3.4.4)$$

Taking commutants in $B(H_0)$ and using Haag duality, this is, equivalently,

$$N \overset{\mathrm{def}}{=} (P_0\,\mathcal{A}_{I^c}P_0)'' \;\supset\; \pi_0(\mathrm{Diff}_{I^c}S^1)'' \overset{\mathrm{def}}{=} M. \qquad (3.3.4.5)$$

We already have $N \subset M$ and we now want to show $N = M$.

By conjugating with diffeomorphism group elements, we see that it is sufficient to consider the case when $I$ is the upper or lower half of the circle. In this case, the modular automorphism group $\sigma_t$ on $M$ is just conjugation by the Möbius transformations $\psi_t$ that fix $\partial\overline{I} = \{-1, 1\}$. If $x = \phi_1(f_1)\cdots\phi_n(f_n)$, $f_i \in L^2_{I^c}(S^1)$, is a chain of smeared primary fields in $P_0\,\mathcal{A}_{I^c}P_0$, then $\sigma_t(x)$ also lies in $P_0\,\mathcal{A}_{I^c}P_0$, since

$$\pi_0(\psi_t)\,\phi_i(f_i)\,\pi_0(\psi_t)^* \;=\; \phi_i(f_i^{\psi_t}) \qquad (3.3.4.6)$$

and $\psi_t$ preserves $I$ and $I^c$. Hence $\sigma_t(N) \subset N$ and we can apply Takesaki's theorem.

To show that $N = M$, it is sufficient to show that $\overline{N\Omega} = \overline{M\Omega}$, i.e. that $N\Omega$ is dense in $H_0$. We let $\xi \in (N\Omega)^\perp$ and show that $\xi = 0$. We have

$$\langle\,\phi_1(f_1)\cdots\phi_n(f_n)\Omega,\,\xi\,\rangle \;=\; 0 \qquad (3.3.4.7)$$

for all chains $\phi_1(f_1)\cdots\phi_n(f_n) \in P_0\,\mathcal{A}_{I^c}P_0$. To begin with, the $f_i$ have support in $I^c$. However, since the $\phi_i(f_i)$ are bounded operators, following the proof of the Reeh-Schlieder theorem, we can arbitrarily rotate the support of each $f_i$ and still preserve (3.3.4.7). By taking a finite partition of unity on $S^1$, we see that (3.3.4.7) holds for all $f_i \in L^2(S^1)$. We now show that the vectors

$$\phi_1(f_1)\cdots\phi_n(f_n)\,\Omega, \quad f_i \in L^2(S^1), \qquad (3.3.4.8)$$

are total in $H_0$.

In fact, their linear span contains the dense subspace of finite energy vectors. Let $\psi = \phi^h_{h\,0}$ with $h = h_{1,2}$ (or $h_{2,2}$), and $K$ the linear span of the vectors

$$\psi(f_1)^*\psi(g_1)\cdots\psi(f_n)^*\psi(g_n)\,\Omega, \qquad (3.3.4.9)$$

where the $f_i$, $g_i$ have finite Fourier series. Then $\Omega \in K \subset H_0^{fin}$. Recall that $H_0^{fin}$ is the linear span of the vectors $L_{-i_1}\cdots L_{-i_n}\,\Omega$. Since

$$\begin{aligned}[L_m, \psi(g)] \;&=\; \psi(h), \quad h \;=\; e^{mi\theta}(mh\,g + i\,g');\\ L_m\Omega \;&=\; 0 \quad (m \geq -1),\end{aligned} \qquad (3.3.4.10)$$

the subspace $K$ is invariant under the $sl(2,\mathbb{C})$ subalgebra spanned by $L_0$, $L_1$ and $L_{-1}$; and is invariant under the full Virasoro algebra if and only if $L_{-m}\,\Omega \in K$ for all $m \geq 2$. Since

$$(m-2)!\,L_{-m}\Omega \;=\; L_{-1}^{m-2}L_{-2}\Omega, \qquad (3.3.4.11)$$

it is sufficent to check that $L_{-2}\,\Omega \in K$. Let $g = 1$ and $f = e^{2i\theta}$; then $\psi(f)\Omega = 0$ and

$$\langle\,\psi(f)^*\psi(g)\Omega,\,L_{-2}\Omega\,\rangle \;=\; 2(1+h)\,\|\psi(g)\Omega\|^2 \;\neq\; 0. \qquad (3.3.4.12)$$

Since the $L_0$-eigenspace of $H_0$ with eigenvalue 2 is one-dimensional, it follows that

$$\psi(f)^*\psi(g)\Omega \;=\; L_{-2}\Omega \qquad (3.3.4.13)$$

up to normalisation. Hence $K = H_0^{fin}$, $\xi = 0$, $(N\Omega)^\perp = 0$ and $N = M$.

strong additivity

**3.3.5. Von Neumann density.** Let $I \subset S^1$ be an interval, and $I_1$, $I_2$ the intervals obtained by omitting an interior point of $I$. Then

$$\mathcal{A}_I'' \;=\; \mathcal{A}_{I_1}'' \vee \mathcal{A}_{I_2}'' \qquad (3.3.5.1)$$

or, equivalently, $\mathcal{A}_I' = \mathcal{A}_{I_1}' \cap \mathcal{A}_{I_2}'$. This follows because $\mathcal{A}_I$ is generated by $\mathcal{A}_{I_1}$ and $\mathcal{A}_{I_2}$. To see this, just write

$$\phi(f) \;=\; \phi(f\,\chi_1) + \phi(f\,\chi_2), \qquad (3.3.5.2)$$

where $\chi_i$ is the characteristic function of $I_i$, when $\phi \in C$ and $f \in L^2_I(S^1)$. Hence

$$\begin{aligned} \rho(\mathrm{Diff}_{I^c} S^1)'' &= \mathcal{A}_I{}' \\ &= \mathcal{A}_{I_1}{}' \cap \mathcal{A}_{I_2}{}' \\ &= \rho(\mathrm{Diff}_{I_1^c} S^1)'' \cap \rho(\mathrm{Diff}_{I_2^c} S^1)''. \end{aligned} \tag{3.3.5.3}$$

In particular,

$$\pi_0(\mathrm{Diff}_{I^c} S^1)'' = \pi_0(\mathrm{Diff}_{I_1^c} S^1)'' \cap \pi_0(\mathrm{Diff}_{I_2^c} S^1)''. \tag{3.3.5.4}$$

Taking commutants and using Haag duality, we have *von Neumann density*:

$$\pi_0(\mathrm{Diff}_I S^1)'' = \pi_0(\mathrm{Diff}_{I_1} S^1)'' \vee \pi_0(\mathrm{Diff}_{I_2} S^1)''. \tag{3.3.5.5}$$

Since the restrictions $\pi_i|_{\mathrm{Diff}_I S^1}$ are unitarily equivalent, we have for all $i$ that

$$\pi_i(\mathrm{Diff}_I S^1)'' = \pi_i(\mathrm{Diff}_{I_1} S^1)'' \vee \pi_i(\mathrm{Diff}_{I_2} S^1)''. \tag{3.3.5.6}$$

Von Neumann density has the following important corollary. Let $A \subset S^1$ be a finite subset. Then, for each $i$, the restrictions $\pi_i|_{\mathrm{Diff}^A S^1}$ are *irreducible* and *distinct* (i.e. not unitarily equivalent) representations of $\mathrm{Diff}^A S^1$. To see this, it is sufficient to observe that

$$\pi_i(\mathrm{Diff}^A S^1)' = \mathbb{C}. \tag{3.3.5.7}$$

Let $J_1, \ldots, J_n \subset S^1$ be consecutive intervals obtained by omitting the points in $A$. Let $I_k = \overline{J_k \cup J_{k+1}}$, $k = 1, \ldots, n$, with $J_{n+1} = J_1$. Then

$$\pi_i(\mathrm{Diff}_{I_k} S^1)'' = \pi_i(\mathrm{Diff}_{J_k} S^1)'' \vee \pi_i(\mathrm{Diff}_{J_{k+1}} S^1)''. \tag{3.3.5.8}$$

Clearly, the open intervals $I_k \backslash \partial I_k$ cover $S^1$. Noting Proposition 1.2, it follows that

$$\begin{aligned} \pi_i(\mathrm{Diff}^A S^1)'' &= \bigvee_k \pi_i(\mathrm{Diff}_{J_k} S^1)'' \\ &= \bigvee_k \pi_i(\mathrm{Diff}_{I_k} S^1)'' \\ &= \pi_i(\mathrm{Diff}^+ S^1)''. \end{aligned} \tag{3.3.5.9}$$

The result follows since the $\pi_i$ are irreducible and distinct. In particular,

$$\begin{aligned} \pi_i(\mathrm{Diff}_I S^1)' \cap \pi_i(\mathrm{Diff}_{I^c} S^1)' &= (\pi_i(\mathrm{Diff}_I S^1)'' \vee \pi_i(\mathrm{Diff}_{I^c} S^1)'')' \\ &= \pi_i(\mathrm{Diff}^+ S^1)' \\ &= \mathbb{C}. \end{aligned} \tag{3.3.5.10}$$

It follows that the inclusion $\pi_i(\mathrm{Diff}_I S^1)'' \subset \pi_i(\mathrm{Diff}_{I^c} S^1)'$ is irreducible.

**3.3.6. Remarks.** From the above considerations,

$$\pi_{h,c}(\mathrm{Diff}_I S^1)'' \subset \pi_{h,c}(\mathrm{Diff}_{I^c} S^1)' \tag{3.3.6.1}$$

is an irreducible inclusion of hyperfinite Type III factors. In fact, the ergodicity of the modular automorphism group implies here that we have Type III$_1$ factors [Co].

VI + VII

## Chapter VI

# Connes fusion of discrete series representations

By restriction, a discrete series representation $H_{h,c}$ can be regarded as a representation of $\mathrm{Diff}_I S^1 \times \mathrm{Diff}_{I^c} S^1$ and therefore, by local equivalence and Haag duality, as an $(M_I, M_I)$-bimodule, where $M_I = \pi_0(\mathrm{Diff}_I S^1)''$. The discrete series representations remain irreducible and distinct when so regarded, so that the basic objects of study remain the same. This artifice, however, provides us with additive (direct sums) and multiplicative (a tensor product operation) structures on the corresponding category of representations. The latter, Connes fusion, is a specialised form of a quite general relative tensor product on bimodules over von Neumann algebras [Sa]. More specifically, the category $\mathbf{Bimod}_M$ of bimodules over a Type III factor $M$ is a C* monoidal category. When $M = M_I$, the discrete series representations at a fixed central charge are the simple objects of a semi-simple subcategory $\mathbf{Pos}_c$, closed under the tensor product operation. The subcategory $\mathbf{Pos}_c$ has considerable more structure; in fact, it is a modular category [Tu]. That is to say, a monoidal category equipped with a braiding, a twist, and a compatible duality; together with some conditions that ensure finite decompositions. A key ingredient is the construction, from localised fields, of bounded intertwiners that satisfy braiding relations, following a general prescription due to Wassermann [Wa2]. We also compute the representation ring associated to Connes fusion of the discrete series representations.

## 1. Direct sums of discrete series representations

**1.1. Absence of naive direct sums.** If $H$ is a positive energy representation of $\mathrm{Diff}^+S^1$, then, by complete reducibility, $H = \oplus_i H_i \otimes V_i$, where the $H_i$ are irreducible and distinct, and the $V_i$ are multiplicity spaces. But since the $L_0$-eigenvalues are integrally-spaced, the corresponding highest weights $h_i$ can only differ by integers.

**1.2. Direct sums of discrete series representations.** We would like to define the direct sum of arbitrary discrete series representations at a fixed central charge $c$. Clearly, this cannot in general be a positive energy representation of $\mathrm{Diff}^+S^1$ in the usual sense. However, since the discrete series representations remain irreducible and distinct when restricted to

$\mathrm{Diff}^A S^1$, for [illegible]nite subset $A \subset S^1$, we can take the direct sum to be that of representations of $\mathrm{Diff}^A S^1$, at least if $A$ has cardinality $|A| \geq 2$. It must be that the obstacle to taking naive direct sums is the fact that the cocycles of the projective representations do not in general coincide; so we should be able to think of a direct sum as a representation of a universal central extension of $\mathrm{Diff}^+S^1$ [Seg]. This is difficult to prove but we can note the following.

Let $(H_i, \pi_i)$ be discrete series representations, not necessarily distinct, at a fixed central charge $c$. Let $A \subset S^1$ be a finite subset with $n \geq 2$ points, and $J_1, \ldots, J_n$ the open intervals obtained by deleting these points. For each $k$, the restrictions $\pi_i|_{\mathrm{Diff}_{J_k} S^1}$ are unitarily equivalent; it follows that there is a continuous projective unitary representation $\varrho$ of $\mathrm{Diff}^A S^1 = \times_{k=1}^{n} \mathrm{Diff}_{J_k} S^1$ on $H = \oplus_i H_i$. If $\upsilon$ is the representation of $\mathrm{Diff}^B S^1$ on $H$ similarly constructed for the finite subset $B \subset S^1$, $|B| \geq 2$, then $\varrho$ and $\upsilon$ coincide when restricted to $\mathrm{Diff}^{A \cup B} S^1$. Moreover, by von Neumann density, $\varrho|_{\mathrm{Diff}^{A\cup B} S^1} = \upsilon|_{\mathrm{Diff}^{A \cup B} S^1}$ can be re-extended to a representation of $\mathrm{Diff}^A S^1$ or $\mathrm{Diff}^B S^1$. In this sense, $\varrho$ does not depend on the choice of the subset $A$.

## 2. Intertwiners for local diffeomorphism groups

**2.1. Principal intertwiners.** Let $(H_i, \pi_i)$ be the discrete series representations at a fixed central charge $c$; and $I \subset S^1$ an interval. Let

$u: e \to e_-$
$e_- \in A(I^c)$

$$M_I = \pi_0(\mathrm{Diff}_I S^1)''; \quad M_{I^c} = \pi_0(\mathrm{Diff}_{I^c} S^1)'' = M_I'. \tag{2.1.1}$$

By local equivalence, the $H_i$ are unitarily equivalent $M_I$-modules, as well as unitarily equivalent $M_{I^c}$-modules; moreover, these actions commute. We can identify $M_{I^c} = M_I'$ with $M_I^{op}$, the opposite algebra of $M_I$, via the map $x \mapsto Jx^*J$, and regard the $H_i$ as irreducible and distinct $(M_I, M_I)$-bimodules. Let

// $\mathcal{F}(I^c) = \{e, u^*a\}$

$$\mathfrak{X}_i = \mathrm{Hom}_{M_I}(H_0, H_i) = \{T \in B(H_0, H_i) : Tx = xT,\ x \in M_I\} \tag{2.1.2}$$

be the space of intertwiners for $M_I$ from the vacuum sector $H_0$ to $H_i$. We call these the principal intertwiners for $M_I$. Similarly, let

$$\mathfrak{Y}_i = \mathrm{Hom}_{M_{I^c}}(H_0, H_i) = \{T \in B(H_0, H_i) : Ty = yT,\ y \in M_{I^c}\}. \tag{2.1.3}$$

// $\mathcal{F}(I) = \{e, u_+^* a_+\}$

Let $U_{i0} \in \mathfrak{X}_i$ be unitary. By Haag duality, we must have

$$\mathfrak{X}_i = U_{i0} M_{I^c}. \tag{2.1.4}$$

$\{e, a\}^{\perp} \{e, a\} = a^* a$

Since the vacuum vector $\Omega$ is cyclic for $M_I$, we can identify the linear space $\mathfrak{X}_i$ with the dense subspace $\mathfrak{X}_i\Omega \subset H_i$. Observe that there is a right, as well as a left, action of $M_{I^c}$ on the subspace $\mathfrak{X}_i$: if $T \in \mathfrak{X}_i$ and $y_1, y_2 \in M_{I^c}$, then $y_1 T y_2 \in \mathfrak{X}_i$.

**2.2. Other bounded intertwiners.** More generally, the bounded intertwiners

$$\mathrm{Hom}_{M_I}(H_i, H_j) = U_{j0}\, M_{I^c}\, U_{i0}^*, \tag{2.2.1}$$

if $U_{i0} \in \mathfrak{X}_i$ and $U_{j0} \in \mathfrak{X}_j$ are unitary.

## 3. Construction of bounded intertwiners from localised fields

In the following, let $(H_i, \pi_i)$ be the discrete series representations at a fixed central charge $c$; and $I \subset S^1$ an interval. Let $H = \oplus_i H_i$, regarded as an $(M_I, M_I)$-bimodule or, equivalently, as a representation $\pi = \oplus_i \pi_i|_{\mathrm{Diff}^a T_{S^1}}$ of $\mathrm{Diff}_I S^1 \times \mathrm{Diff}_{I^c} S^1$. Let $P_i : H \to H_i$ be the corresponding projections onto the irreducible summands. We recall that

$$M_I = \pi_0(\mathrm{Diff}_I S^1)'' = P_0\, \mathcal{A}_I''\, P_0. \tag{3.1}$$

$\vartheta$ Let $\vartheta_i : \pi_0(\mathrm{Diff}_I S^1)'' \to \pi_i(\mathrm{Diff}_I S^1)''$ be the spatial isomorphism corresponding to the unitary equivalence $\pi_0|_{\mathrm{Diff}_I S^1} \cong \pi_i|_{\mathrm{Diff}_I S^1}$. When there is no confusion, we also use the same symbols $\vartheta_i$ when dealing with the complementary interval $I^c$.

**3.1. Lemma.** *Let $\phi : H_i^\infty \otimes V_{\lambda,\mu} \to H_j^\infty$ be a primary field, $f \in C_I^\infty(S^1)$, and $x = \phi(f)$ the (closeable) smeared primary field. Then $\overline{x}$ is affiliated to $\pi(\mathrm{Diff}_{I^c} S^1)' = \mathcal{A}_I''$. The same is true for any chain $x = \phi_1(f_1)\cdots\phi_n(f_n)$ of smeared primary fields localised in $I$.*

*Proof.* Clearly, for all $u \in \pi(\mathrm{Diff}_{I^c} S^1)$, $u\,x = x\,u$ on $H_i^\infty$. If $v$ is a unitary in $\pi(\mathrm{Diff}_{I^c} S^1)''$, there are unitaries $v_n$ in the unital $*$-algebra generated by $\pi(\mathrm{Diff}_{I^c} S^1)$ such that $v_n \to v$ in the strong operator topology (unitary density theorem). And if $\xi \in \mathcal{D}(\overline{x})$, there are $\xi_n \in H_i^\infty$ such that $\xi_n \to \xi$ and $x\,\xi_n \to \overline{x}\,\xi$. Then $v_n\,\xi_n \to v\,\xi$ and $x\,v_n\,\xi_n = v_n\,x\,\xi_n \to v\,\overline{x}\,\xi$, whence $v\,\xi \in \mathcal{D}(\overline{x})$ and $\overline{x}\,v\,\xi = v\,\overline{x}\,\xi$, i.e. $v^*\,\overline{x}\,v = \overline{x}$. We note that the chain $\phi_1(f_1)\cdots\phi_n(f_n)$ is closeable since it is extended by the closed operator $\{\phi_n^*(f_n^*)\cdots\phi_1^*(f_1^*)\}^*$. □

**3.2. Preliminaries.** Let $f \in C_{I^c}^\infty(S^1)$ and $g \in C_I^\infty(S^1)$. For each $i, j$, let $\phi_{ji}$ and $\psi_{ji}$ denote respectively the primary fields of conformal dimension $h_\phi$ and $h_\psi$ connecting the $H_i$ sector to the $H_j$ sector (whenever they exist); and let

$$x_{ji} = \bar{\phi}_{ji}(f); \quad y_{ji} = \bar{\psi}_{ji}(g) \tag{3.2.1}$$

be the corresponding localised fields mapping $H_i^\infty$ into $H_j^\infty$. Then $\overline{x}_{ji}$ is affiliated to $\mathcal{A}_{I^c}''$, and $\overline{y}_{ji}$ to $\mathcal{A}_I''$. By Proposition IV.2.2, we have the braiding relations

$$x_{kj}\, y_{ji} = \sum_l y_{kl}\, x_{li}\, C_{lj} \tag{3.2.1}$$

on $H_i^\infty$, where

$$C_{lj} = C_{h_l h_j}^{h_k h_\phi h_\psi h_i}. \tag{3.2.2}$$

More generally, and this is important, this relation holds *with insertions*,

$$\{\vartheta_k(g_1)\, x_{kj}\, \vartheta_j(g_2)\}\, \{\vartheta_j(h_1)\, y_{ji}\, \vartheta_i(h_2)\} = \sum_l \{\vartheta_k(h_1)\, y_{kl}\, \vartheta_l(h_2)\}\, \{\vartheta_l(g_1)\, x_{li}\, \vartheta_i(g_2)\}\, C_{lj} \tag{3.2.3}$$

on $H_i^\infty$, for $g_1, g_2 \in \pi_0(\mathrm{Diff}_{I^c} S^1)$ and $h_1, h_2 \in \pi_0(\mathrm{Diff}_I S^1)$. Aim: replace the localised fields by bounded intertwiners. Let $h \in C_{I^c}^\infty(S^1)$ and

$$z_{ji} = \bar{\phi}_{ji}(h) \tag{3.2.4}$$

denote another collection of localised fields, localised like $x_{ji}$ in the interval $I^c$. In the following, the braiding relations we consider also hold with insertions of local diffeomorphisms; for simplicity of notation, we usually do not write these down explicitly. Equivalently, we can regard the $x_{ji}$'s (resp. the $z_{ji}$'s) to be pre- or post-multiplied by local diffeomorphisms with support in $I^c$, i.e. of the form $\vartheta_j(g_1)\, x_{ji}\, \vartheta_i(g_2)$ with $g_1, g_2 \in \pi_0(\mathrm{Diff}_{I^c} S^1)$. Clearly, a chain of localised fields with insertions is also closeable and satisfies Lemma 3.1.

**3.3. Construction of unitary principal intertwiners.** For $\varepsilon > 0$, let

$$v_\varepsilon = [\overline{y}_{j0}^*\, \overline{y}_{j0} + \varepsilon]^{-\frac{1}{2}}. \tag{3.3.1}$$

Since $\overline{y}_{j0}^*\, \overline{y}_{j0}$ is affiliated to $M_I$, $v_\varepsilon^2$ and hence $v_\varepsilon$ lie in $M_I$. Let

$$\overline{y}_{j0} = u_{j0}\, [\overline{y}_{j0}^*\, \overline{y}_{j0}]^{\frac{1}{2}} \tag{3.3.2}$$

be the polar decomposition; so $u_{j0}$ is a partial isometry with initial space $(\ker \overline{y}_{j0})^\perp \subset H_0$ and final space $\overline{\mathrm{im}\, y_{j0}} \subset H_j$. By properties of the functional calculus, $\overline{y}_{j0}\, v_\varepsilon$ is bounded and converges in the strong operator topology to $u_{j0}$ as $\varepsilon \to 0$. Since $\overline{y}_{j0}\, v_\varepsilon$ lies in $\mathfrak{Y}_j$, so does $u_{j0}$. Consider the braiding relation

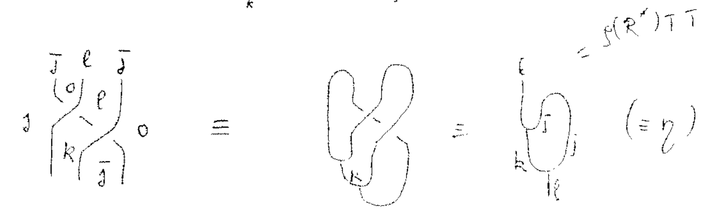
$$\varepsilon(\rho, \sigma\bar{\sigma}) R_\sigma \overset{!}{=} \rho(R_\sigma)\, \varepsilon(\rho, \mathrm{id})$$

$$\overset{!}{\|}$$

$$\varepsilon(\rho, \sigma) \cdot \sigma(\varepsilon(\rho, \bar{\sigma}))$$

$$(z_{l0}^*\, x_{l0})\, y_{j0}^*\, u_{j0} = y_{j0}^* \sum_k \mu_k\, (z_{kj}^*\, x_{kj})\, u_{j0} \tag{3.3.3}$$

$\mu =$



on $H_j^\infty$, where

$$\mu_k = C^{0\,h_\bullet h_\bullet h_k}_{h_\bullet h_\bullet} C^{h_\bullet h_\bullet h_k h_\bullet}_{h_k 0}. \tag{3.3.4}$$

We now show that there exists a unitary $U_{j0} \in \mathfrak{Y}_j$ such that $U_{j0}^* H_j^\infty \subset \mathcal{D}(\overline{z_{I0}^* x_{I0}})$ and

$$\overline{z_{I0}^* x_{I0}}\, U_{j0}^* = U_{j0}^* \sum_k \mu_k z_{kj}^* x_{kj} \tag{3.3.5}$$

on $H_j^\infty$. Using Lemma 3.1, it would then follow that what is true for $U_{j0}$ above is also true for an arbitrary intertwiner in $\mathfrak{Y}_j$. We proceed by first replacing the localised field $y_{j0}$ by its phase $u_{j0}$, which is then modified to make it unitary.

(a) Let $\xi \in H_j^\infty$; then

$$\begin{aligned} v_\epsilon^* y_{j0}^* \xi &\to u_{j0}^* \xi. \\ \overline{z_{I0}^* x_{I0}}\, v_\epsilon^* y_{j0}^* \xi &= v_\epsilon^* y_{j0}^* \sum_k \mu_k z_{kj}^* x_{kj} \xi \\ &\to u_{j0}^* \sum_k \mu_k z_{kj}^* x_{kj} \xi \end{aligned} \tag{3.3.6}$$

as $\varepsilon \to 0$. It follows that $u_{j0}^* H_j^\infty \subset \mathcal{D}(\overline{z_{I0}^* x_{I0}})$ and

$$\overline{z_{I0}^* x_{I0}}\, u_{j0}^* = u_{j0}^* \sum_k \mu_k z_{kj}^* x_{kj} \tag{3.3.7}$$

on $H_j^\infty$. We have therefore replaced $y_{j0}$ by its phase $u_{j0} \in \mathfrak{Y}_j$, in (3.3.3).

(b) Claim: we can replace $u_{j0}$ in the braiding relation (3.3.7) with

$$w_{j0} = \sum_n \vartheta_j(a_n)\, u_{j0}\, b_n \in \mathfrak{Y}_j, \tag{3.3.8}$$

a strong operator convergent sum, where $a_n, b_n \in M_I$: i.e. $w_{j0}^* H_j^\infty \subset \mathcal{D}(\overline{z_{I0}^* x_{I0}})$ and

$$\overline{z_{I0}^* x_{I0}}\, w_{j0}^* = w_{j0}^* \sum_k \mu_k x_{kj}^* x_{kj} \tag{3.3.9}$$

on $H_j^\infty$. It is sufficient by linearity to prove this for $w_{j0} = \vartheta_j(a)\, u_{j0}\, b$, $a, b \in M_I$. By Lemma 3.1, we can certainly replace $u_{j0}$ by $u_{j0}\, b$. So let $a_n$ be a sequence of elements in the linear span of $\pi_0(\mathrm{Diff}_I S^1)$, converging to $a \in M_I$ in the strong operator topology. If $\xi \in H_j^\infty$, then $\vartheta_j(a_n)\xi \in H_j^\infty$ and, since $u_{j0}$ is a bounded operator,

$$\begin{aligned} u_{j0}^* \vartheta_j(a_n)\xi &\to u_{j0}^* \vartheta_j(a)\xi, \\ \overline{z_{I0}^* x_{I0}}\, u_{j0}^* \vartheta_j(a_n)\xi &= u_{j0}^* \vartheta_j(a_n) \sum_k \mu_k z_{kj}^* x_{kj} \xi \\ &\to u_{j0}^* \vartheta_j(a) \sum_k \mu_k z_{kj}^* x_{kj} \xi. \end{aligned} \tag{3.3.10}$$

It follows that $u_{j0}^* \vartheta_j(a)\, H_j^\infty \subset \mathcal{D}(\overline{z_{I0}^* x_{I0}})$ and

$$\overline{z_{I0}^* x_{I0}}\, u_{j0}^* \vartheta_j(a) = u_{j0}^* \vartheta_j(a) \sum_k \mu_k z_{kj}^* x_{kj} \tag{3.3.11}$$

on $H_j^\infty$. This proves the claim.

(c) Repeat the arguments of Part (a) with (3.3.9) in place of (3.3.3) to replace $w_{j0}$ by its phase $v_{j0}$. Since $v_{j0}^* v_{j0}$ is a projection in $M_I$, a Type III factor, there is a partial isometry $v \in M_I$ such that $vv^* = v_{j0}^* v_{j0}$ and $v^* v = 1$. Following Part (b), replace $v_{j0}$ by the partial isometry $U_{j0} = v_{j0} v$ satisfying $U_{j0} U_{j0}^* = v_{j0} v_{j0}^*$, $U_{j0}^* U_{j0} = 1$. Then $U_{j0}$ is unitary if $v_{j0} v_{j0}^* = 1$, i.e. if $\overline{\mathrm{im}\, w_{j0}} = H_j$.

(d) Wassermann [Wa2] has given a construction of a $w_{j0}$ (3.3.8) with dense image. Let $q = u_{j0} u_{j0}^*$. Let $\{g_1, g_2, \ldots\}$ be a countable dense subgroup of the unitary group of $M_I$; $\{p_1, p_2, \ldots\}$ an orthogonal family of projections in $M_I$, $p_n p_m = p_n \delta_{nm}$; $\{v_1, v_2, \ldots\}$ a family of partial isometries in $M_I$ such that $v_n v_n^* = 1$, $v_n^* v_n = p_n$, so that $v_n v_m^* = \delta_{nm}$; and

$$w_{j0} = \sum_n 2^{-n} \vartheta_j(g_n)\, u_{j0}\, v_n, \tag{3.3.12}$$

a norm-convergent series. Here, we note that: the unitary group of $M_I$ is a closed subgroup of the separable metrisable unitary group $U(H_0)$, and is therefore separable and metrisable, so has a countable dense subgroup; $M_I$ is a Type III factor, so all its projections are equivalent. Then

$$w_{j0} w_{j0}^* = \sum_n 2^{-2n} \vartheta_j(g_n)\, q\, \vartheta_j(g_n)^*, \tag{3.3.13}$$

so that $\xi \in (\mathrm{im}\, w_{j0})^\perp = \ker w_{j0}^*$ if and only if $\vartheta_j(g_n)\, q\, \vartheta_j(g_n)^* \xi = 0$ for all $n$. Hence

$$\ker w_{j0}^* = \bigcap_n \ker \vartheta_j(g_n)\, q\, \vartheta_j(g_n)^* \tag{3.3.14}$$

and, if $Q$ is the projection onto $\ker w_{j0}^*$, then $Q = \bigwedge_n \vartheta_j(g_n)\, q\, \vartheta_j(g_n)^* \in \pi_j(\mathrm{Diff}_{I^c} S^1)'$. Hence $\vartheta_j(g_n)\, Q\, \vartheta_j(g_n)^* = Q$ for all $n$. Since the $\vartheta_j(g_n)$ are dense in the unitary group of $\pi_j(\mathrm{Diff}_I S^1)''$, we have $Q \in \{\vartheta_j(g_1), \vartheta_j(g_2), \ldots\}' = \pi_j(\mathrm{Diff}_I S^1)'$. Hence

$$Q \in \pi_j(\mathrm{Diff}_I S^1)' \cap \pi_j(\mathrm{Diff}_{I^c} S^1)' = \mathbb{C} \tag{3.3.15}$$

by irreducibility of the inclusion $\pi_j(\mathrm{Diff}_I S^1)'' \subset \pi_j(\mathrm{Diff}_{I^c} S^1)'$. Since $Q \neq 1$, $Q = 0$.

**3.4. Positivity property of braiding coefficients.** From (3.3.5), for $\xi \in H_j^\infty$,

$$\|\overline{x}_{l0} U_{j0}^* \xi\|^2 = \sum_k \mu_k \|x_{kj} \xi\|^2 \geq 0, \tag{3.4.1}$$

and this holds also for all $f$ (in $x_{kj} = \tilde{\phi}_{kj}(f)$) with support in $I^c$. Since the $\mu_k$ are independent of $f$ and $\xi$, this suggests that each $\mu_k \geq 0$. Note that the braiding matrices depend on the conventions for smearing the primary fields, but the positivity property $\mu_k \geq 0$, if it holds. does not: a braiding matrix $C^{h_1 h_* h_* h_2}_{h_3 h_4}$ would only change by a factor $\epsilon^{\alpha i\{h_1+h_2-h_3-h_4\}}$. so that $\mu_k$, given by (3.3.4). is an invariant combination.

**3.4.1. Lemma.** *We necessarily have $\mu_k \geq 0$.*

*Proof.* Let $a \in \pi_0(\mathrm{Diff}_I S^1)$ and $b \in \pi_0(\mathrm{Diff}_{I^c} S^1)$. Let $z_{l0} = x_{l0}$ in (3.3.5) and write explicitly the insertion of local diffeomorphisms:

$$U_{j0}\, \overline{x}_{l0}^* \,\vartheta_l(b)\, \overline{x}_{l0}\, U_{j0}^*\, \vartheta_j(a) = \sum_k \mu_k\, x_{kj}^*\, \vartheta_k(b)\, x_{kj}\, \vartheta_j(a) \tag{3.4.1.1}$$

on $H_j^\infty$, where $a \in \pi_0(\mathrm{Diff}_I S^1)$ and $b \in \pi_0(\mathrm{Diff}_{I^c} S^1)$. Let $\xi \in H_j^\infty$ and write $\eta = U_{j0}^* \xi$: $\varrho_j(a) = U_{j0}^*\, \vartheta_j(a)\, U_{j0}$, an element in the unitary group of $\pi_0(\mathrm{Diff}_I S^1)''$. Then

$$\langle \overline{x}_{l0}^*\, \vartheta_l(b)\, \overline{x}_{l0}\, \varrho_j(a)\, \eta, \eta\rangle = \sum_k \mu_k \langle \vartheta_k(b)\, \vartheta_k(a)\, x_{kj} \xi, x_{kj} \xi\rangle. \tag{3.4.1.2}$$

By Lemma 3.1, $\vartheta_l(\varrho_j(a))^* \overline{x}_{l0}\, \varrho_j(a) = \overline{x}_{l0}$, so that

$$\langle \vartheta_l(b)\, \vartheta_l(\varrho_j(a))\, \overline{x}_{l0}\, \eta, \overline{x}_{l0}\, \eta\rangle = \sum_k \mu_k \langle \vartheta_k(b)\, \vartheta_k(a)\, x_{kj} \xi, x_{kj} \xi\rangle. \tag{3.4.1.3}$$

Let $\mathcal{B}$ be the unital $*$-algebra generated by $\pi(\mathrm{Diff}^{\delta 7} S^1)$. The right-hand-side of (3.4.1.3) is a linear functional on $\mathcal{B}$, which we shall denote as $\mathcal{B} \ni b \mapsto \varphi(b) = \sum_k \mu_k \varphi_k(b)$, where $\varphi_k(b) = \langle b\, x_{kj} \xi, x_{kj} \xi\rangle$ is a vector state (up to normalisation) on $\mathcal{B}$. Then $\varphi$ extends (and we denote the extension by the same symbol) to a linear functional on the von Neumann algebra $\mathcal{B}'' = (\oplus_k \pi_k(\mathrm{Diff}^{\delta 7} S^1))''$. Since the representations $(H_k, \pi_k)$ are irreducible and distinct, $\mathcal{B}'' = \oplus_k B(H_k)$. Now, by the left-hand-side of (3.4.1.3), $\varphi(b^* b) \geq 0$ for all $b \in \mathcal{B}$. Since $\varphi$ is a linear combination of vector states, if $b_n \in \mathcal{B}$ converges to $x \in \mathcal{B}''$ in the strong operator topology, then $\varphi(x^* x) = \lim_{n \to \infty} \varphi(b_n^* b_n)$. But $\mathcal{B}''$ coincides with the strong/weak operator closure of $\mathcal{B}$, so that $\varphi(x^* x) \geq 0$ for all $x \in \oplus_k B(H_k)$, and in particular for $x \in B(H_k)$. It follows that each $\mu_k \geq 0$. □

**3.5. Construction of bounded intertwiners.** The identity

$$U_{j0}\, \overline{x}_{l0}^*\, \overline{x}_{l0}\, U_{j0}^* = \sum_k \mu_k\, x_{kj}^*\, x_{kj} \tag{3.5.1}$$

holds on $H_j^\infty$. The left-hand-side is the restriction of the self-adjoint $R = U_{j0}\, \overline{x}_{l0}^*\, \overline{x}_{l0}\, U_{j0}^*$ to $H_j^\infty$; the right-hand-side is the restriction of the self-adjoint $\overline{T}^* \overline{T}$, where $T = \sum_k \mu_k^{\frac{1}{2}} x_{kj}$ is defined on $H_j^\infty$ and closeable. Ideally, (3.5.1) extends to an operator equality

$$R = \overline{S} = \overline{T}^* \overline{T}. \tag{3.5.2}$$

This is the case if and only if the symmetric operator $S = \sum_k \mu_k\, x_{kj}^*\, x_{kj}$ is essentially self-adjoint on $H_j^\infty$, since a self-adjoint operator has no proper symmetric extension. Or, equivalently, if and only if $1 + S$ has dense image. For if $\xi \in \mathcal{D}(R)$, then by density and because $S \subset R$, there are $\xi_n \in \mathcal{D}(S)$ such that $(1+S)\xi_n = (1+R)\xi_n \to (1+R)\xi$, whence $\xi_n \to \xi$ because $(1+R)^{-1}$ is bounded. Hence $\xi \in \mathcal{D}(\overline{S})$ and we have $R = \overline{S}$. In particular, and trivially, this is the case when the $x_{ij}$'s are bounded operators. We are unable to prove (3.5.2), but we shall not need to use it either.

Suppose that (3.5.2) holds. For $\varepsilon > 0$, let $u_\varepsilon = (\overline{x}_{l0}^*\, \overline{x}_{l0} + \varepsilon)^{-\frac{1}{2}} \in M_{I^c}$, and $\vartheta_j(u_\varepsilon) = U_{j0}\, u_\varepsilon\, U_{j0}^* = (\overline{T}^* \overline{T} + \varepsilon)^{-\frac{1}{2}}$. Then $\vartheta_j(u_\varepsilon)$ maps $H_j$ into $\mathcal{D}(\overline{T})$. For each $k$, $\mathcal{D}(\overline{T}) \subset \mathcal{D}(\overline{x}_{kj})$, the operators $P_k \overline{T} (\overline{T}^* \overline{T} + \varepsilon)^{-\frac{1}{2}}$ are bounded with norm $\leq 1$ (for all $\varepsilon > 0$), and coincides with $\mu_k^{\frac{1}{2}}\, \overline{x}_{kj}\, \vartheta_j(u_\varepsilon)$. It follows that the strong operator limit

$$\text{s.o.}\lim_{\varepsilon \to 0}\, \overline{x}_{kj}\, \vartheta_j(u_\varepsilon) \tag{3.5.3}$$

exists, but it clearly does not in general converge to the phase of $\overline{x}_{kj}$. Note that the adjoint of $\overline{x}_{kj}\, \vartheta_j(u_\varepsilon)$ also converges, with

$$\text{s.o.}\lim_{\varepsilon \to 0} \{\overline{x}_{kj}\, \vartheta_j(u_\varepsilon)\}^* = \{\text{s.o.}\lim_{\varepsilon \to 0}\, \overline{x}_{kj}\, \vartheta_j(u_\varepsilon)\}^*. \tag{3.5.4}$$

**3.5.1. The bounded case.** We consider the case when the smeared primary fields $x_{ij}$ in the braiding relation (3.3.5) are bounded operators. In this case, and writing explicitly the insertion of local diffeomorphisms,

$$z_{l0}^*\, \vartheta_l(b)\, x_{l0}\, U_{j0}^* = U_{j0}^* \sum_k \mu_k\, z_{kj}^*\, \vartheta_k(b)\, x_{kj} \tag{3.5.1.1}$$

on $H_j$, where $b \in \pi_0(\mathrm{Diff}_{I^c} S^1)$. By linearity and continuity, this holds in fact for all $b \in M_{I^c}$. Now we can replace each $x_{ij}$ in (3.5.1.1) by a strong operator convergent sum

$\sum_n \vartheta_l(b_n)\, x_{ij}\, \vartheta_j(c_n)$, where $b_n, c_n \in M_{I^c}$. We can also replace $x_{l0}$ by its phase, with a corresponding modification of the $x_{kj}$'s, in (3.5.1.1). To see this, note that we can certainly replace $x_{l0}$ by $x_{l0}\, u_\varepsilon$, where $u_\varepsilon = [x_{l0}^* x_{l0} + \varepsilon]^{-\frac{1}{2}} \in M_{I^c}$; and $x_{kj}$ by $x_{kj}\, \vartheta_j(u_\varepsilon)$. By previous considerations, the strong operator limit, as $\varepsilon \to 0$, of $x_{kj}\, \vartheta_j(u_\varepsilon)$ exists, with $x_{l0}\, u_\varepsilon$ converging to the phase of $x_{l0}$. Arguing as in § 3.3, it follows that we can replace $x_{l0}$ by a unitary, and hence an arbitrary element in $\mathfrak{X}_l$.

We have therefore assigned to each $a_{l0} \in \mathfrak{X}_l$ a collection $\{a_{kj} : H_j \to H_k\}$ of bounded intertwiners for $M_I$, such that the braiding relations

$$z_{l0}^*\, \vartheta_l(y)\, a_{l0}\, b_{j0}^* \;=\; b_{j0}^* \sum_k \mu_k\, z_{kj}^*\, \vartheta_k(y)\, a_{kj} \qquad (3.5.1.2)$$

are satisfied, for all $b_{j0} \in \mathfrak{Y}_j$ and $y \in M_{I^c}$. The assignment $a_{l0} \mapsto \{a_{kj}\}$ respects the left and right actions of $M_{I^c}$ on $M_I$-intertwiners, i.e.

$$\vartheta_l(y_1)\, a_{l0}\, y_2 \mapsto \{\vartheta_k(y_1)\, a_{kj}\, \vartheta_j(y_2)\} \qquad (3.5.1.3)$$

for all $y_1, y_2 \in M_{I^c}$. This is easy to see. The assigment is determined completely by the assignment of a unitary element $V_{l0} \mapsto \{V_{kj}\}$, for we define that $V_{l0}\, y$ is mapped to $\{V_{kj} \vartheta_j(y)\}$, for each $y \in M_{I^c}$. Now, if $y_1 \in M_{I^c}$, we certainly have

$$\vartheta_l(y_1)\, V_{l0} = V_{l0}\, y_2 \qquad (3.5.1.4)$$

for a unique $y_2 \in M_{I^c}$. But

$$\begin{aligned} z_{l0}^*\, \vartheta_l(y_1)\, V_{l0}\, U_{j0}^* \;&=\; U_{j0}^* \sum_k \mu_k\, z_{kj}^*\, \vartheta_k(y_1)\, V_{kj} \\ z_{l0}^*\, V_{l0}\, \vartheta_l(y_2)\, U_{j0}^* \;&=\; U_{j0}^* \sum_k \mu_k\, z_{kj}^*\, V_{kj}\, \vartheta_j(y_2). \end{aligned} \qquad (3.5.1.5)$$

It follows that

$$\sum_k \mu_k \left( \{\vartheta_k(y_1)\, V_{kj} - V_{kj}\, \vartheta_j(y_2)\}\xi,\, z_{kj}\, \eta \right) \;=\; 0 \qquad (3.5.1.6)$$

for all $\xi, \eta \in H_j$, and $z_{kj}$ a localised field with arbitrary insertions. But then, arguing as in Lemma 3.4.1, this means that

$$\vartheta_k(y_1)\, V_{kj} \;=\; V_{kj}\, \vartheta_j(y_2), \qquad (3.5.1.7)$$

which proves the claim. We can regard the assigment $a_{l0} \mapsto \{a_{kj}\}$, which intertwines the natural $(M_{I^c}, M_{I^c})$-action, as the smeared analogue of the state-field correspondence.

Similarly, set $z_{l0} = x_{l0}$, and replace it by the unitary $V_{l0} \in \mathfrak{X}_l$. Then

$$\alpha_{l0}^*\, \vartheta_l(y)\, a_{l0}\, b_{j0}^* \;=\; b_{j0}^* \sum_k \mu_k\, \alpha_{kj}^*\, \vartheta_k(y)\, a_{kj} \qquad (3.5.1.8)$$

for all $b_{j0} \in \mathfrak{Y}_j$ and $\alpha_{l0}, a_{l0} \in \mathfrak{X}_l$. In particular,

$$\vartheta_j(\alpha_{l0}^*\, a_{l0}) \;=\; \sum_k \mu_k\, \alpha_{kj}^*\, a_{kj}. \qquad (3.5.1.9)$$

The braiding relation (3.5.1.8) is the basic result that we shall require to compute the Connes fusion of discrete series representations; from Chapter IV, we know that it holds when the highest weight $h_j$ is arbitrary, and $h_l = h_{1,2}$ or $h_{2,2}$; and this is all that we need.

**3.5.2. Remark.** The methods of § 3.3 can likewise be applied to the braiding relation

$$x_{l0}\, w_{j0}^*\, y_{j0} \;=\; \sum_k \gamma_k\, w_{kl}^*\, y_{kl}\, x_{l0} \qquad (3.5.2.1)$$

on $H_0^\infty$, where $x_{l0}$ and the $y_{kl}$'s are as before, $w_{kl} = \bar{\psi}_{kl}(h)$ for some $h \in C_I^\infty(S^1)$, and

$$\gamma_k \;=\; C^{h_k h_\phi h_\psi 0}_{h_\phi h_\psi}\, C^{h_\phi h_\phi h_\psi h_\psi}_{h_k 0}, \qquad (3.5.2.2)$$

provided that the smeared primary fields $\bar{\psi}_{kl}(f)$ are bounded operators. Using the same techniques, we can replace $x_{l0}$ by a unitary element $V_{l0} \in \mathfrak{X}_l$. As in § 3.4, we can also show that $\gamma_k \geq 0$; however, since

$$C^{h_k h_\phi h_\psi 0}_{h_\phi h_\psi} \;=\; C^{0 h_\phi h_\psi h_k}_{h_\psi h_\phi}, \qquad (3.5.2.3)$$

we really have $\gamma_k = \mu_k \geq 0$. Finally, arguing as in § 3.5.1, we obtain an analogous assignment $b_{j0} \mapsto \{b_{kl}\}$ of $M_{I^c}$-intertwiners, satisfying

$$\vartheta_j(x_1)\, b_{j0}\, x_2 \;\mapsto\; \{\vartheta_k(x_1)\, b_{kl}\, \vartheta_l(x_2)\} \qquad (3.5.2.4)$$

for all $x_1, x_2 \in M_I$; and such that

$$a_{l0}\, \beta_{j0}^*\, \vartheta_j(x)\, b_{j0} \;=\; \sum_k \gamma_k\, \beta_{kl}^*\, \vartheta_j(x)\, b_{kl}\, a_{l0} \qquad (3.5.2.5)$$

for all $a_{l0} \in \mathfrak{X}_l$ and $x \in M_I$.

**3.5.3. Summary.** By equations (3.5.1.8) and (3.5.2.5) respectively, the sequence

$$\{a_{l0}^* \vartheta_l(y)\, \alpha_{l0}\}\, \{b_{j0}^* \vartheta_j(x)\, \beta_{j0}\}, \tag{3.5.3.1}$$

where $a_{l0}, \alpha_{l0} \in \mathfrak{X}_l$; $b_{j0}, \beta_{j0} \in \mathfrak{Y}_j$; and $y \in M_{I^c}$, $x \in M_I$, can be re-expressed as

$$\sum_k \mu_k \, \{a_{kj} b_{j0}\}^* \vartheta_k(y)\, \vartheta_k(x)\, \{\alpha_{kj} \beta_{j0}\} \tag{3.5.3.2}$$

when $(h_\phi =)\; h_l = h_{1,2}$ or $h_{2,2}$; and as

$$\sum_k \gamma_k \, \{b_{kl} a_{l0}\}^* \vartheta_k(x)\, \vartheta_k(y)\, \{\beta_{kl} \alpha_{l0}\} \tag{3.5.3.3}$$

when $(h_\psi =)\; h_j = h_{1,2}$ or $h_{2,2}$. The restrictions on the values of the highest weights reflects what we are able to prove, rather than the true state of affairs (cf. § 3.5.4).

**3.5.4. The general case.** We outline the arguments that proceed from the equality (3.5.2); we shall only use the following results when this equality is already known to hold, i.e. when the relevant localised fields are bounded.

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We replace each of the localised fields $x_{ji}$ by $v_{ji} = \text{s.o.}\lim_{\epsilon \to 0} \overline{x}_{ji}\, \vartheta_i(u_\epsilon)$, a bounded intertwiner for $M_I$; and each $y_{ji}$ by $u_{ji} = \text{s.o.}\lim_{\epsilon \to 0} \overline{y}_{ji}\, \vartheta_i(v_\epsilon)$, a bounded intertwiner for $M_{I^c}$ in the braiding relations. Since $\overline{x}_{ji}\, \vartheta_i(u_\epsilon)$ and $\overline{y}_{ji}\, \vartheta_i(v_\delta)$ are bounded operators, if

$$x_{kj}\, y_{ji} = \sum_l y_{kl}\, x_{li}\, C_{lj}, \tag{3.5.4.1}$$

on $H_j^\infty$, then, for $\xi \in H_j^\infty$ and $\eta \in H_k^\infty$, we have

$$\begin{aligned}
\langle \{\overline{x}_{kj}\, \vartheta_j(u_\epsilon)\}\, \{\overline{y}_{ji}\, \vartheta_i(v_\delta)\}\, \xi, \eta \rangle &= \langle \overline{y}_{ji}\, \vartheta_i(v_\delta)\, \xi, \{\overline{x}_{kj}\, \vartheta_j(u_\epsilon)\}^* \eta \rangle \\
&= \langle \overline{y}_{ji}\, \vartheta_i(v_\delta)\, \xi, \vartheta_j(u_\epsilon)^*\, \overline{x}_{kj}^* \eta \rangle \\
&= \langle \vartheta_i(u_\epsilon)\, \vartheta_i(v_\delta)\, \xi, y_{ji}^*\, x_{kj}^* \eta \rangle \\
&= \sum_l C_{lj} \langle \vartheta_i(v_\delta)\, \vartheta_i(u_\epsilon)\, \xi, x_{li}^*\, y_{kl}^* \eta \rangle \\
&= \sum_l C_{lj} \langle \xi, \{\overline{x}_{li}\, \vartheta_i(u_\epsilon)\}^* \{\overline{y}_{kl}\, \vartheta_l(v_\delta)\}^* \eta \rangle \\
&= \sum_l C_{lj} \langle \{\overline{y}_{kl}\, \vartheta_l(v_\delta)\}\, \{\overline{x}_{li}\, \vartheta_i(u_\epsilon)\}\, \xi, \eta \rangle.
\end{aligned} \tag{3.5.4.2}$$

Here, we used the fact that, if $T$ is a closed, unbounded operator, and $A$ is a bounded operator such that $TA$ is bounded, then $(TA)^* = A^* T^*$ on $\mathcal{D}(T^*)$; and also that $\overline{x}_{kj}$

(resp. $\overline{y}_{ji}$) is affiliated to $\pi(\mathrm{Diff}_I S^1)'$ (resp. $\pi(\mathrm{Diff}_{I^c} S^1)'$). Taking the $\epsilon, \delta \to 0$ limit gives the equality

$$v_{kj}\, u_{ji} = \sum_l u_{kl}\, v_{li}\, C_{lj} \tag{3.5.4.3}$$

of bounded operators on $H_j$. Similar arguments apply for the braiding relations

$$\begin{aligned}
x_{kj}\, y_{ij}^* &= \sum_l y_{lk}^*\, x_{lj}\, C_{lj} \\
x_{jk}^*\, y_{ji} &= \sum_l y_{kl}\, x_{il}^*\, C_{lj} \\
x_{jk}^*\, y_{ij}^* &= \sum_l y_{lk}^*\, x_{il}^*\, C_{lj}.
\end{aligned} \tag{3.5.4.4}$$

Note that we can immediately replace, say, $y_{ij}^*$ in the first relation with $y_{ij}\, \vartheta_j(v_\epsilon)$, by multiplying on the left by $\vartheta_k(v_\epsilon)$, and using $\vartheta_k(v_\epsilon)\, \overline{x}_{kj} = \overline{x}_{kj}\, \vartheta_k(v_\epsilon)$ on $\mathcal{D}(\overline{x}_{kj})$. Finally, in the same way as before, we can correct the $v_{ji}$'s and $u_{ji}$'s such that the principal parts $v_{l0}$, $u_{k0}$ are unitary.

## 4. Computing the positive braiding coefficients

We outline an algorithm for computing the coefficients

$$\mu_k = C^{0\, h_\phi h_\psi h_k}_{h_\psi h_\phi}\; C^{h_\phi h_\phi h_\psi h_\psi}_{h_k\, 0} \tag{4.1}$$

using explicit expressions for the braiding matrices $C^{h_1 h_2 h\, h_3}_{h_4 h_5}$, $h = h_{1,2}$ or $h_{2,1}$, that we obtained in Chapter II. The precise values are not important: we only want to check that there is a strict inequality: $\mu_k > 0$. Of course, if $h_\phi = h_{p_1,q_1}$ and $h_\psi = h_{p_2,q_2}$, then $\mu_k$ is defined only for $h_k = h_{p_3,q_3}$ with

$$\begin{aligned}
p_3 &= |p_1 - p_2| + 1, |p_1 - p_2| + 3, \ldots, \min(p_1 + p_2 - 1,\, 2m - p_1 - p_2 - 1) \\
q_3 &= |q_1 - q_2| + 1, |q_1 - q_2| + 3, \ldots, \min(q_1 + q_2 - 1,\, 2(m+1) - q_1 - q_2 - 1)
\end{aligned} \tag{4.2}$$

when the central charge $c = 1 - 6/m(m+1)$. We have already computed in § II.5.3 the values of $\mu_k$ when $h_\psi = h_{1,2}$ or $h_{2,1}$:

$$\begin{aligned}
C^{0\, h_{p,q} h_{1,2} h_{p,q\pm1}}_{h_{1,2} h_{p,q}}\; C^{h_{p,q} h_{p,q} h_{1,2} h_{1,2}}_{h_{p,q\pm1}\, 0} &= \frac{\Gamma(\frac{2}{m+1})\, \Gamma(\pm\{p - \frac{qm}{m+1}\})}{\Gamma(\frac{1}{m+1})\, \Gamma(\frac{1}{m+1} \pm \{p - \frac{qm}{m+1}\})} \\
C^{0\, h_{p,q} h_{2,1} h_{p\pm1,q}}_{h_{2,1} h_{p,q}}\; C^{h_{p,q} h_{p,q} h_{2,1} h_{2,1}}_{h_{p\pm1,q}\, 0} &= \frac{\Gamma(-\frac{2}{m})\, \Gamma(\pm\{q - \frac{p(m+1)}{m}\})}{\Gamma(-\frac{1}{m})\, \Gamma(-\frac{1}{m} \pm \{q - \frac{p(m+1)}{m}\})}.
\end{aligned} \tag{4.3}$$

We recall that $\Gamma(z) = \int_0^\infty t^{z-1}\epsilon^{-t}\,dt$ and $\Gamma(z+1) = z\,\Gamma(z)$. Let $n$ be a strictly negative integer. We have (a) $\Gamma(z) > 0$ when $z > 0$ and $2n < z < 2n+1$; (b) $\Gamma(z) < 0$ when $2n+1 < z < 2n+2$. Since $1 \le p \le m-1$, $1 \le q \le m$ and $m = 3, 4, \ldots$, we see explicitly that the coefficients (4.3) are strictly positive.

We recall from § 3.3 that

$$\bar{x}_{I0}^{*}\,\bar{x}_{I0}\,a_{j0}^{*} = a_{j0}^{*}\sum_k \mu_k\, x_{kj}^{*}\, x_{kj} \tag{4.4}$$

on $H_j^\infty$, for all $a_{j0} \in \mathfrak{D}_j$. Now factorise $a_{j0} = b_{jk}c_{k0}$ as a product of bounded intertwiners for $M_{I^c}$. Here $b_{jk} = \tilde{\chi}_{jk}(g) : H_k - H_j$ is a bounded localised field, $g \in L_I^2(S^1)$, and $c_{k0} \in \mathfrak{D}_k$ is unitary. Then $b_{kj}^{*}$ maps smooth vectors to smooth vectors and, on $H_j^\infty$,

$$\begin{aligned}\bar{x}_{I0}^{*}\,\bar{x}_{I0}\,(b_{jk}c_{k0})^{*} &= \bar{x}_{I0}^{*}\,\bar{x}_{I0}\,c_{k0}^{*}\,b_{jk}^{*}\\ &= c_{k0}^{*}\sum_m a_m\, x_{mk}^{*}\, x_{mk}\, b_{jk}^{*}\\ &= c_{k0}^{*}\sum_m a_m\, x_{mk}^{*}\sum_n \beta_n\, b_{nm}^{*}\, x_{nj}\\ &= c_{k0}^{*}\sum_m a_m\sum_n \beta_n\sum_p \gamma_p\, b_{pk}^{*}\, x_{np}^{*}\, x_{nj}.\end{aligned} \tag{4.5}$$

Hence

$$b_{jk}^{*}\sum_i \mu_i\, x_{ij}^{*}\, x_{ij} = \sum_m a_m\sum_n \beta_n\sum_p \gamma_p\, b_{pk}^{*}\, x_{np}^{*}\, x_{nj}. \tag{4.6}$$

Of course, $\gamma_p$ and the sum in $p$ depend on $n$, $m$; $\beta_n$ and the sum in $n$ depend on $m$.

**4.1. The case $h_\psi = h_{2,2}$.** Let $h_\psi = h_j = h_{2,2}$, $h_k = h_{2,1}$, $\chi$ have conformal dimension $h_\chi = h_{1,2}$, and $h_l = h_{p,q}$. The coefficients $\beta_n$, $\gamma_p$ (likewise $\alpha_m > 0$) are known, being braiding matrices $C^{h_1 h_2 h\, h_3}_{h_4 h_5}$ with $h = h_{1,2}$ (resp. $h_{2,1}$). We obtain

$$\mu_{r,s} = C^{0\,h_{p,q}h_{2,2}h_{r,s}}_{h_{2,2}h_{p,q}}\,C^{h_{p,q}h_{p,q}h_{2,2}h_{2,2}}_{h_{r,s}\,0} = \alpha_{r,q}\,C^{h_{r,q}h_{p,q}h_{1,2}h_{2,2}}_{h_{r,s}h_{2,1}}\,C^{h_{2,1}h_{p,q}h_{1,2}h_{r,s}}_{h_{2,2}h_{r,q}} \tag{4.1.1}$$

with

$$\alpha_{r,q} = C^{0\,h_{p,q}h_{2,1}h_{r,q}}_{h_{2,1}h_{p,q}}\,C^{h_{p,q}h_{p,q}h_{2,1}h_{2,1}}_{h_{r,q}\,0} > 0, \tag{4.1.2}$$

where $1 \le r = p \pm 1 \le m-1$, $1 \le s = q \pm 1 \le m$. It is straightforward to verify that

$$\mu_{r,q\pm1} = \alpha_{r,q}\,\frac{\Gamma(1+\frac{1}{m+1})\,\Gamma(1+\frac{2}{m+1})\,\Gamma(\pm\{r-\frac{qm}{m+1}\})\,\Gamma(\pm\{r-\frac{qm}{m+1}\}+\frac{1}{m+1})}{\Gamma(1\pm\frac{1}{2}\{r-\frac{qm}{m+1}\})^2\,\Gamma(\pm\frac{1}{2}\{r-\frac{qm}{m+1}\}+\frac{2}{m+1})^2} > 0. \tag{4.1.3}$$

The cases $h_\psi = h_{2,2}$ and $h_{1,2}$, given by (4.1.3) and (4.3) respectively, are all we need.

**4.2. The general case.** More generally, let $h_\chi = h_{1,2}$, $h_\phi = h_l = h_{p,q}$ as before, and $h_k = h_{a,b}$, $h_\psi = h_j = h_{a,b+1}$, and suppose that we have an explicit expression for

$$a_m = C^{0\,h_{p,q}h_{a,b}h_m}_{h_{a,b}h_{p,q}}\,C^{h_{p,q}h_{p,q}h_{a,b}h_{a,b}}_{h_m\,0} > 0. \tag{4.2.1}$$

For each $n$, we can explicitly compute the coefficient of $b_{pk}^{*}\,x_{np}^{*}\,x_{nj}$ on the right-hand-side of (4.6), and check that it vanishes when $p \ne n$, and is positive when $p = n$. In particular, we can compute

$$\mu_i = C^{0\,h_{p,q}h_{a,b+1}h_i}_{h_{a,b+1}h_{p,q}}\,C^{h_{p,q}h_{p,q}h_{a,b+1}h_{a,b+1}}_{h_i\,0} \tag{4.2.2}$$

in terms of (4.2.1) and braiding matrices $C^{h_1 h_2 h\, h_3}_{h_4 h_5}$ with $h = h_{1,2}$. More precisely, for each $h_{x,y}$, we are required to compute the following:

$$\begin{aligned}&\sum_{\pm} \alpha_{x,y\pm1}\,C^{h_{x,y\pm1}h_{p,q}h_{1,2}h_{a,b+1}}_{h_{x,y}h_{a,b}}\,C^{h_{a,b}h_{p,q}h_{1,2}h_{x,y}}_{h_{a,b-1}\,h_{x,y\pm1}} = 0\\ &\sum_{\pm} \alpha_{x,y\pm1}\,C^{h_{x,y\pm1}h_{p,q}h_{1,2}h_{a,b+1}}_{h_{x,y}h_{a,b}}\,C^{h_{a,b}h_{p,q}h_{1,2}h_{x,y}}_{h_{a,b+1}\,h_{x,y\pm1}} > 0.\end{aligned} \tag{4.2.3}$$

The number of terms in the former sum is either 0 or 2; and, in the latter, either 1 or 2. The product

$$C^{h_{x,y\pm1}h_{p,q}h_{1,2}h_{a,b+1}}_{h_{x,y}h_{a,b}}\,C^{h_{a,b}h_{p,q}h_{1,2}h_{x,y}}_{h_{a,b+1}\,h_{x,y\pm1}} \tag{4.2.4}$$

is a positive number, and given by

$$\frac{\Gamma(1\pm\{\frac{ym}{m+1}-x\})\,\Gamma(\frac{m}{m+1}\pm\{\frac{ym}{m+1}-x\})\,\Gamma(a-\frac{bm}{m+1})\,\Gamma(a-\frac{bm}{m+1}+\frac{1}{m+1})}{\Gamma(\frac{1+a+p}{2}-\frac{(b+q)m}{2(m+1)}\pm\frac{1}{2}\{\frac{ym}{m+1}-x\})^2\,\Gamma(\frac{1+a-p}{2}-\frac{(b-q)m}{2(m+1)}\pm\frac{1}{2}\{\frac{ym}{m+1}-x\})^2} > 0. \tag{4.2.5}$$

Since each $\alpha_{x,y\pm1} > 0$ by assumption, the latter inequality of (4.2.3) follows. The former can be successively (albeit laboriously) checked by explicit computation using the recursion relations above (together with those obtained with $h_{2,1}$ in place of $h_{1,2}$; see the following paragraph). Finally, to deduce that

$$\mu_{x,y} = \sum_{\pm} \alpha_{x,y\pm1}\,C^{h_{x,y\pm1}h_{p,q}h_{1,2}h_{a,b+1}}_{h_{x,y}h_{a,b}}\,C^{h_{a,b}h_{p,q}h_{1,2}h_{x,y}}_{h_{a,b+1}\,h_{x,y\pm1}}, \tag{4.2.6}$$

we note that, in (4.6), if

$$b_{jk}^{*}\sum_i \epsilon_i\, x_{ij}^{*}\, x_{ij} = 0 \tag{4.2.7}$$

on $H_j^\infty$, then this holds for all $g$ with support in $I$ (here, $b_{jk} = \tilde{\chi}_{jk}(g)$). Taking matrix elements with smooth vectors and applying the Reeh-Schlieder theorem, we deduce that

$$\sum_i \epsilon_i x_{ij}^* x_{ij} = 0 \tag{4.2.8}$$

on $H_j^\infty$. Arguing as in Lemma 3.4.1, we deduce that each $\epsilon_i = 0$.

An analogous situation occurs for $h_\chi = h_{2,2}$, since the smeared primary fields with conformal dimension $h_{2,2}$ are also bounded operators. However, we do not have explicit expressions for braiding matrices $C_{h_4 h_5}^{h_1 h_2 h\, h_3}$ with $h = h_{2,2}$. We have them for $h = h_{2,1}$, but do not know that the corresponding smeared primary fields are bounded. So we have first to replace smeared primary fields by bounded intertwiners in the relevant braiding relations, and this is possible only if (3.5.3) holds. The rest then proceeds as before.

## 5. Connes fusion of bimodules over a Type III factor

We describe Connes fusion of $(M, M)$-bimodules in the case when $M$ is a Type III factor. In the form that is developed here, we have learnt it from Wassermann [Wa2]. The general construction can be found in [Sa]. We obtain the $C^*$ monoidal structure on the category **Bimod**$_M$ of $(M, M)$-bimodules.

**5.1. Connes' theory of bimodules.** We outline the ideas underlying Connes fusion. Let $M$ (resp. $A, B, \ldots$) be a Type III factor and $L^2(M)$ its standard representation with cyclic, separating vector $\Omega$. The Tomita-Takesaki theory applies, $M' = JMJ$; and we identify the opposite algebra $M^{op}$ with its commutant $M'$ by the $*$-anti-isomorphism $M \to M'$, $x \mapsto J x^* J$. An $(A, B)$-*bimodule* is a pair of commuting normal representations of $A$ and $B^{op}$ on a Hilbert space $H$. In particular, $L^2(M)$ is an $(M, M)$-bimodule.

A representation of $M$ on $H$ is a unital $*$-homomorphism $\pi_H^M : M \to B(H)$. Normality of $\pi_M^H$ means there is a family of vectors $\mathcal{F} \subset H$ such that $\pi_H^M(M)\mathcal{F}$ is total in $H$, and $x \mapsto \langle \pi_H^M(x)\xi, \xi \rangle$ is a normal state on $M$ for all $\xi \in \mathcal{F}$. We may write $x \cdot \eta$ for $\pi_H^M(x)\eta$. A direct sum of normal representations is normal. If $\pi_H^M : M \to B(H)$ is normal, its image $\pi_H^M(M)$ is a von Neumann algebra (Proposition 7.1.15 of [KaR2]). A unitary equivalence of $(A, B)$-bimodules $H$ and $K$ is a unitary map $H \to K$ that intertwines $A$ and $B^{op}$.

Since $M$ is a Type III factor, the $M$-modules are all unitarily equivalent. If the representation $\pi_H^M : M \to B(H)$ is normal, it is implemented by a unitary $U \in \mathrm{Hom}_M(L^2(M), H)$, i.e. $\pi_H^M(x) = U x U^*$ $(x \in M)$. We also define an *embedding* of $B$ in $A$ to be a unital $*$-homomorphism $B \to A$ that is implemented by a unitary map $L^2(B) \to L^2(A)$.

An $(A, B)$-bimodule $H$ defines, up to an inner automorphism of $A$, an embedding $\varrho : B \to A$, given by $B' \to A'$, $b' \mapsto U^* V b' V^* U$, where $U \in \mathrm{Hom}_A(L^2(A), H)$ and $V \in \mathrm{Hom}_{B^{op}}(L^2(B), H)$ are unitary. Conversely, if $\varrho : B \to A$ is an embedding, it defines an $(A, B)$-bimodule structure on $L^2(A)$ through the representation of $B^{op}$ on $L^2(A)$ given by $b' \mapsto J_A \varrho(\underset{(b'^{*})^{*}}{J_B b' J_B}) J_A$; we denote this $(A, B)$-bimodule by $\varrho L^2(A)$. This gives a 1-1 correspondence $\underset{b'^{op} \mapsto \varrho(b)^{op}}{\text{between}}$ isomorphism classes (w.r.t. unitary equivalence) of $(A, B)$-bimodules and embeddings $\varrho : B \to A$ modulo inner automorphisms of $A$. This is significant because bimodules have an additive structure, i.e. direct sums, while embeddings have a multiplicative one, i.e. composition of maps; these structures clearly induce corresponding ones on the equivalence classes. $\mathcal{B} = A = \mathcal{A}(I)$, $\varrho \in \Delta_I$, $\bar{\varrho} := j\varrho j \in \Delta_{I'}$ $\Rightarrow$ $a' \cdot \xi_\varrho = j\varrho j(a')\xi_\varrho = \bar{\varrho}(a')\xi_\varrho$
$a \cdot \xi_\varrho = a\xi_\varrho = \bar{\varrho}(a)\xi_\varrho$

$b'\xi = j\varrho j($

**5.2. Definition of Connes fusion.** We consider $(M, M)$-bimodules. Let $H_0 = L^2(M)$. Let $H_i$ and $H_j$ be $(M, M)$-bimodules, $\mathfrak{X}_i = \mathrm{Hom}_M(H_0, H_i)$ and $\mathfrak{Y}_j = \mathrm{Hom}_{M^{op}}(H_0, H_j)$. The algebraic tensor product $\mathfrak{X}_i \otimes \mathfrak{Y}_j \ni x \otimes y$ transforms under $M^{op} \otimes M \ni b' \otimes a$ by left-multiplication, $x \otimes y \mapsto b' \cdot x \otimes a \cdot y$. There is a contravariant sesquilinear form $(\,.\,,\,.\,)$ on $\mathfrak{X}_i \otimes \mathfrak{Y}_j$, given by

$\overline{\mathfrak{X}_i\Omega} = H_i$
$\overline{\mathfrak{Y}_j\Omega} = H_j$

$a : H_j \to H_j$ commutes with the action of $M^{op}$ on $H_j$ $\} \subset M'$

$x^*x \in \mathrm{Hom}_M(H_0, H_0) = M'$
$y^*y \in \ldots\ldots\ M$

$$(x_1 \otimes y_1, x_2 \otimes y_2) = \langle x_2^* x_1 y_2^* y_1 \Omega, \Omega \rangle. \tag{5.2.1}$$

If $U \in \mathfrak{X}_i$ and $V \in \mathfrak{Y}_j$ are unitary, then $\mathfrak{X}_i = U M'$, $\mathfrak{Y}_j = V M$; and it is easy to see that the form is positive semi-definite. The quotient of $\mathfrak{X}_i \otimes \mathfrak{Y}_j$ by the kernel of this form is a pre-Hilbert space which we denote by $[\mathfrak{X}_i \otimes \mathfrak{Y}_j] \ni [x \otimes y]$. When there is no confusion, we may dispense with the parentheses.

The *Connes fusion* of the bimodules $H_i$, $H_j$ is defined as the Hilbert space completion of $\mathfrak{X}_i \otimes \mathfrak{Y}_j$; we write this as $H_i \boxtimes H_j$. $M^{op}$ and $M$ each acts by bounded operators on $\mathfrak{X}_i \otimes \mathfrak{Y}_j$, since

$$(b' \cdot x_1 \otimes a \cdot y_1, b' \cdot x_2 \otimes a \cdot y_2) = \langle T V^* y_1 U^* x_1 \Omega, T V^* y_2 U^* x_2 \Omega \rangle, \tag{5.2.2}$$

where $T = U^* b' \cdot U V^* a \cdot V$, so that $H_i \boxtimes H_j$ supports commuting representations of $M^{op}$ and $M$. Let $U_i \in \mathfrak{X}_i$ (resp. $U_j \in \mathfrak{X}_j$) be unitary intertwiners, and $\varrho_i : M \to M$ (resp. $\varrho_j$) be the embedding given by $M' \to M'$, $b' \mapsto U_i^* b' \cdot U_i$. The map $H_i \to \overset{H_0 \text{ as Hilbert space}}{\varrho_i H_0}$, $\xi \mapsto U_i^* \xi$, is a unitary

$b'x = b'U(U^*x)^{\in M'}$
$ay = aV(V^*y)_{\in M}$

$\langle (U^*b'^*)(b'U)(U^*x_1)\ (y_2^*V)(V^*a^*)(aV)(V^*y_1)\Omega,\ U^*x_2\Omega \rangle$

$= \langle (U^*b'^*)(b'U)(V^*a^*)(aV)\ U^*x_1 V^*y_1\Omega,\ V^*y_2 U^*x_2 \ldots$ [illegible] equivalence of $(M, M)$-bimodules. Furthermore, $\mathfrak{X}_i \otimes \mathfrak{Y}_j \rightarrow \varrho_j \varrho_i H_0$, $x \otimes y \mapsto U_j^* y U_i^* x \Omega$, defines a unitary map $H_i \boxtimes H_j \rightarrow \varrho_j \varrho_i H_0$ that intertwines $M$ and $M^{op}$. It follows that $H_i \boxtimes H_j$ is an $(M, M)$-bimodule unitarily equivalent to $\varrho_j \varrho_i H_0$. ✓

Since $\Omega \in H_0$ is cyclic and separating for $M$, $\mathfrak{X}_i$ (resp. $\mathfrak{Y}_j$) can be identified with the dense subspace $\mathfrak{X}_i \Omega \subset H_i$. We have the inclusion of linear spaces $[\mathfrak{X}_i \otimes \mathfrak{Y}_j] \subset H_i \boxtimes H_j$. More generally, we also have $[H_i \otimes \mathfrak{Y}_j] \subset H_i \boxtimes H_j$ (resp. $[\mathfrak{X}_i \otimes H_j] \subset H_i \boxtimes H_j$) in the sense that, if $\mathfrak{X}_i \Omega \ni x_n \Omega \rightarrow \xi$, is a convergent sequence in $H_i$, then $[x_n \otimes y]$ is a convergent sequence in $H_i \boxtimes H_j$, for each $y \in \mathfrak{Y}_j$; let $[\xi \otimes y]$ be the limit. To see this, note that

$$\langle y^* y x^* x \Omega, \Omega \rangle = \langle \pi_{H_i}^M (y^* y) x \Omega, x \Omega \rangle \qquad (= \langle \pi_{H_i}^{M'} (x^* x) y \Omega, y \Omega \rangle) \tag{5.2.3}$$

for all $y \in \mathfrak{Y}_j$, $x \in \mathfrak{X}_i$. It follows that we can equivalently define $H_i \boxtimes H_j$ as the Hilbert space completion of $H_i \otimes \mathfrak{Y}_j$ (resp. $\mathfrak{X}_i \otimes H_j$) with respect to the positive semi-definite form

$$(\xi_1 \otimes y_1, \xi_2 \otimes y_2) = \langle \pi_{H_i}^M (y_2^* y_1) \xi_1, \xi_2 \rangle \tag{5.2.4}$$

(resp. $(x_1 \otimes \eta_1, x_2 \otimes \eta_2) = \langle \pi_{H_j}^{M'} (x_2^* x_1) \eta_1, \eta_2 \rangle$).

More generally, if $D_i$ is a dense subspace of $H_i$ (resp. $D_j \subset H_j$ dense), then $[D_i \otimes \mathfrak{Y}_j]$ (resp. $[\mathfrak{X}_i \otimes D_j]$) is dense in $H_i \boxtimes H_j$. In particular, $[\mathfrak{X}_i \otimes \mathfrak{X}_j]$, $[\mathfrak{Y}_i \otimes \mathfrak{Y}_j] \subset H_i \boxtimes H_j$ are dense subspaces. Moreover, as elements of $H_i \boxtimes H_j$,

$$\begin{aligned} [x \otimes b' \cdot \eta] &= [x b' \otimes \eta] \\ [a \cdot \xi \otimes y] &= [\xi \otimes y a] \end{aligned} \tag{5.2.5}$$

for all $x \in \mathfrak{X}_i$, $y \in \mathfrak{Y}_j$; $b' \in M'$, $a \in M$; and $\eta \in H_j$, $\xi \in H_i$.

**5.3. Additive structures on Bimod$_M$.** We consider the category **Bimod**$_M$ of $(M, M)$-bimodules and bimodule maps, i.e. bounded intertwiners for $M$ and $M^{op}$.

**5.3.1. Lemma.** **Bimod**$_M$ *is a* $C^*$ *category.*

A $\mathbb{C}$-linear category is an abelian category such that the Hom-sets $\mathrm{Mor}(a, b)$ are $\mathbb{C}$-linear spaces, and composition $\mathrm{Mor}(b, c) \times \mathrm{Mor}(a, b) \rightarrow \mathrm{Mor}(a, c)$ is $\mathbb{C}$-linear. A $C^*$ category is a $\mathbb{C}$-linear category such that: (a) the Hom-sets $\mathrm{Mor}(a, b)$ are Banach spaces; and there is a conjugate-linear involution $* : \mathrm{Mor}(a, b) \rightarrow \mathrm{Mor}(b, a)$; (b) if $f \in \mathrm{Mor}(a, b)$, $g \in \mathrm{Mor}(b, c)$, then $\|gf\| \le \|g\| \|f\|$, $\|f\|^2 = \|f^* f\| = \|f^*\|^2$ and $(gf)^* = f^* g^*$.

*Proof.* This is immediate. □

**5.4. Lemma (Functoriality of Connes fusion).** *Connes fusion defines a* $\mathbb{C}$*-bilinear functor* $\boxtimes$ : **Bimod**$_M \times$ **Bimod**$_M \rightarrow$ **Bimod**$_M$.

*Proof.* To a pair $(H_i, H_j)$ of bimodules, we assign the bimodule $H_i \boxtimes H_j$. Let $(H_k, H_l)$ be another pair of bimodules, and $(f, g) : (H_i, H_j) \rightarrow (H_k, H_l)$ a pair of bimodule maps. Define $f \boxtimes g$ to be the map $H_i \boxtimes H_j \rightarrow H_k \boxtimes H_l$ given by $\mathfrak{X}_i \otimes \mathfrak{Y}_j \rightarrow \mathfrak{X}_k \otimes \mathfrak{Y}_l$, $x \otimes y \mapsto fx \otimes gy$ (this is bounded and densely defined): then $f \boxtimes g$ is a bimodule map. We easily see that $1_H \boxtimes 1_K = 1_{H \boxtimes K}$; $hf \boxtimes kg = (h \boxtimes k)(f \boxtimes g)$ when the composites $hf$ and $kg$ are defined; and that the assignment $(f, g) \mapsto f \boxtimes g$ is $\mathbb{C}$-bilinear. This proves the lemma. In addition to compatibility with the abelian and $\mathbb{C}$-linear structures of **Bimod**$_M$, we have further that $(f \boxtimes g)^* = f^* \boxtimes g^*$ and $\|f \boxtimes g\| \le \|f\| \|g\|$, i.e. compatibility with the $C^*$ structure. □

**5.5. Multiplicative structure of Bimod$_M$.** If $(H_i, H_j, H_k)$ is a triplet of bimodules, define a unitary equivalence $\alpha_{ijk} : H_i \boxtimes (H_j \boxtimes H_k) \rightarrow (H_i \boxtimes H_j) \boxtimes H_k$ by

$$[\mathfrak{X}_i \otimes [H_j \otimes \mathfrak{Y}_k]] \ni [x \otimes [\xi \otimes y]] \mapsto [[x \otimes \xi] \otimes y] \in [[\mathfrak{X}_i \otimes H_j] \otimes \mathfrak{Y}_k]. \tag{5.5.1}$$

This is well-defined since

$$\begin{aligned} \langle [x_1 \otimes [\xi_1 \otimes y_1]], [x_2 \otimes [\xi_2 \otimes y_2]] \rangle &= \langle (y_2^* y_1)(x_2^* x_1) \cdot \xi_1, \xi_2 \rangle \\ &= \langle [[x_1 \otimes \xi_1] \otimes y_1], [[x_2 \otimes \xi_2] \otimes y_2] \rangle. \end{aligned} \tag{5.5.2}$$

It maps a dense subspace of $H_i \boxtimes (H_j \boxtimes H_k)$ onto a dense subspace of $(H_i \boxtimes H_j) \boxtimes H_k$ isometrically, and intertwines $M$ and $M^{op}$. Hence it defines a unitary bimodule map $\alpha_{ijk}$. It is easy to see that

$$\begin{aligned} [H_i \otimes [\mathfrak{Y}_j \otimes \mathfrak{Y}_k]] &\subset H_i \boxtimes (H_j \boxtimes H_k) \\ [[\mathfrak{X}_i \otimes \mathfrak{X}_j] \otimes H_k] &\subset (H_i \boxtimes H_j) \boxtimes H_k; \end{aligned} \tag{5.5.3}$$

and that, under the map $\alpha_{ijk}$,

$$\begin{aligned} [H_i \otimes [\mathfrak{Y}_j \otimes \mathfrak{Y}_k]] \ni [\eta \otimes [y_1 \otimes y_2]] &\mapsto [[\eta \otimes y_1] \otimes y_2] \in [[H_i \otimes \mathfrak{Y}_j] \otimes \mathfrak{Y}_k] \\ [\mathfrak{X}_i \otimes [\mathfrak{X}_j \otimes H_k]] \ni [x_1 \otimes [x_2 \otimes \zeta]] &\mapsto [[x_1 \otimes x_2] \otimes \zeta] \in [[\mathfrak{X}_i \otimes \mathfrak{X}_j] \otimes H_k]. \end{aligned} \tag{5.5.4}$$

Moreover, if $(f, g, h) : (H_i, H_j, H_k) \rightarrow (H_p, H_q, H_r)$ is a triplet of bimodule maps, then the following diagram commutes:

$$\begin{tikzcd} H_i \boxtimes (H_j \boxtimes H_k) \arrow[r, "\alpha_{ijk}"] \arrow[d, "f \boxtimes (g \boxtimes h)"'] & (H_i \boxtimes H_j) \boxtimes H_k \arrow[d, "(f \boxtimes g) \boxtimes h"] \\ H_p \boxtimes (H_q \boxtimes H_r) \arrow[r, "\alpha_{pqr}"] & (H_p \boxtimes H_q) \boxtimes H_r. \end{tikzcd} \tag{5.5.5}$$

We have therefore obtained a natural isomorphism $\alpha : \boxtimes(1 \times \boxtimes) \to \boxtimes(\boxtimes \times 1)$ of functors $\mathbf{Bimod}_M^{\times 3} \to \mathbf{Bimod}_M$. The bifunctor $\boxtimes$ is therefore associative up to a natural isomorphism $\alpha$.

For each bimodule $H_i$, we define a unitary equivalence $\mu_i : H_0 \boxtimes H_i \to H_i$ by

$$[M' \otimes H_i] \ni [b' \otimes \xi] \mapsto b' \cdot \xi \in H_i, \tag{5.5.6}$$

and another unitary equivalence $\nu_i : H_i \boxtimes H_0 \to H_i$ by

$$[H_i \otimes M] \ni [\xi \otimes a] \mapsto a \cdot \xi \in H_i. \tag{5.5.7}$$

It is easy to see that

$$\begin{aligned} \mu_i &: [H_0 \otimes \mathfrak{Y}_i] \ni [\eta \otimes y] \mapsto y\eta \in H_i \\ \nu_i &: [\mathfrak{X}_i \otimes H_0] \ni [x \otimes \eta] \mapsto x\eta \in H_i. \end{aligned} \tag{5.5.8}$$

If $f : H_i \to H_p$ is a bimodule map, then the diagrams

$$\begin{tikzcd} H_0 \boxtimes H_i \arrow[r, "\mu_i"] \arrow[d, "1\boxtimes f"'] & H_i \arrow[d, "f"] \\ H_0 \boxtimes H_p \arrow[r, "\mu_p"] & H_p \end{tikzcd} \qquad \begin{tikzcd} H_i \boxtimes H_0 \arrow[r, "\nu_i"] \arrow[d, "f\boxtimes 1"'] & H_i \arrow[d, "f"] \\ H_p \boxtimes H_0 \arrow[r, "\nu_p"] & H_p \end{tikzcd} \tag{5.5.9}$$

manifestly commute. The bimodule $H_0 = L^2(M)$, is a left (resp. right) unit for $\boxtimes$ up to a natural isomorphism $\mu$ (resp. $\nu$).

ς , ∘ , id , (ass) , 1x , x1

**5.5.1. Lemma.** *($\mathbf{Bimod}_M$, $\boxtimes$, $L^2(M)$, $\alpha$, $\mu$, $\nu$) is a monoidal category.*

*Proof.* Commutativity of the pentagonal diagram

$$\begin{tikzcd} H_i \boxtimes (H_j \boxtimes (H_k \boxtimes H_l)) \arrow[r, "\alpha"] \arrow[d, "1\boxtimes\alpha"'] & (H_i \boxtimes H_j) \boxtimes (H_k \boxtimes H_l) \arrow[r, "\alpha"] & ((H_i \boxtimes H_j) \boxtimes H_k) \boxtimes H_l \\ H_i \boxtimes ((H_j \boxtimes H_k) \boxtimes H_l) \arrow[rr, "\alpha"] & & (H_i \boxtimes (H_j \boxtimes H_k)) \boxtimes H_l \arrow[u, "\alpha\boxtimes 1"'] \end{tikzcd} \tag{5.5.1.1}$$

follows by tracking the path of the dense subspace of vectors $[\mathfrak{X}_i \otimes [\mathfrak{X}_j \otimes [\mathfrak{Y}_k \otimes \mathfrak{Y}_l]]]$. To check commutativity of the triangular diagram

$$\begin{tikzcd} H_i \boxtimes (H_0 \boxtimes H_j) \arrow[r, "\alpha"] \arrow[d, "1\boxtimes\mu"'] & (H_i \boxtimes H_0) \boxtimes H_j \arrow[d, "\nu\boxtimes 1"] \\ H_i \boxtimes H_j \arrow[r, phantom, "="] & H_i \boxtimes H_j, \end{tikzcd} \tag{5.5.1.2}$$

note that an element $[x \otimes [a \otimes y]] \in [\mathfrak{X}_i \otimes [M \otimes \mathfrak{Y}_j]]$ is mapped to $[x \otimes ya]$ and $[a \cdot x\Omega \otimes y]$ respectively by $1 \boxtimes \mu$ and the composite $(\nu \boxtimes 1)\,\alpha$. By (5.2.5), they coincide. □

additivity of inductions

**5.6. Compatibility of additive and multiplicative structures.** $\mathrm{Bimod}_M$ is both abelian and monoidal, and the bi-additivity of $\boxtimes$ makes these structures compatible; and similarly with the $\mathbb{C}$-linear and $C^*$ structures. It is usual to call such a functor $\boxtimes$ a *tensor product*. Let the bimodule $H$ be a direct sum (binary bi-product) of the bimodules $H_1$ and $H_2$, $H \cong H_1 \oplus H_2$; i.e. there are bimodule maps $H_k \xrightarrow{i_k} H \xrightarrow{p_k} H_k$, $k = 1, 2$, such that $p_k i_k = 1_k$ and $i_1 p_1 + i_2 p_2 = 1_H$. By the functoriality and bi-additivity of $\boxtimes$, $H \boxtimes K$ is a direct sum of $H_1 \boxtimes K$ and $H_2 \boxtimes K$ (and similarly for $K \boxtimes H$),

$$\begin{tikzcd} H_1 \arrow[d, shift right, "i_1"'] & H_1 \boxtimes K \arrow[d, shift right, "i_1\boxtimes 1"'] \\ H \arrow[u, shift right, "p_1"'] \arrow[d, shift right, "p_2"'] & H \boxtimes K \arrow[u, shift right, "p_1\boxtimes 1"'] \arrow[d, shift right, "p_2\boxtimes 1"'] \\ H_2 \arrow[u, shift right, "i_2"'] & H_2 \boxtimes K. \arrow[u, shift right, "i_2\boxtimes 1"'] \end{tikzcd} \tag{5.6.1}$$

In this sense, the tensor product operation $\boxtimes$ is *distributive* over the direct sum $\oplus$.

**5.6.1. Grothendieck ring of $\mathrm{Bimod}_M$.** Let $[H]$ be the isomorphism class of the bimodule $H$. The Grothendieck ring of the abelian monoidal category $\mathrm{Bimod}_M$ is the ring (with unit) generated by the elements $[H]$; with addition given by $[H] + [K] = [H \oplus K]$; and multiplication by $[H] \cdot [K] = [H \boxtimes K]$. The defining properties of a ring are easily verified.

## 6. Connes fusion of discrete series representations

We consider the discrete series representations of $\mathrm{Diff}^+S^1$ at a fixed central charge $c$. By local equivalence and Haag duality they can be regarded as $(M_I, M_I)$-bimodules, where $M_I = \pi_{0,c}(\mathrm{Diff}_I S^1)''$, where $I \subset S^1$ is a fixed interval. By von Neumann density they remain irreducible and distinct as $(M_I, M_I)$-bimodules. We consider $\mathrm{Pos}_c$, the full subcategory of $\mathrm{Bimod}_{M_I}$, whose objects are finite direct sums of discrete series representations. For reasons already given in § 1.2, the choice of the interval $I$ is not essential; for definiteness, we take $I$ to be the upper or lower half-circle in the plane. The $C^*$ structure is obviously preserved, and $\mathrm{Pos}_c$ is a $C^*$ category. By definition, $\mathrm{Pos}_c$ is semi-simple, with the discrete series representations as the simple objects. We want to show that $\mathrm{Pos}_c$ is a monoidal subcategory, i.e. that the monoidal structure on $\mathrm{Bimod}_{M_I}$ restricts to $\mathrm{Pos}_c$. This means that $\mathrm{Pos}_c$ is closed under the tensor product operation $\boxtimes$, and $H_0 = L^2(M_I)$ is an object in $\mathrm{Pos}_c$ (but this is clear). By the bi-additivity of $\boxtimes$, it is necessary and sufficient to show

not yet monoidal

that the tensor product of a pair of discrete series representations is a direct sum of the same, i.e. that

$$[H_i \boxtimes H_j] = \sum_{k \in \mathbf{Y}_c} N_{ij}^k [H_k], \tag{6.1}$$

where $\{[H_i], i \in \mathbf{Y}_c\}$ are the isomorphism classes of the discrete series representations at central charge $c$, and $(N_{ij}^k)$ is an ordered set of non-negative integers (to be determined), the so-called *fusion rules*. Of course, $\mathbf{Y}_c$ is a finite set, with cardinality $|\mathbf{Y}_c| = \frac{1}{2}m(m-1)$ when $c = 1 - 6/m(m+1)$. As usual, we fix a set $\{H_i, i \in \mathbf{Y}_c\}$ of discrete series representations, and take the indexing set $\mathbf{Y}_c$ to be the set of highest weights $(h_{p,q}, c)$, $p = 1, 2, \ldots, m-1$; $q = 1, 2, \ldots, m$, modulo the relation $(p,q) \sim (m-p, m+1-q)$.

**6.1. Computation of Connes fusion.** We compute the Connes fusion of a generating discrete series representation, viz. $H_{1,2}$ or $H_{2,2}$, with an arbitrary one. Let $a_{i0}, \alpha_{i0} \in \mathfrak{X}_i$; $b_{j0}, \beta_{j0} \in \mathfrak{Y}_j$; and $y_1, y_2 \in M_I'$, $x_1, x_2 \in M_I$. By (3.5.3.2), p.57

$$\begin{aligned}\langle [y_1 \cdot \alpha_{i0} \otimes x_1 \cdot \beta_{j0}], [y_2 \cdot a_{i0} \otimes x_2 \cdot b_{j0}] \rangle &= \langle a_{i0}^* y_2^* y_1 \cdot \alpha_{i0}\, b_{j0}^* x_2^* x_1 \cdot \beta_{j0}\, \Omega, \Omega \rangle \\ &= \sum_{k \in \langle i,j \rangle} \mu_k \langle y_1 x_1 \cdot \alpha_{kj}\, \beta_{j0}\, \Omega, y_2 x_2 \cdot a_{kj}\, b_{j0}\, \Omega \rangle,\end{aligned} \tag{6.1.1}$$

when the highest weight $h_i = h_{1,2}$ or $h_{2,2}$. For these values, we have computed directly the braiding coefficients $\mu_k$ — (4.3) and (4.1.3) respectively — and in particular the indexing set $\langle i,j \rangle$. It follows from (6.1.1) that the map

p.98 p.95

$$[\mathfrak{X}_i \otimes \mathfrak{Y}_j] \to \bigoplus_{k \in \langle i,j \rangle} H_k, \quad [a_{i0} \otimes b_{j0}] \mapsto \sum_{k \in \langle i,j \rangle} \mu_k^{\frac{1}{2}}\, a_{kj}\, b_{j0}\, \Omega \tag{6.1.2}$$

defines an isometric bimodule map $H_i \boxtimes H_j \to \oplus_{k \in \langle i,j \rangle} H_k$. Its image is therefore a sub-bimodule, and decomposes as a direct sum of irreducible pieces. Since the $H_k$'s, $k \in \langle i,j \rangle$, are *distinct* irreducible bimodules, this decomposition is unique; by inspection of (6.1.2), the map must be surjective, and therefore a unitary equivalence of bimodules. We obtain

$$[H_{1,2}] \cdot [H_{p,q}] = [H_{1,2} \boxtimes H_{p,q}] = \sum_{1 \leq s = q \pm 1 \leq m} [H_{p,s}] \tag{6.1.3a}$$

$$[H_{2,2}] \cdot [H_{p,q}] = [H_{2,2} \boxtimes H_{p,q}] = \sum_{\substack{1 \leq r = p \pm 1 \leq m-1 \\ 1 \leq s = q \pm 1 \leq m}} [H_{r,s}] \tag{6.1.3b}$$

and of course $[H_0 \boxtimes H_{p,q}] = [H_{p,q} \boxtimes H_0] = [H_{p,q}]$, where $H_0 = H_{1,1}$.

**6.2. Proposition.** *$\mathrm{Pos}_c$ is a monoidal sub-category of $\mathrm{Bimod}_{M_I}$.*

*Proof.* In the previous section, we obtained (6.1) in important special cases by explicit computation. The general result now follows by induction, by using the associative and distributive properties of $\boxtimes$. Suppose that

$$[H_{a,b}] \cdot [H_{p,q}] = \sum_{(r,s)} N_{(a,b)(p,q)}^{(r,s)} [H_{r,s}]; \tag{6.2.1}$$

then left-multiplication by $[H_{1,2}]$ and $[H_{2,2}]$ respectively give, generically,

$$\left( [H_{a,b-1}] + [H_{a,b+1}] \right) \cdot [H_{p,q}] \tag{6.2.2}$$

and

$$\left( [H_{a-1,b-1}] + [H_{a-1,b+1}] + [H_{a+1,b-1}] + [H_{a+1,b+1}] \right) \cdot [H_{p,q}] \tag{6.2.3}$$

on the left-hand-side and, by (6.1.3), a sum of terms $[H_k]$ on the right-hand-side. It follows that the tensor product $H_{p',q'} \boxtimes H_{p,q}$ is completely reducible to a direct sum of discrete series representations, and hence (6.2.1) holds with $(a,b)$ replaced by $(p',q')$, for $(p',q') = (a, b \pm 1)$, $(a \pm 1, b \pm 1)$ and $(a \pm 1, b \mp 1)$. □

**6.3. Lemma.** *The fusion rules are given by*

$$[H_{p',q'} \boxtimes H_{p,q}] = \sum_{\substack{r = |p-p'|+1 \\ r+p+p' \text{ odd}}}^{\min(p+p'-1,\, 2m-p-p'-1)} \; \sum_{\substack{s = |q-q'|+1 \\ s+q+q' \text{ odd}}}^{\min(q+q'-1,\, 2(m+1)-q-q'-1)} [H_{r,s}]. \tag{6.3.1}$$

*Proof.* Observe that

$$[H_{1,2}] \cdot [H_{2,1} \boxtimes H_{p,q}] = [H_{2,2}] \cdot [H_{p,q}] = \sum_{\substack{1 \leq r = p \pm 1 \leq m-1 \\ 1 \leq s = q \pm 1 \leq m}} [H_{r,s}]. \tag{6.3.2}$$

From equation (6.1.3a), it follows that

$$[H_{2,1} \boxtimes H_{p,q}] = \sum_{1 \leq r = p \pm 1 \leq m-1} [H_{r,q}]. \tag{6.3.3}$$

This is obvious except when $(m, q) = (3,1)$, $(3,3)$ and $(5,3)$, but it is a simple exercise to check these cases by hand. The triplet of equations

$$\begin{aligned}[H_{1,1}] \cdot [H_{p,q}] &= [H_{p,q}] \\ [H_{1,2}] \cdot [H_{p,q}] &= \sum_{1 \leq s = q \pm 1 \leq m} [H_{p,s}] \\ [H_{2,1}] \cdot [H_{p,q}] &= \sum_{1 \leq r = p \pm 1 \leq m-1} [H_{r,q}]\end{aligned} \tag{6.3.4}$$

now allows us to prove the lemma by induction on $p' + q'$. Let (6.3.1) hold for all $(p', q')$ with $p' + q' \leq n$: by (6.3.4), it holds when $n \leq 3$. Generically, we have

$$[H_{p'+1,q'} \boxtimes H_{p,q}] = [H_{2,1}] \cdot [H_{p',q'} \boxtimes H_{p,q}] - [H_{p'-1,q'} \boxtimes H_{p,q}],$$
$$[H_{p',q'+1} \boxtimes H_{p,q}] = [H_{1,2}] \cdot [H_{p',q'} \boxtimes H_{p,q}] - [H_{p',q'-1} \boxtimes H_{p,q}]. \qquad (6.3.5)$$

Using (6.3.4) and the induction hypothesis, we can compute the right-hand-sides when $p' + q' = n$, so we can verify the lemma for $p' + q' = n + 1$. □

**6.4. Remark.** It follows from Lemma 6.3 that $N_{ij}^k = N_{ji}^k$, i.e. the Grothendieck ring of **Pos**$_c$ is commutative; and that $N_{ij}^0 = \delta_{ij}$. We were fortunate that we have been able to compute the fusion coefficients so readily, in so much that these properties now appear accidental. We shall see that the former is a consequence of *braiding*, and that the latter is related to *duality*, in the monoidal category **Pos**$_c$.

## 7. Ribbon and modular categories

We show that the monoidal category **Pos**$_c$ is a ribbon category, i.e. a monoidal category with a braiding, a twist and a compatible duality. Although we do not prove it here, the ribbon category **Pos**$_c$ is modular [Tu], and indeed all but one of the defining axioms of a modular category are manifestly satisfied.

**7.1. Ribbon categories.** Let $(\mathbf{C}, \boxtimes, H_0, \alpha, \mu, \nu)$ be a monoidal category, where

$$\begin{aligned} \alpha &= \alpha_{X,Y,Z} : X \boxtimes (Y \boxtimes Z) \to (X \boxtimes Y) \boxtimes Z; \\ \mu &= \mu_X : H_0 \boxtimes X \to X; \\ \nu &= \nu_X : X \boxtimes H_0 \to X, \end{aligned} \qquad (7.1.1)$$

are the natural isomorphisms, and $H_0$ is the unit for the tensor product $\boxtimes$. A *braiding* in a monoidal category **C** is a natural isomorphism

$$\beta = \beta_{X,Y} : X \boxtimes Y \to Y \boxtimes X \qquad (7.1.2)$$

that satisfies the cabling relations:

$$\beta_{X,Y \boxtimes Z} = \alpha_{Y,Z,X} (\mathrm{Id}_Y \boxtimes \beta_{X,Z})\, \alpha_{Y,X,Z}{}^{-1} (\beta_{X,Y} \boxtimes \mathrm{Id}_Z)\, \alpha_{X,Y,Z}; \qquad (7.1.3a)$$
$$\beta_{X \boxtimes Y,Z} = \alpha_{Z,X,Y}{}^{-1} (\beta_{X,Z} \boxtimes \mathrm{Id}_Y)\, \alpha_{X,Z,Y} (\mathrm{Id}_X \boxtimes \beta_{Y,Z})\, \alpha_{X,Y,Z}{}^{-1}. \qquad (7.1.3b)$$

Then $\mu_X\, \beta_{X,H_0} = \nu_X$; $\nu_X\, \beta_{H_0,X} = \mu_X$; and the braid relation

$$\alpha_{Z,Y,X} (\mathrm{Id}_Z \boxtimes \beta_{X,Y})\, \alpha_{Z,X,Y}{}^{-1} (\beta_{X,Z} \boxtimes \mathrm{Id}_Y)\, \alpha_{X,Z,Y} (\mathrm{Id}_X \boxtimes \beta_{Y,Z}) =$$
$$(\beta_{Y,Z} \boxtimes \mathrm{Id}_X)\, \alpha_{Y,Z,X} (\mathrm{Id}_Y \boxtimes \beta_{X,Z})\, \alpha_{Y,X,Z}{}^{-1} \cdot (\beta_{X,Y} \boxtimes \mathrm{Id}_Z)\, \alpha_{X,Y,Z} \qquad (7.1.4)$$

is satisfied. In this case, **C** is called a braided monoidal category. A *twist* in such a category is a natural isomorphism

$$\theta = \theta_X : X \to X \qquad (7.1.5)$$

that satisfies

$$\theta_{X \boxtimes Y} = \beta_{Y,X}\, \beta_{X,Y} (\theta_X \boxtimes \theta_Y). \qquad (7.1.6)$$

$\beta\,\dot{\beta} = \frac{\theta}{\theta\,\theta}$

It follows that $\theta_{H_0} = \mathrm{Id}_{H_0}$. A *duality* in a monoidal category is an assignment, to each (7.2.2), $\frac{\kappa\,\kappa}{\kappa} = \underline{\xi}^* \xi^*$ object $X$, of a dual object $\overline{X}$ and a pair of morphisms

$$\eta = \eta_X : H_0 \to X \boxtimes \overline{X}; \qquad (7.1.7a)$$
$$\varepsilon = \varepsilon_X : \overline{X} \boxtimes X \to H_0, \qquad (7.1.7b)$$

( $\dot{\beta} = \xi^*$ ) ✓

that satisfy

$$\nu_X (\mathrm{Id}_X \boxtimes \varepsilon_X)\, \alpha_{X,\overline{X},X}{}^{-1} (\eta_X \boxtimes \mathrm{Id}_X) = \mu_X; \qquad (7.1.8a)$$
$$\mu_{\overline{X}} (\varepsilon_X \boxtimes \mathrm{Id}_{\overline{X}})\, \alpha_{\overline{X},X,\overline{X}} (\mathrm{Id}_{\overline{X}} \boxtimes \eta_X) = \nu_{\overline{X}}. \qquad (7.1.8b)$$

For each morphism $f : X \to Y$, we have the dual morphism

$$\overline{f} : \overline{Y} \xrightarrow{\mathrm{Id} \boxtimes \eta} \overline{Y} \boxtimes X \boxtimes \overline{X} \xrightarrow{\mathrm{Id} \boxtimes f \boxtimes \mathrm{Id}} \overline{Y} \boxtimes Y \boxtimes \overline{X} \xrightarrow{\varepsilon \boxtimes \mathrm{Id}} \overline{X}, \qquad (7.1.9)$$

where we have omitted the standard isomorphisms. A *ribbon category* [Tu], or *balanced rigid braided monoidal category* [JS], is a monoidal category with a braiding, a twist and a compatible duality, i.e.

$$(\theta_X \boxtimes \mathrm{Id}_{\overline{X}})\, \eta_X = (\mathrm{Id}_X \boxtimes \theta_{\overline{X}})\, \eta_X. \qquad (7.1.10)$$

This is equivalent to the condition: $\theta_{\overline{X}} = \overline{\theta_X}$.

**7.2. Braiding and twist in Pos$_c$.** For the remainder of this chapter, let $I \subset S^1$ be the upper half-circle. Let $H_i$ and $H_j$ be $(M_I, M_I)$-bimodules in **Pos**$_c$. Wassermann [Wa2] has proposed the braiding map

$y\Omega$

$$\beta_{ij} : H_i \boxtimes H_j \to H_j \boxtimes H_i, \quad [a_{i0} \otimes b_{j0}] \mapsto r_\pi^* \cdot [r_\pi b_{j0} r_\pi^* \otimes r_\pi a_{i0} r_\pi^*] \qquad (7.2.1)$$

( $\dot{\beta} = \xi^*$ )

ou $\mathcal{X}_i \otimes \mathcal{Y}_j$ only ?

([illegible] fields as ... VI.2 ?) $\infty \cdot \frac{I = I_+}{I = I_-}$

Note: $r_\pi \mathcal{X}_i r_\pi^* = \mathcal{Y}_i$ , $r_\pi \mathcal{Y}_j r_\pi^* = \mathcal{X}_j$

where $r_\pi = e^{\pi i L_0}$. This is clearly a unitary equivalence of bimodules. It implies that the Grothendieck ring of **Pos**$_c$ is commutative. We also define the twisting map

$$\theta_i : H_i \rightarrow H_i, \quad \xi \mapsto e^{-2\pi i L_0}\xi. \tag{7.2.2}$$

This is a unitary equivalence, since $H_i$ is semi-simple and $\theta_i$ is just multiplication by the factor $e^{-2\pi i h}$ on an irreducible summand $H \subset H_i$ when it is isomorphic to the discrete series representation $H_{h,c}$ with highest weight $(h, c)$. It is trivial to check that $\beta$ and $\theta$ are natural isomorphisms; and that the relation (7.1.6) is satisfied. It remains only to check the cabling properties of $\beta$.

**Proposition 7.2.1.** *The natural isomorphism $\beta$ is a braiding in* **Pos**$_c$.

*Proof.* Since the objects of **Pos**$_c$ are just finite direct sums of the simple objects; the tensor product $\boxtimes$ is bi-additive; and the relevant maps are natural, it suffices to check the cabling relations on the discrete series representations. We do this by checking the cabling relations on larger bimodules that contain the discrete series representations as sub-bimodules, viz. on the bimodules

$$H_{1,2}^{\boxtimes m} \boxtimes H_{2,2}^{\boxtimes n} \tag{7.2.1.1}$$

consisting of $(m, n)$-fold tensor products of the generating bimodules; from the fusion rules in Lemma 6.2, we know that every discrete series representation occurs as a sub-bimodule of some tensor product of this form. Since the method is identical in each case, we only prove the first of the two cabling relations (7.1.3a), i.e. we prove that the diagram

$$\begin{tikzcd}
H_i \boxtimes (H_j \boxtimes H_k) \arrow[r, "\beta"] \arrow[d, "\alpha"] & (H_j \boxtimes H_k) \boxtimes H_i \\
(H_i \boxtimes H_j) \boxtimes H_k \arrow[d, "\beta\boxtimes\mathrm{Id}"] & H_j \boxtimes (H_k \boxtimes H_i) \arrow[u, "\alpha"] \\
(H_j \boxtimes H_i) \boxtimes H_k \arrow[r, "\alpha^{-1}"] & H_j \boxtimes (H_i \boxtimes H_k) \arrow[u, "\mathrm{Id}\boxtimes\beta"]
\end{tikzcd} \tag{7.2.1.2}$$

commutes, when the bimodules $H_i$, $H_j$ and $H_k$ have the form (7.2.1.1). To begin with, let them be just the generating representations, $H_{1,2}$ or $H_{2,2}$. The proof of the general case will be a straightforward generalisation.

Let $[a_{i0} \otimes [b_{j0}\Omega \otimes c_{k0}]]$ be an element of $[\mathfrak{X}_i \otimes [\mathfrak{Y}_j\Omega \otimes \mathfrak{Y}_k]]$. Moreover, we choose $b_{j0} \in \mathfrak{Y}_j$ to have support only in the sub-interval of $I$ corresponding to $\frac{1}{2}\pi < \theta < \pi$; and $c_{k0} \in \mathfrak{Y}_k$ to have support only in the sub-interval corresponding to $0 < \theta < \frac{1}{2}\pi$. The linear span of these vectors is clearly dense in $H_i \boxtimes (H_j \boxtimes H_k)$. Under the composite map $(\beta \boxtimes \mathrm{Id})\,\alpha$, this element is mapped to

$$[r_\pi^* \cdot [r_\pi b_{j0} r_\pi^* \otimes r_\pi a_{i0} r_\pi^*] \otimes c_{k0}] \in (H_j \boxtimes H_i) \boxtimes H_k \tag{7.2.1.3}$$

Similarly, let $[[\beta_{j0} \otimes \alpha_{i0}] \otimes \gamma_{k0}]$ be an element of the dense subspace $[[\mathfrak{X}_j \otimes \mathfrak{Y}_i] \otimes \mathfrak{Y}_k] \subset (H_j \boxtimes H_i) \boxtimes H_k$. Under the composite map $(\mathrm{Id} \boxtimes \beta)\,\alpha^{-1}$, it is mapped to

$$[\beta_{j0} \otimes r_\pi^* \cdot [r_\pi \gamma_{k0} r_\pi^* \otimes r_\pi \alpha_{i0}\Omega]] \in H_j \boxtimes (H_k \boxtimes H_i). \tag{7.2.1.4}$$

Question: where is (7.2.1.3) mapped to by $(\mathrm{Id} \boxtimes \beta)\,\alpha^{-1}$?

Define the following unitary equivalences of bimodules. Let

$$T : (H_j \boxtimes H_i) \boxtimes H_k \rightarrow \bigoplus_{m\in\langle i,j\rangle} H_m \boxtimes H_k \rightarrow \bigoplus_{m\in\langle i,j\rangle} \bigoplus_{n\in\langle m,k\rangle} H_n \tag{7.2.1.5a}$$

be given by, cf. (6.1.3),

$$\begin{aligned} T : [[\beta_{j0} \otimes \alpha_{i0}] \otimes \gamma_{k0}] &\mapsto \sum_m C_{ji}^m\, [\beta_{mi}\, \alpha_{i0}\Omega \otimes \gamma_{k0}] \\ &\mapsto \sum_m C_{ji}^m \sum_n \mathcal{D}_{mk}^n\, \gamma_{nm}\, \beta_{mi}\, \alpha_{i0}\Omega. \end{aligned} \tag{7.2.1.5b}$$

Also let

$$U : H_j \boxtimes (H_k \boxtimes H_i) \rightarrow \bigoplus_{p\in\langle k,i\rangle} H_j \boxtimes H_p \rightarrow \bigoplus_{p\in\langle k,i\rangle} \bigoplus_{q\in\langle j,p\rangle} H_q \tag{7.2.1.6a}$$

be given on $[B_{j0} \otimes [C_{k0} \otimes A_{i0}\Omega]] \in [\mathfrak{X}_j \otimes [\mathfrak{X}_k \otimes \mathfrak{X}_i\Omega]]$ by

$$\begin{aligned} U : [B_{j0} \otimes [C_{k0} \otimes A_{i0}\Omega]] &\mapsto \sum_p C_{ki}^p\, [B_{j0} \otimes C_{pi}\, A_{i0}\Omega] \\ &\mapsto \sum_p C_{ki}^p \sum_q C_{jp}^q\, B_{qp}\, C_{pi}\, A_{i0}\Omega. \end{aligned} \tag{7.2.1.6b}$$

Trivially, we have the commutative diagram

$$\begin{tikzcd}
(H_j \boxtimes H_i) \boxtimes H_k \arrow[r, "(\mathrm{Id}\boxtimes\beta)\,\alpha^{-1}"] \arrow[d, "T"'] & H_j \boxtimes (H_k \boxtimes H_i) \arrow[d, "U"] \\
\bigoplus_{m\in\langle i,j\rangle} \bigoplus_{n\in\langle m,k\rangle} H_n \arrow[r, "U\,(\mathrm{Id}\boxtimes\beta)\,\alpha^{-1}\,T^*"] & \bigoplus_{p\in\langle k,i\rangle} \bigoplus_{q\in\langle j,p\rangle} H_q .
\end{tikzcd} \tag{7.2.1.7}$$

The vector

$$T\,[[\beta_{j0}\otimes\alpha_{i0}]\otimes\gamma_{k0}] = \sum_m C^m_{ji}\sum_n \mathcal{D}^n_{mk}\,\gamma_{nm}\,\beta_{mi}\,\alpha_{i0}\Omega \tag{7.2.1.8}$$

is mapped by $U\,(\mathrm{Id}\boxtimes\beta)\,\alpha^{-1}\,T^*$ to

$$\sum_p C^p_{ki}\sum_q C^q_{jp}\,\beta_{qp}\,\gamma_{pi}\,\alpha_{i0}\Omega. \tag{7.2.1.9}$$

To see this, it suffices to note that: (i) Bimodule maps intertwine the operators $e^{itL_0}$; and (ii) Since $\gamma_{k0}\in\mathfrak{Y}_k$ and $r_\pi\,\gamma_{k0}\,r_\pi^*\in\mathfrak{X}_k$, we have the following assignment of non-principal intertwiners:

$$r_\pi\,\gamma_{k0}\,r_\pi^* \;\longmapsto\; \{r_\pi\,\gamma_{rs}\,r_\pi^*\}. \tag{7.2.1.10}$$

In contrast, since $\beta_{j0}\in\mathfrak{X}_i$ and $r_\pi\,\beta_{j0}\,r_\pi^*\in\mathfrak{Y}_j$, we have (but not yet used) instead

$$r_\pi\,\beta_{j0}\,r_\pi^* \;\longmapsto\; \{e^{2\pi i(h_j+h_s-h_r)}\,r_\pi\,\beta_{rs}\,r_\pi^*\}. \tag{7.2.1.11}$$

These assertions follow from our smearing conventions; i.e. from (IV.2.1.4). The construction of intertwiners $\{a_{ij}\}$ for $M_I$ from localised fields $\{\tilde{\phi}_{ji}(f)\}$, $f\in C^\infty_{I'}(S^1)$; and that of intertwiners $\{b_{ij}\}$ for $M_{I'}$ from localised fields $\{\tilde{\phi}_{ji}(f\circ r_\pi)\}$, preserves these relations. By similar considerations, we see that $T$ maps (7.2.1.3) to

$$\sum_m C^m_{ji}\sum_n \mathcal{D}^n_{mk}\,c_{nm}\,b_{mi}\,a_{i0}\Omega. \tag{7.2.1.12}$$

Claim: this is mapped by $U\,(\mathrm{Id}\boxtimes\beta)\,\alpha^{-1}\,T^*$ to

$$\sum_p C^p_{ki}\sum_q C^q_{jp}\,b_{qp}\,c_{pi}\,a_{i0}\Omega. \tag{7.2.1.13}$$

This is certainly the naive result obtained by replacing the intertwiners $\alpha$, $\beta$, $\gamma$ in (7.2.1.8) and (7.2.1.9) by $a$, $b$, $c$ respectively. However, the intertwiners $a$, $\alpha$ (resp. $b$, $\beta$; and $c$, $\gamma$) have support in different intervals, so we have to justify the claim.

We proceed in the following way. Consider the graph of the map $U\,(\mathrm{Id}\boxtimes\beta)\,\alpha^{-1}\,T^*$, and the dense subspaces corresponding to different supports for the intertwiners $\alpha$, $\beta$ and $\gamma$ in (7.2.1.8) and (7.2.1.9). Using continuity arguments, modify these supports until they coincide with those of the intertwiners $a$, $b$ and $c$; then observe that the result is the required one. We perform the following operations on the pair of expressions (7.2.1.8) and (7.2.1.9), which lie in the graph of $U\,(\mathrm{Id}\boxtimes\beta)\,\alpha^{-1}\,T^*$ (draw some pictures!): (i) We can certainly restrict the respective supports $J_\alpha$, $J_\beta$, $J_\gamma$ of the intertwiners $\alpha$, $\beta$, $\gamma$. We choose $J_\gamma$ to correspond to $0<\theta<\frac{1}{2}\pi$; and $J_\beta$ to correspond to $\pi<\theta<\frac{3}{2}\pi$. (ii) Since the intertwiner $\alpha$ occurs as $\alpha_{i0}\Omega$ in both expressions, we can use continuity to replace this vector by an arbitrary one in $H_i$, and therefore change the support of $\alpha_{i0}$ arbitrarily. We choose $J_\alpha$ to correspond to $\frac{3}{2}\pi<\theta<2\pi$. The intertwiners $\alpha$, $\beta$ and $\gamma$ now have disjoint supports. (iii) In (7.2.1.8) we braid $\beta$ with $\alpha$; and in (7.2.1.9) we braid $\beta$ first with $\gamma$ and then $\alpha$, so that $\beta$ acts on the vacuum vector in each case. Now change its support to correspond to $\frac{1}{2}\pi<\theta<\pi$; and then undo the braidings. The braiding coefficients do *not* change even though the support of $\beta$ has been altered, because the *relative* positions of the supports remain the same. (iv) Finally, extend the support of $\alpha$ to all of the lower half-circle. This proves the claim.

Define also the unitary equivalence

$$V:(H_j\boxtimes H_k)\boxtimes H_i \to \bigoplus_{r\in(j,k)} H_r\boxtimes H_i \to \bigoplus_{r\in(j,k)}\bigoplus_{s\in(r,i)} H_s, \tag{7.2.1.14a}$$

given on $[[B_{j0}\otimes C_{k0}\Omega]\otimes D_{i0}]\in[[\mathfrak{X}_j\otimes\mathfrak{X}_k\Omega]\otimes\mathfrak{Y}_i]$ by

$$\begin{aligned} V:[[B_{j0}\otimes C_{k0}\Omega]\otimes D_{i0}] &\longmapsto \sum_r C^r_{jk}\,[B_{rk}\,C_{k0}\Omega\otimes D_{i0}]\\ &\longmapsto \sum_r C^r_{jk}\sum_s \mathcal{D}^s_{ri}\,D_{sr}\,B_{rk}\,C_{k0}\Omega. \end{aligned} \tag{7.2.1.14b}$$

So we have the commutative diagram

$$\begin{tikzcd} H_j\boxtimes(H_k\boxtimes H_i) \arrow[r, "\alpha"] \arrow[d, "U"'] & (H_j\boxtimes H_k)\boxtimes H_i \arrow[d, "V"] \\ \displaystyle\bigoplus_{p\in(k,i)}\bigoplus_{q\in(j,p)} H_q \arrow[r, "V\alpha U^*"] & \displaystyle\bigoplus_{r\in(j,k)}\bigoplus_{s\in(r,i)} H_s. \end{tikzcd} \tag{7.2.1.15}$$

The vector

$$U\,[B_{j0}\otimes[C_{k0}\otimes D_{i0}]] = \sum_p C^p_{ki}\sum_q C^q_{jp}\,B_{qp}\,C_{pi}\,D_{i0}\Omega \tag{7.2.1.16}$$

is mapped by the composite $V\alpha U^*$ to

$$\sum_r C^r_{jk}\sum_s \mathcal{D}^s_{ri}\,D_{sr}\,B_{rk}\,C_{k0}\Omega. \tag{7.2.1.17}$$

Claim: the composite $V \circ U^*$ maps (7.2.1.13) to

$$\sum_r C^r_{jk} \sum_s \mathcal{D}^s_{ri}\, e^{2\pi i\{h_i+h_r-h_s\}}\, a_{sr}\, b_{rk}\, c_{k0}\Omega. \tag{7.2.1.18}$$

The proof is along the same lines. We consider (7.2.1.16) and (7.2.1.17) as an ordered pair in the graph of $V \circ U^*$, and use continuity arguments to change the support of the intertwiners $D$, $B$, $C$: (i) Let the support of $D$ be restricted to $0 < \theta < \frac{1}{2}\pi$; that of $B$ to $\pi < \theta < \frac{3}{2}\pi$; and that of $C$ to $\frac{3}{2}\pi < \theta < 2\pi$. (ii) By braiding $B$ with $C$, and then $D$, in (7.2.1.16); and with $C$ in (7.2.1.17); then changing the support of $B$ when it acts on $\Omega$, and finally unbraiding, we can change the support of $B$ so as to correspond to $\frac{1}{2}\pi < \theta < \pi$. (iii) By first braiding and then unbraiding $C$ with $D$ in (7.2.1.16); and changing the support of $C$ on each side when it is acting on the vacuum vector, we can change its support so as to also correspond to $\frac{1}{2}\pi < \theta < \pi$. (iv) Now change the support of $D$ to $\pi < \theta < 2\pi$. This involves braiding $D$ with $B$ and $C$ in (7.2.1.17), changing its support, and then unbraiding. Now, because the relative position of the support of $D$, with respect to those of $B$ and $C$, is changed, a phase factor

$$e^{-2\pi i\{h_s+h_k-h_r-h'\}}\, e^{-2\pi i\{h'+0-h_k-h_i\}} = e^{2\pi i\{h_i+h_r-h_s\}} \tag{7.2.1.19}$$

is introduced, cf. the remarks following the proof of Proposition IV.2.2. (v) Finally, correct the support of $C$ to $0 < \theta < \frac{1}{2}\pi$. This establishes the claim.

However, the image of $[\, a_{i0} \otimes [b_{j0}\Omega \otimes c_{k0}]\,]$ under the braiding map $\beta$ is

$$r_\pi^* \cdot \{ r_\pi \cdot [b_{j0}\Omega \otimes c_{k0}] \otimes r_\pi a_{i0} r_\pi^* \}, \tag{7.2.1.20}$$

which is mapped by $V$, through the intermediary

$$r_\pi^* \cdot \left\{ \sum_r C^r_{jk} \, [r_\pi b_{rk} c_{k0}\Omega \otimes r_\pi a_{i0} r_\pi^*] \right\}, \tag{7.2.1.21}$$

to (7.2.1.18). Here we note that we have the assignment

$$r_\pi a_{i0} r_\pi^* \mapsto \{ e^{2\pi i\{h_i+h_r-h_s\}} r_\pi a_{sr} r_\pi^* \}. \tag{7.2.1.22}$$

Hence the diagram (7.2.1.2) commutes when $H_i$, $H_j$ and $H_k$ are the generating representations $H_{1,2}$ and $H_{2,2}$. The proof in the case when they are tensor products of the generating representations follows from the results of following section. □

**7.3. More explicit computations of Connes fusion.** We compute explicitly the arbitrary tensor products of the generating representations, viz. $H_{1,2}$ and $H_{2,2}$. That is, we show how to produce explicit unitary bimodule maps from the tensor product space into a direct sum of irreducibles. This often compensates for not being [illegible] (at present) to rigorously justify the explicit computation of Connes fusion of arbitrary discrete series representations, cf. § 6.1. However, the presentation shall be with a view to completing the proof of Proposition 7.2.1. Consider the $M$-fold tensor product of generating representations,

[illegible]

$$H_i = H_{i_1} \boxtimes (\cdots(H_{i_{M-1}} \boxtimes H_{i_M})\cdots), \tag{7.3.1}$$

where $i = (i_1, \ldots, i_M)$. The linear span of the vectors

$$[a^{(1)}_{i_1 0} \otimes [\cdots [a^{(M-1)}_{i_{M-1}0} \otimes a^{(M)}_{i_M 0}\Omega]\cdots]] \in [\mathfrak{X}_{i_1} \otimes [\cdots [\mathfrak{X}_{i_{M-1}} \otimes \mathfrak{X}_{i_M}\Omega]\cdots]] \tag{7.3.2}$$

is dense in $H_i$, even when we arbitrarily restrict the support of each intertwiner $a^{(m)}$ to a sub-interval of $\{\theta : \pi < \theta < 2\pi\}$. In particular, we shall take $n$ *disjoint* sub-intervals, $J_1, \cdots, J_M$, and let $a^{(m)}$ be supported in $J_m$. Using the unitary equivalences of bimodules

$$H_{i_m} \boxtimes H_r \to \bigoplus_{s\in(i_m, r)} H_s, \quad [a_{i_m 0} \otimes \xi] \mapsto \sum_s C^s_{i_m r}\, a_{sr}\xi, \tag{7.3.3}$$

where $H_r$ is an arbitrary discrete series representation, we obtain the unitary equivalence

$$H_i \to \bigoplus_{s_{M-1}\in(i_{M-1}, i_M)} \cdots \bigoplus_{s_1 \in (i_1, s_2)} H_{s_1} \tag{7.3.4}$$

which maps the vector (7.3.2) to

$$\sum_s C^s_i\, a_s\Omega = \sum_{s_{M-1}} C^{s_{M-1}}_{i_{M-1}i_M} \cdots \sum_{s_1} C^{s_1}_{i_1 s_2}\, a^{(1)}_{s_1 s_2} \cdots a^{(M-1)}_{s_{M-1}i_M}\, a^{(M)}_{i_M 0}\Omega. \tag{7.3.5}$$

In particular, this has the form $a_{i0}\Omega$, with $a_{i0} \in \mathfrak{X}_i$, the bounded intertwiners for $M_I$ mapping $H_0$ to $H_i$. Similarly, let

$$H_j = (\cdots(H_{j_N} \boxtimes H_{j_{N-1}})\cdots) \boxtimes H_{j_1}; \tag{7.3.6a}$$

$$H_k = (\cdots(H_{k_P} \boxtimes H_{k_{P-1}})\cdots) \boxtimes H_{k_1} \tag{7.3.6b}$$

be tensor products of generating representations and consider the dense subspaces spanned respectively by the vectors

$$[[\cdots[b^{(N)}_{j_N 0}\Omega \otimes b^{(N-1)}_{j_{N-1}0}]\cdots] \otimes b^{(1)}_{j_1 0}] \in [[\cdots[\mathfrak{Y}_{j_N}\Omega \otimes \mathfrak{Y}_{j_{N-1}}]\cdots] \otimes \mathfrak{Y}_{j_1}]; \tag{7.3.7a}$$

$$[[\cdots[c^{(P)}_{k_P 0}\Omega \otimes c^{(P-1)}_{k_{P-1}0}]\cdots] \otimes c^{(1)}_{k_1 0}] \in [[\cdots[\mathfrak{Y}_{k_P}\Omega \otimes \mathfrak{Y}_{k_{P-1}}]\cdots] \otimes \mathfrak{Y}_{k_1}], \tag{7.3.7b}$$

where the intertwiners $b^{(m)}$ (resp. $c^{(m)}$) have mutually disjoint supports. We likewise have unitary equivalences:

$$H_j \to \bigoplus_{t_{N-1}\in(j_{N-1},j_N)} \cdots \bigoplus_{t_1\in(j_1,t_2)} H_{t_1} \qquad (7.3.8a)$$

$$H_k \to \bigoplus_{u_{N-1}\in(k_{P-1},k_P)} \cdots \bigoplus_{u_1\in(k_1,u_2)} H_{u_1} \qquad (7.3.8b)$$

which respectively map (7.3.7a) and (7.3.7b) to

$$\begin{aligned} b_{j0}\Omega &= \sum_t \mathcal{D}^t_j\, b_t\Omega \\ &= \sum_{t_{N-1}} \mathcal{D}^{t_{N-1}}_{j_{N-1}j_N} \cdots \sum_{t_1} \mathcal{D}^{t_1}_{j_1t_2}\, b^{(1)}_{t_1t_2} \cdots b^{(N-1)}_{t_{N-1}j_N}\, b^{(N)}_{j_N0}\Omega; \qquad (7.3.9a) \\ c_{k0}\Omega &= \sum_u \mathcal{D}^u_k\, c_u\Omega \\ &= \sum_{u_{N-1}} \mathcal{D}^{u_{P-1}}_{k_{P-1}k_P} \cdots \sum_{u_1} \mathcal{D}^{u_1}_{k_1u_2}\, c^{(1)}_{u_1u_2} \cdots c^{(P-1)}_{u_{P-1}k_P}\, c^{(P)}_{k_P0}\Omega. \qquad (7.3.9b) \end{aligned}$$

Here $b_{j0} \in \mathfrak{D}_j$ and $c_{k0} \in \mathfrak{D}_k$; by choosing the supports of the $b^{(m)}$'s and $c^{(m)}$'s accordingly, we can clearly also shrink the supports of $b_{j0}$ and $c_{k0}$ arbitrarily. By definition, the vector $[a_{i0} \otimes b_{j0}]$ has norm

$$\begin{aligned} \langle a_{i0}^*a_{i0}\, b_{j0}^*b_{j0}\Omega,\, \Omega\rangle &= \langle \vartheta_j(a_{i0}^*a_{i0})\, b_{j0}\Omega,\, b_{j0}\Omega\rangle \\ &= \langle \vartheta_i(b_{j0}^*b_{j0})\, a_{i0}\Omega,\, a_{i0}\Omega\rangle. \end{aligned} \qquad (7.3.10)$$

This has the form

$$\sum_{s,t} C^s_i \mathcal{D}^t_j \sum_{s',t'} C^{s'}_i \mathcal{D}^{t'}_j \,\langle \vartheta_{t_1}(a_{s'}^*a_s)\, b_t\Omega,\, b_{t'}\Omega\rangle \qquad (7.3.11)$$

Let $H_m$ be a discrete series representation, i.e. an irreducible bimodule; and let $U_{m0} \in \mathfrak{D}_m$. Then

$$\vartheta_m(a_{s'}^*a_s) = U_{m0}\,(a^{(1)}_{s'_1s'_2} \cdots a^{(M-1)}_{s'_{M-1}i_M}\, a^{(M)}_{i_M0})^*\,(a^{(1)}_{s_1s_2} \cdots a^{(M-1)}_{s_{M-1}i_M}\, a^{(M)}_{i_M0})\, U_{m0}^*. \qquad (7.3.12)$$

This can be evaluated when the $a^{(m)}$ are (bounded) localised fields, exactly as in § 3.3, where we had $M = 1$; and therefore also when the $a^{(m)}$ are arbitrary intertwiners for $M_I$, using the methods of § 3.5 which construct intertwiners from localised fields.

We sketch the $M = 2$ case. It is straightforward to fill in the details. We have:

$$U_{m0}\,(a^{(1)}_{s_1i_2}a^{(2)}_{i_20})^*\,(a^{(1)}_{s_1i_2}a^{(2)}_{i_20})\,U_{m0}^* = \sum_n \sum_p \sum_q \overline{\mathcal{S}^m_{nq}}\,\mathcal{R}^m_{np}\,(a^{(1)}_{nq}a^{(2)}_{qm})^*\,(a^{(1)}_{np}a^{(2)}_{pm}), \qquad (7.3.13)$$

where

$$\begin{aligned} \mathcal{R}^m_{np} &= C^{s_1i_2m\,p}_{n\,i_2}\, C^{i_2i_2m\,m}_{p0}; \\ \overline{\mathcal{S}^m_{nq}} &= C^{0\,i_2m\,q}_{0\,i_2}\, C^{i_2i_2m\,n}_{q\,s_1} \end{aligned} \qquad (7.3.14)$$

are the braiding coefficients. By inserting von Neumann algebra elements, we have

$$\sum_{n,t_1} \sum_p \mathcal{R}^{t_1}_{np} \sum_q \overline{\mathcal{S}^{t_1}_{nq}}\,\langle x\,a^{(1)}_{np}\,y\,a^{(2)}_{pt_1}\,z\,b_{j0}\Omega,\; x\,a^{(1)}_{nq}\,y\,a^{(2)}_{qt_1}\,z\,b_{j0}\Omega\rangle \ge 0 \qquad (7.3.15)$$

for all $x$, $y$, $z$ in the algebra generated by $M_I$ and $M_{I^c}$; or at least by the respective dense sub-algebras corresponding to dividing each of the intervals $I$, $I^c$ into a disjoint union of sub-intervals. Arguing as in Lemma 3.4.1, this inequality also holds for fixed $n$, $t_1$; and when we restrict the sums such that $p$, $q$ range over the same (but arbitrary) subset of the original values. In particular, this implies that

$$\mathcal{S}^{t_1}_{np} = \mathcal{T}^2_n\,\mathcal{R}^{t_1}_{np} \qquad (7.3.16)$$

for some positive constant $\mathcal{T}^2_n$ depending only on $n$, and for all $p$, $t_1$. It follows that

$$\|[a_{i0} \otimes b_{j0}]\|^2 = \|\xi\|^2, \qquad (7.3.17)$$

where

$$\begin{aligned} \xi &= \sum_{n,t_1,p} \mathcal{T}_n\,\mathcal{R}^{t_1}_{np}\, a^{(1)}_{np}a^{(2)}_{pt_1}\, b_{j0}\Omega \\ &= \sum_{n,p} \sum_t \mathcal{T}_n\,\mathcal{R}^{t_1}_{np}\,\mathcal{D}^t_j\, a^{(1)}_{np}a^{(2)}_{pt_1}\, b_t\Omega; \end{aligned} \qquad (7.3.18)$$

and

$$[a_{i0} \otimes b_{j0}] \mapsto \sum_{n,p} \sum_t \mathcal{T}_n\,\mathcal{R}^{t_1}_{np}\,\mathcal{D}^t_j\, a^{(1)}_{np}a^{(2)}_{pt_1}\, b_t\Omega \qquad (7.3.19)$$

is the required unitary equivalence. The arguments generalise to arbitrary $M$.

In summary, there is a unitary equivalence from $H_i \boxtimes H_j$ to a direct sum of irreducibles, which has the form

$$\begin{aligned} [a_{i0} \otimes b_{j0}] &\mapsto \sum_p \mathcal{K}_p\, a^{(1)}_{p_1p_2} \cdots a^{(M-1)}_{p_{M-1}p_M}\, a^{(M)}_{p_Mp_{M+1}}\, b_{j0}\Omega \\ &= \sum_q \mathcal{L}_q\, b^{(1)}_{q_1q_2} \cdots b^{(N-1)}_{q_{N-1}q_N}\, b^{(N)}_{q_Nq_{N+1}}\, a_{i0}\Omega. \end{aligned} \qquad (7.3.20)$$

This generalises the special case $M = N = 1$ obtained before. But now the same must be true for $[a_{i0} \otimes b_{j0} \otimes c_{k0}] \in H_i \boxtimes H_j \boxtimes H_k$ (and so on), wherever we choose to place the brackets, provided that the elementary intertwiners have disjoint supports. The proof of Proposition 7.2.1 is equally valid for tensor products of the generating representations.

**7.4. Duality in $\mathbf{Pos}_c$.** Let $X$ be an irreducible bimodule in $\mathbf{Pos}_c$. An irreducible bimodule $Y$, for which the tensor product $X \boxtimes Y$ contains a summand isomorphic to the vacuum sector $H_0$, will be called a conjugate of $X$. From the fusion rules, it is clear that $Y$ always exists, is unique up to isomorphism, and $X \boxtimes Y$ contains the vacuum sector precisely once; moreover, $Y$ is isomorphic to $X$.

In fact, given the existence of conjugates, the following argument of Wassermann's shows that uniqueness follows, and $X \boxtimes Y$ necessarily contains only a single copy of $H_0$. The proof proceeds by showing that

$$\mathrm{Hom}(H_1 \boxtimes X, H_2) \cong \mathrm{Hom}(H_1, H_2 \boxtimes Y) \tag{7.4.1}$$

as linear spaces, for all bimodules $H_1, H_2$. Setting $H_1 = H_0$ and $H_2 = X$ shows that $\mathrm{Hom}(H_0, X \boxtimes Y) = \mathbb{C}$, by irreduciblity of $X$. If $Z$ is another conjugate of $X$, then set $H_1 = Z$ and $H_2 = H_0$ to deduce that $\mathrm{Hom}(Z, Y) = \mathbb{C}$, by the previous result. Let

$$\begin{aligned} \eta &: \; H_0 \to X \boxtimes Y; \\ \varepsilon &: \; Y \boxtimes X \to H_0 \end{aligned} \tag{7.4.2}$$

be non-zero bimodule maps; they exist because $Y$ is a conjugate of $X$. For each pair of bimodules $H_1, H_2$, define linear maps

$$\begin{aligned} f_{H_1,H_2} &: \; \mathrm{Hom}(H_1 \boxtimes X, H_2) \to \mathrm{Hom}(H_1, H_2 \boxtimes Y); \\ g_{H_1,H_2} &: \; \mathrm{Hom}(H_1, H_2 \boxtimes Y) \to \mathrm{Hom}(H_1 \boxtimes X, H_2) \end{aligned} \tag{7.4.3}$$

by

$$\begin{aligned} f_{H_1,H_2}(T) &= (T \boxtimes \mathrm{Id}_Y)\, \alpha_{H_1,X,Y}\, (\mathrm{Id}_{H_1} \boxtimes \eta)\, \nu_{H_1}{}^{-1}; \\ g_{H_1,H_2}(S) &= \nu_{H_2}\, (\mathrm{Id}_{H_2} \boxtimes \varepsilon)\, \alpha_{H_2,Y,X}{}^{-1}\, (S \boxtimes \mathrm{Id}_X). \end{aligned} \tag{7.4.4}$$

It is easy to show that

$$g_{H_1,H_2} f_{H_1,H_2}(T) = T\,(\mathrm{Id}_{H_1} \boxtimes \gamma); \tag{7.4.5a}$$

$$f_{H_1,H_2} g_{H_1,H_2}(S) = (\mathrm{Id}_{H_2} \boxtimes \sigma)\, S, \tag{7.4.5b}$$

where

$$\gamma = \nu_X\, (\mathrm{Id}_X \boxtimes \varepsilon)\, \alpha_{X,Y,X}{}^{-1}\, (\eta \boxtimes \mathrm{Id}_X)\, \mu_X{}^{-1} \in \mathrm{End}(X); \tag{7.4.6a}$$

$$\sigma = \mu_X\, (\varepsilon \boxtimes \mathrm{Id}_Y)\, \alpha_{Y,X,Y}\, (\mathrm{Id}_Y \boxtimes \eta)\, \nu_Y{}^{-1} \in \mathrm{End}(Y). \tag{7.4.6b}$$

Since the bimodules $X, Y$ are irreducible, $\gamma = k\,\mathrm{Id}_X$ and $\sigma = k'\,\mathrm{Id}_Y$ for some scalars $k$, $k'$ not depending on $H_1, H_2$. Then $g_{H_1,H_2} f_{H_1,H_2}(T) = kT$, $f_{H_1,H_2} g_{H_1,H_2}(S) = k'T$ and we must have $k = k'$. To prove (7.4.1), it suffices to show that $k \neq 0$, whence we can choose the maps $\eta$, $\varepsilon$ such that $k = k' = 1$.

We show that $(\mathrm{Id}_X \boxtimes \varepsilon)\, \alpha_{X,Y,X}{}^{-1}\, (\eta \boxtimes \mathrm{Id}_X) \neq 0$. By changing the maps $\varepsilon$, $\eta$ by suitable non-zero scalars, we may assume that $\varepsilon^* \varepsilon$ and $\eta\eta^*$ are projections, onto submodules of $Y \boxtimes X$ and $X \boxtimes Y$ respectively. Let $e_1 = \eta\eta^* \boxtimes \mathrm{Id}_X$ and $e_2 = \mathrm{Id}_X \boxtimes \varepsilon^* \varepsilon$. In the following, we suppress the isomorphism $\alpha$. Let

$$\begin{aligned} \mathrm{End}_{M_I}(X \boxtimes Y \boxtimes X) &= M \\ &\cup \\ \mathrm{End}_{M_I}(Y \boxtimes X) &= N \\ &\cup \\ \mathrm{End}_{M_I}(X) &= P. \end{aligned} \tag{7.4.7}$$

Then $M \cap N' = \mathbb{C}$ and $N \cap P' = \mathbb{C}$. The latter, and also the former by the same argument, follows by regarding $X$ as an $(M_I, \mathrm{End}_{M_I}(X)^{op})$-bimodule, which can be identified with $H_0$, and $\mathrm{End}_{M_I}(X)$ with $M_I^{op}$; then use the irreducibility of $Y$. Now

$$\begin{aligned} e_1 M e_1 &= \mathrm{End}_{M_I}(\, e_1(X \boxtimes Y) \boxtimes X\,) \\ &= \mathrm{End}_{M_I}(H_0 \boxtimes X) \\ &= P e_1, \end{aligned} \tag{7.4.8}$$

so that $e_1 e_2 e_1 = x\, e_1$ for some $x \in P$. However, $e_2 \in \mathrm{End}_{M_I}(X \boxtimes Y \boxtimes X)_{M_I}$, and

$$e_1\, \mathrm{End}_{M_I}(X \boxtimes Y \boxtimes X)_{M_I}\, e_1 = \mathrm{End}_{M_I}(H_0 \boxtimes X)_{M_I} = \mathbb{C} e_1 \tag{7.4.9}$$

by irreducibility of $X$, so that

$$e_1 e_2 e_1 = \lambda e_1 \tag{7.4.10}$$

for some $\lambda \in \mathbb{C}$. Now $e_1 \in P'$ and

$$q = \bigvee_{u \in U(P)} u\, e_2\, u^*, \tag{7.4.11}$$

where $U(P)$ is the unitary group of $P$, is a non-zero projection in $N \cap P' = \mathbb{C}$; hence $q = 1$. It follows that $\lambda \neq 0$. If $k = 0$, then $e_2 e_1 = 0$ and $\lambda = 0$. Hence $k \neq 0$. Choosing the maps $\eta$, $\varepsilon$ such that $k = 1$, we obtain the identities

$$\nu_X\, (\mathrm{Id}_X \boxtimes \varepsilon)\, \alpha_{X,Y,X}{}^{-1}\, (\eta \boxtimes \mathrm{Id}_X)\, \mu_X{}^{-1} = \mathrm{Id}_X; \tag{7.4.12a}$$

$$\mu_Y\, (\varepsilon \boxtimes \mathrm{Id}_Y)\, \alpha_{Y,X,Y}\, (\mathrm{Id}_Y \boxtimes \eta)\, \nu_Y{}^{-1} = \mathrm{Id}_Y. \tag{7.4.12b}$$

We define the dual object $\overline{H}$ of a bimodule $H$ to be the same bimodule $H$. The maps

$$\begin{aligned} \eta_H :&\ \ H_0 \to H \boxtimes \overline{H}; \\ \varepsilon_H :&\ \ \overline{H} \boxtimes H \to H_0 \end{aligned} \tag{7.4.13}$$

are chosen such that the identities

$$\nu_H\,(\mathrm{Id}_H \boxtimes \varepsilon_H)\,{\alpha_{H,\overline{H},H}}^{-1}\,(\eta_H \boxtimes \mathrm{Id}_H)\,{\mu_H}^{-1} \;=\; \mathrm{Id}_H; \tag{7.4.14a}$$

$$\mu_{\overline{H}}\,(\varepsilon_H \boxtimes \mathrm{Id}_{\overline{H}})\,\alpha_{\overline{H},H,\overline{H}}\,(\mathrm{Id}_{\overline{H}} \boxtimes \eta_H)\,{\nu_{\overline{H}}}^{-1} \;=\; \mathrm{Id}_{\overline{H}} \tag{7.4.14b}$$

hold. When $H$ is irreducible, we have already seen above how this can be done. When $H$ is the direct sum $H = \oplus_i H_i$ of irreducibles, we define $\eta_H$ to be the map from $H_0$ into the summand $\oplus_i H_i \boxtimes \overline{H_i}$ of $H \boxtimes \overline{H}$, given by $\eta_H = \oplus_i \eta_{H_i}$. The map $\varepsilon_{\overline{H}}^{*}$ is similarly defined.

**7.5. Compatible duality.** The identity

$$(\theta_H \boxtimes \mathrm{Id}_{\overline{H}})\,\eta_H \;=\; (\mathrm{Id}_H \boxtimes \theta_{\overline{H}})\,\eta_H \tag{7.5.1}$$

is clearly satisified because $\overline{H} = H$, $\theta_H = e^{-2\pi i L_0}$, and, for $H = \oplus_i H_i$, a direct sum of irreducibles, the image of $\eta_H$ sits inside $\oplus_i H_i \boxtimes H_i$.

yet, in the generality, it holds
also if $H \neq \overline{H}$!

# Chapter VII
# Open problems and further directions

We list some open problems and possible directions for further investigation.

**1.** It remains to study the properties of the category $\mathbf{Pos}_c$. As a ribbon category, it has a canonical *trace*: positivity, $\mathrm{tr}(x^*x) \geq 0$, and the Jones-Markov property should be checked. $\mathbf{Pos}_c$ should also be a modular category, and thus should give rise to a 3-dimensional topological field theory. It remains also to study and classify the subfactors that arise.

**2.** An analogous coset construction exists for the unitary highest weight representations of the super-Virasoro algebra. Several new features in these models make them interesting to study, such as the construction of the (super-) Lie algebra elements and new primary fields, while enough remains similar to the Virasoro algebra case considered here to suggest that the same methods will continue to be useful.

**3.** Some technical problems still remain unsolved in our work. If $\pi : \mathrm{Diff}^+S^1 \to PU(H)$ is a positive energy representation, the pull-back $\pi^*U(H)$ of the circle bundle $U(H) \to PU(H)$ is a topological central extension of $\mathrm{Diff}^+S^1$. This should be isomorphic to a smooth central extension.

**4.** The braiding relations satisfied by localised fields hold on the dense subspace of smooth vectors. We would like to know that they held as operator identities.

**5.** Constructive conformal field theory possesses the main features of "general" and "algebraic" quantum field theory, satisfying both the modified Wightmann and Haag-Kastler axioms. It would be interesting to understand the role, if any, of path-integrals.

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$\subset \mathcal{F}(I_-)$ commutes with $\mathcal{A}(I_+)$.

$$\mathcal{X}_i = \{ F(e, u_-^* a_-) \} \quad \text{where} \quad e : \rho_i \to id \circ \rho_i \qquad u_\pm : \rho_i \to \rho_i^\pm \text{ loc. in } \mathcal{A}(I_\pm)$$

$$\mathcal{Y}_i = \{ F(e, u_+^* a_+) \} \qquad a_\pm \in \mathcal{A}(I_\pm)$$

$\cap \mathcal{F}(I_+)$ commutes with $\mathcal{A}(I_-)$.

---

Obs: for $e : \rho_i \to id \circ \rho_i$, $F(e,a)^* F(e,a) = a^* a$

It follows that the Connes fusion product equals

$$\| x_i \otimes y_j \|^2 = \| F(i, u_-^* a_-) \times F(j, u_+^* a_+) \|^2 := \langle F_i^* F_i F_j^* F_j \Omega, \Omega \rangle$$

$$= \langle a_-^* a_- a_+^* a_+ \Omega, \Omega \rangle = \| a_- a_+ \Omega \|^2 = \| \rho_j (u_-^* a_-) u_+^* a_+ \Omega \|^2$$

$$= \| \varepsilon^* \cdot \rho_i (u_+^* a_+) u_-^* a_- \Omega \|^2$$

Thus, up to unitary equivalence in the commutant of $\pi_{ji}$,

$$|\mathcal{X}_i \otimes \mathcal{Y}_j] \ni [ |i, u_-^* a_- \Omega\rangle \times |j, u_+^* a_+ \Omega\rangle ] = |ji, \rho_j (u_-^* a_-) u_+^* a_+ \Omega\rangle$$

$$= |ji, \rho_j (u_-^*) u_+^* a_- a_+ \Omega\rangle$$

$$= |ji, \varepsilon^* \cdot \rho_i (u_+^*) u_-^* a_+ a_- \Omega\rangle$$

$$= |ji, \varepsilon^* \cdot \rho_i (u_+^* a_+) u_-^* a_- \Omega\rangle$$

These are completed in the first (second) entry to

$$[ |i, \xi\rangle \times |j, u_+^* a_+ \Omega\rangle ] = |ji, \varepsilon \cdot \rho_i (u_+^* a_+) \xi \rangle \qquad \text{on } H \times \mathcal{Y}_j$$

$$[ |i, u_-^* a_- \Omega\rangle \times |j, \eta\rangle ] = |ji, \rho_j (u_-^* a_-) \eta \rangle \qquad \text{on } \mathcal{X}_i \times H$$

$$m \cdot |j, u_+^* a_+ \Omega\rangle \cdot n = |j, \rho_j(m) u_+^* a_+ n \Omega\rangle$$