Berkeley, dec. 2014

Chern-Simons Lectures

- Thanks for the invitation
- Given that these are the "Chern-Simons" lectures, I thought that it might be a good idea to tell you about Chern-Simons theory.

Chern-Simons theory is a 3-dimensional TQFT, more precisely it's a certain family of 3d TQFTs: there's one Chern-Simons theory for every choice of gauge group G and level k.

Here, G is just a compact Lie group and k is something a bit complicated, but if we restrict to connected Lie groups, then k is a G-invariant metric on the Lie algebra, subject to (a certain quantization) condition [$\exp(2\pi X) = e \Rightarrow \langle X, X\rangle_k \in 2\mathbb{Z}$ $\forall X \in \mathfrak{g}$]

If G is furthermore simple and simply connected, then there is only one invariant inner product on $\mathfrak{g}$, up to scale, and we may safely treat k as being just an integer $\in \mathbb{N}$.

One for every:
- gauge group G (simply connected)
- level k $(\in \mathbb{N})$

$\langle x, x\rangle_k \in 2\mathbb{Z}$
$\forall X \in \mathfrak{g}$ for which $\exp(2\pi X) = e$

Chern-Simons theory is a TQFT, so let me first tell you what those are.

According to Atiyah's original definition, a TQFT is a functor $\cancel{Z : \mathrm{Cob}_{2,3} \longrightarrow \mathrm{Vect}}$:

it assigns vector spaces to closed surfaces and linear maps to cobordisms (up to diffeo.)

$$Z(M) : Z(\partial_{in} M) \longrightarrow Z(\partial_{out} M)$$

Moreover, such a functor should be symmetric monoidal w.r.t. to operation of $\amalg$ on $\mathrm{Cob}_{2,3}$ and $\otimes$ on Vect.

$$Z(M_1 \amalg M_2) = Z(M_1) \otimes Z(M_2)$$

The modern way of presenting this is via extended TQFTs: those are functors

$$Z : \mathrm{Cob}_{1,2,3} \longrightarrow \mathrm{LinCat}$$

from the 2-category whose objects are closed 1-manifolds (always oriented), whose 1-morphisms are 2-dim cobordisms and whose 2-morphisms are 3-dimensional cobordisms between those 2-dimensional cobordisms.

Linear categories might be a bit less familiar, these are categories of the form $\mathrm{Vect} \oplus \ldots \oplus \mathrm{Vect}$: categories whose objects are n-tuples of vector spaces and whose morphisms are n-tuples of linear maps.

then we have linear functors, and natural transformations between them.

If you don't like doing things so much in coordinates, then you can also say that the objects of LinCat are semisimple categories with finitely many simples. (and that might sound restrictive, but it can actually be shown that any functor $Z: \mathrm{Cob}_{1,2,3} \to \mathrm{LinCat}$ will map to such categories).

Just like above, such a functor should be symmetric monoidal w.r.t. $\amalg$ on $\mathrm{Cob}_{1,2,3}$ and a certain notion of $\boxtimes$ of linear categories which I now describe:

Formal expressions:

$$\mathcal{C} \boxtimes \mathcal{D} := \begin{cases} \bullet \text{ Objects : } \bigoplus_{i=1}^{n} X_i \boxtimes Y_i \\ \bullet \ \mathrm{Hom}\left(\bigoplus_{i=1}^{n} X_i \boxtimes Y_i \,,\, \bigoplus_{j=1}^{m} X_j' \boxtimes Y_j'\right) \\ \quad := \bigoplus_{ij} \mathrm{Hom}(X_i, X_j') \otimes \mathrm{Hom}(Y_i, Y_j') \end{cases}$$

(formal)

(actual)

Let's now assume that we do have a TQFT, and let us see what structure that induces on the category $\mathcal{C} := Z(S^1)$:

First of all:

 $\longmapsto$ product $m : \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C}$

$X \otimes Y := m(X \boxtimes Y)$

 $\longmapsto$ unit object $1 \in \mathcal{C}$

$\mathrm{Vect} \to \mathcal{C}$
$\mathbb{C} \longmapsto 1$

Then

(*) $\,|\!\cong$

$\longmapsto$ natural transformation

$m(m \otimes 1) \simeq m(1 \otimes m)$.

(that's exactly the data of an associator for the product $\otimes$)

and

(*) $\,|\!\cong$

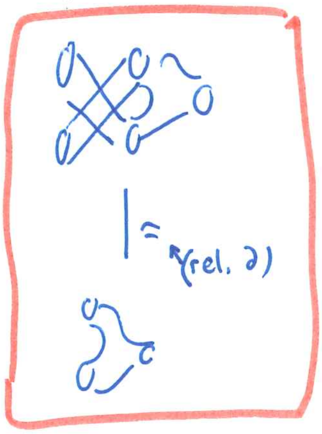
(*) $\,|\!\cong$

$\longmapsto$ $X \otimes 1 \simeq X \simeq 1 \otimes X$

The pentagon axiom and the triangle axiom (for the unit isomorphism) follow from certain equations in $\mathrm{Bord}_{1,2,3}$ satisfied by the diffeomorphisms (*).

$\Rightarrow$ $\mathcal{C}$ is a $\boxed{\text{tensor category}}$ (see handout)

Let us look at the further structure that we have on $\mathcal{C}$:



$\longmapsto$ Natural transformation

$$\beta : m \circ \tau \simeq m$$

$\left(\begin{matrix} \mathcal{C}\boxtimes\mathcal{C} \circlearrowleft \\ X\boxtimes Y \mapsto Y\boxtimes X \end{matrix}\right)$

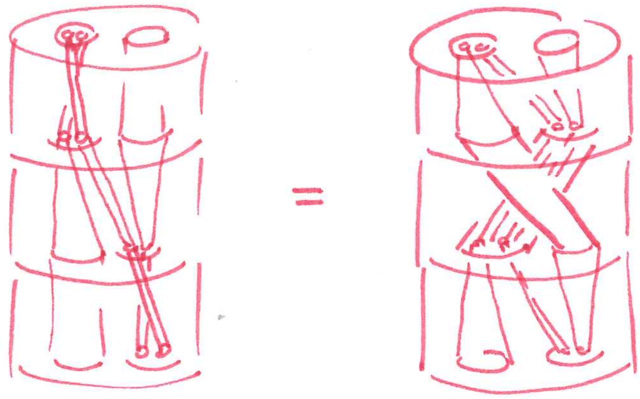
One can also think of this (and that's going to be more convenient) as a diffeomorphism that exchanges the two input circles.

Claim: that's exactly a <u>braiding</u>, i.e. a family of isomorphisms $X\otimes Y \simeq Y\otimes X$, natural in X and Y.

Really, there are no diffeomorphisms in $\mathrm{Cob}_{1,2,3}$. What we have is cobordisms. If we draw the pair of pants as , then β is given by:

This turns out to aslo be more convenient if we want to check the axioms:

PREDRAW →



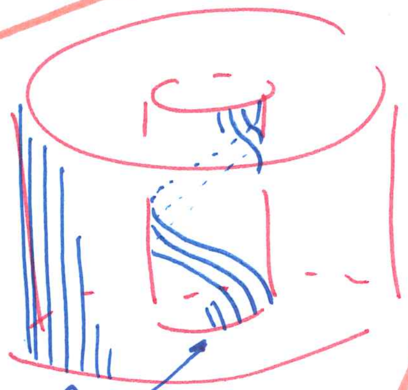
Finally, we have the Dehn twist:

a natural transformation $1_{\mathcal{C}} \Rightarrow 1_{\mathcal{C}}$:

$$\theta_X : X \to X$$

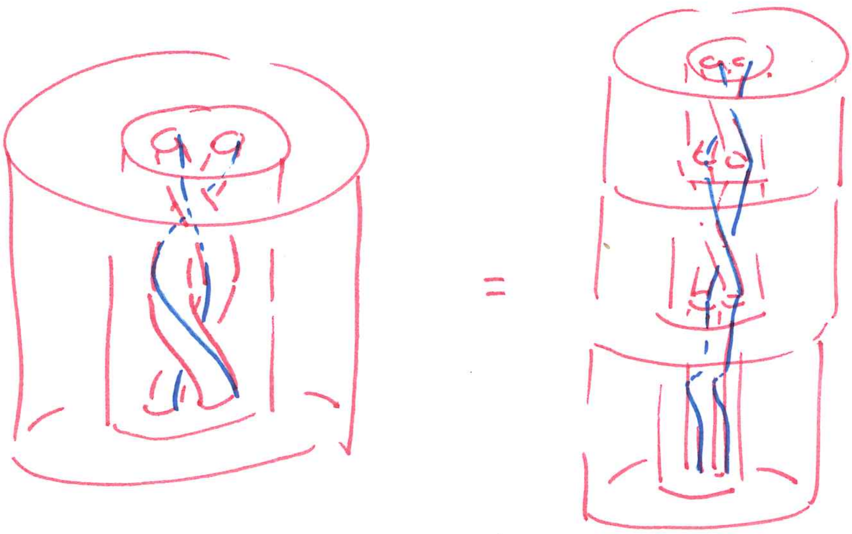
for every $X \in \mathcal{C}$.

As a cobordism:



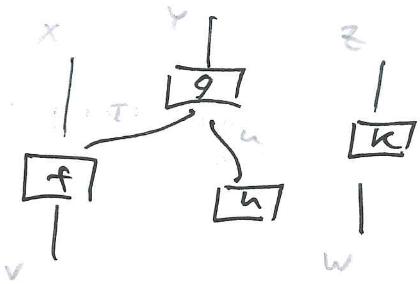
It satisfies $\theta_{X\otimes Y} = (\theta_X \otimes \theta_Y) \circ \beta^2$

PRE-DRAW →



All in all, by looking at the structure that we have so far, we get the concept of a balanced category. (see handout)

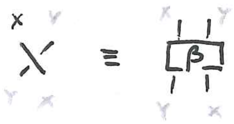
In order to work with these structured categories, the language of string diagrams is extremely convenient. Let's look at the first diagram (top right of handout):



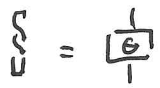
This represents a morphism $X \otimes Y \otimes Z \to V \otimes W$ in the following way ...

(scanning)

(Braided):



(Balanced):



Now let's step back a bit and think about what it means to write down a linear functor between linear categories.

linear map $f: V \to W$ $\overset{\text{pick bases}}{\rightsquigarrow}$ $f: \mathbb{C}^n \to \mathbb{C}^m$ $\rightsquigarrow$ matrix (a_{ij})

$a_{ij} = \langle w_j, f(v_i) \rangle$ $\quad f(v_i) = \sum_j a_{ij} w_j$

linear functor $F: \mathcal{C} \to \mathcal{D}$ $\overset{\text{pick bases}}{\rightsquigarrow}$ $F: \mathrm{Vect}^n \to \mathrm{Vect}^m$ $\rightsquigarrow$ matrix of vector spaces (A_{ij})

$A_{ij} = \mathrm{Hom}(d_j, F(c_i))$ $\quad F(c_i) = \bigoplus_j A_{ij} \otimes d_j$

and a natural transformation is just an entry-wise map between two vector space valued matrices.

Proposition:
If $F: \mathcal{C} \to \mathcal{D}$ corresponds to the matrix (A_{ij}) then its right (also left) adjoint corresponds to the matrix $(A^*_{ij})_{ji}$ (conjugate transpose).

[as in the definition of the adjoint of a map $\mathbb{C}^n \to \mathbb{C}^m$]

proof: Let R be the functor given by the matrix $(A^*_{ij})_{ji}$

$$\boxed{\mathrm{Hom}(F(X), Y) \overset{?}{\cong} \mathrm{Hom}(X, R(Y))}$$

By linearity, it's enough to check it when X and Y are basis elements.

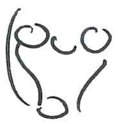
$$\mathrm{Hom}(F(c_i), d_j) = \mathrm{Hom}\Big(\bigoplus_k A_{ik} \otimes d_k, d_j\Big) = \mathrm{Hom}(A_{ij}, \mathbb{C}) = A^*_{ij}$$

$$\mathrm{Hom}(c_i, R(d_j)) = \mathrm{Hom}\Big(c_i, \bigoplus_k A^*_{kj}\, c_k\Big) = \mathrm{Hom}(\mathbb{C}, A^*_{ij}) = A^*_{ij} \quad \boxtimes$$

Let us apply all this to our situation of interest.

$m = Z\Big($  $\Big)$ $\qquad A^k_{ij} = \mathrm{Hom}(c_k, c_i \otimes c_j)$

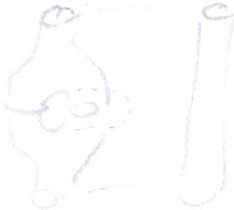
$m: \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C}$

$\Delta = Z\Big($  $\Big)$ $\qquad ?$

$\Delta: \mathcal{C} \to \mathcal{C} \boxtimes \mathcal{C}$ coproduct

m and Δ are adjoint in Bord_{123}

with units and counit of the adjunction

given by η:  . and ε: 

"legs join" "hole shrinks"

Def: Maybe I should tell you that two morphisms F, R in a bicategory are <u>adjoint</u> if there are 2-morphisms

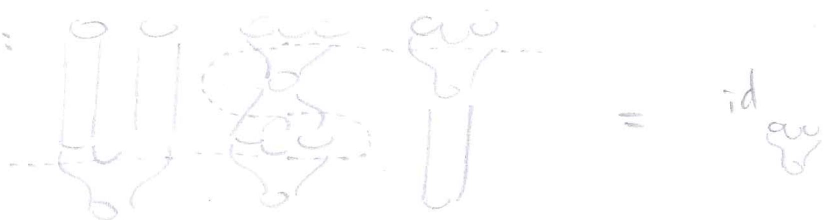
$$\eta : FR \Rightarrow 1 \qquad \varepsilon : 1 \Rightarrow RF$$

s.t. $F \overset{\eta 1}{\Rightarrow} FRF \overset{1\varepsilon}{\Rightarrow} F$

and $R \overset{1\eta}{\Rightarrow} RFR \overset{\varepsilon 1}{\Rightarrow} R$

are identity 2-morphisms.

(Evaluated in Cat, this recovers the usual notion of adjunction between functors)

Check:  ← "straighten the S"

Therefore: $\Delta = Z\Big($  $\Big)$ corresponds to the matrix

$$A_k^{ij} := \mathrm{Hom}(c_k, c_i \otimes c_j)^* = \mathrm{Hom}(c_i \otimes c_j, c_k)$$

Similarly:

$Z($  $)$ $\qquad A^i = \mathrm{Hom}(c_i, 1)$

$Z($  $)$ $\qquad A_i = \mathrm{Hom}(1, c_i)$

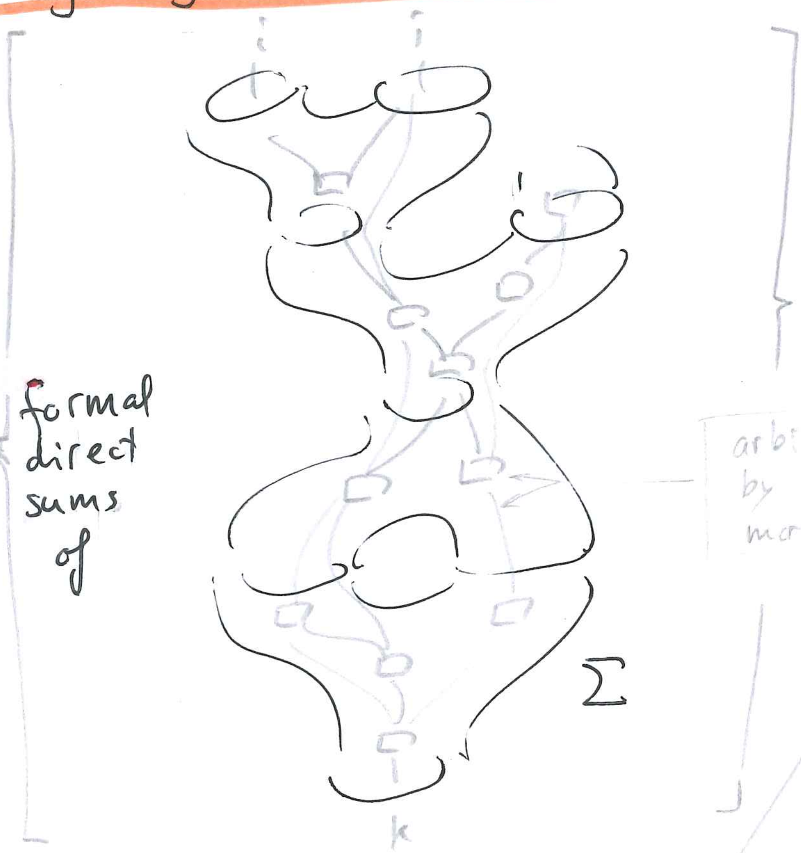
Now given a 2-manifold written as composite of the generating 2-manifolds  m Δ 1 ε , we want a way of understanding the structure spaces of the corresponding functor $\mathcal{C}^{\boxtimes n} \longrightarrow \mathcal{C}^{\boxtimes m}$.

I'll just give you the answer:

Note: Cobordisms are read ↑ whereas internal string diagrams are read ↓ (sorry!)

PREDRAW

$Z(\Sigma)^{ij}_k$ = formal direct sums of

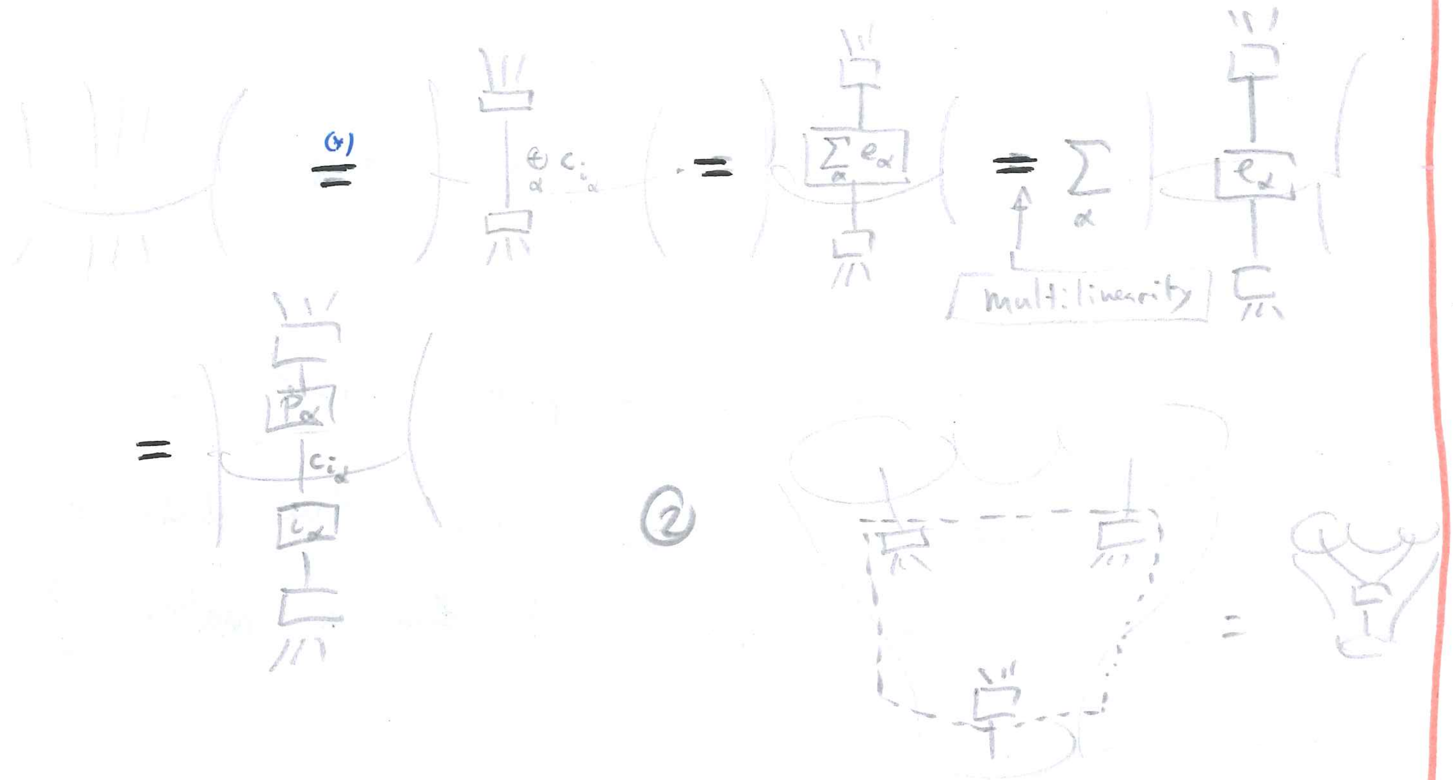


arbitrary labelings by objects and morphisms

- isotopy
- local equivalence 
- multilinearity in the coupon labelings

proof: I'll show you how to simplify an internal string diagram like the one above to one of a particularly simple form:

1



$\Rightarrow$ Any string diagram is

$\sum$ things like this:

$\in \bigoplus \bigotimes \mathrm{Hom}(\ldots,\ldots)$

labellings of circles by basis elements

(co)pants, cups & caps

these are the things that we already computed

that's exactly the structure spaces of the composite functor!

Remains to be done:
check that the resulting element is independent of the
- choice of basis in (x)
- the move  .

$\boxtimes$

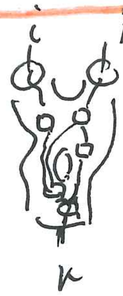
If I have a surface  which is written in such a way that there is a well defined "interior" then

we computed $Z($  $) : \mathcal{C} \to \mathcal{C} \boxtimes \mathcal{C}$

To describe it we had to pick a basis $\{c_i\}$ of $\mathcal{C}$

Once we picked such a basis, then the data of a functor became equivalent to ~~the~~ a matrix of vector spaces.

And we computed those vector spaces

$Z($  $)_k^{ij} = \left\{ \text{internal string diagrams} \right.$  $\left. \right\} / \sim$

$: \mathcal{C} \to \mathcal{C} \boxtimes \mathcal{C}$

⚠ functors go up, whereas string diagrams go down,
(and there's simply no way to fix this.)

Last time, I computed some functors. Now I want to compute some [redacted] nat. transformations

If $F: \mathcal{C} \to \mathcal{D}$ corresponds to the matrix (A_{ij}) and $G: \mathcal{C} \to \mathcal{D}$ corresponds to the matrix (B_{ij}), then a natural transformation $\tau: F \Rightarrow G$ corresponds to linear maps $f_{ij}: A_{ij} \to B_{ij}$.

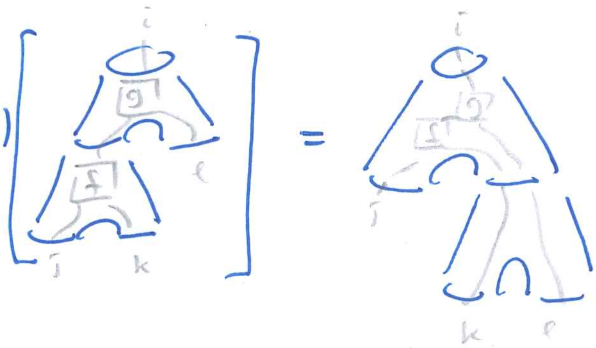
Concretely:

$$f_{ij}: A_{ij} = \mathrm{Hom}(d_j, F(c_i)) \longrightarrow B_{ij} = \mathrm{Hom}(d_j, G(c_i))$$

$$x \longmapsto \tau_{c_i} \circ x$$

$\tau_{c_i}: F(c_i) \to G(c_i)$

Let's figure out what this does in some concrete cases: Recall α: 

Claim $Z(\alpha)$ 

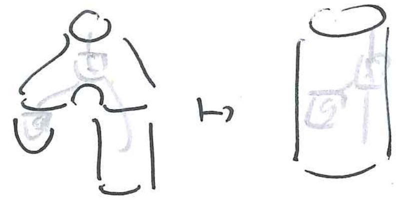
proof: Recall that a vector $v \in A^i_{jkl}$ in the structure space of some functor F is a map $c_i \to F(c_j\ c_k\ c_l)$.

In this case $v = (f \otimes 1) \circ g : c_i \to (c_j \otimes c_k) \otimes c_l$

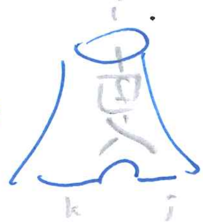
LHS $= Z(\alpha)\Big((f \otimes 1) \circ g\Big) = \alpha \circ (f \otimes 1) \circ g$

RHS $= \left[\, c_i \xrightarrow{\alpha \circ (f \otimes 1) \circ g} c_j \otimes (c_k \otimes c_l) \xrightarrow{1 \otimes \mathrm{id}} c_j \otimes (c_k \otimes c_l) \,\right]$ ✓

☒

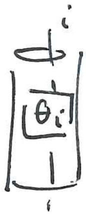
Similarly:  under the unit nat. transf.

Let's do the braiding:

Claim $Z(\beta)\Big($  $\Big) =$ 

proof: Recall that $(f: c_i \to c_j \otimes c_k) \in A^i_{jk}$.

$LHS = Z(\beta)(f) = \beta \circ f = RHS$ $\boxtimes$

Similarly:  $\xmapsto{Z(\theta)}$ 

Dehn twist

Now we'd like to compute the image under Z of some non-invertible 3-cobordisms.

Recall that if a functor F is given by a matrix (A_{ij}) then its adjoint R is given by the "conjugate transpose" $(A^*_{ij})_{ji}$. Let's work out the unit and counit of the adjunction:

$R(d_j) = \bigoplus_k A^*_{kj} \otimes c_k$

$FR(d_j) \to d_j$

$\bigoplus_{k\ell} A^*_{kj} \otimes A_{k\ell} \otimes d_\ell \to d_j$

that's the only reasonable thing $\to$ $\begin{cases} \bigoplus_k A^*_{kj} \otimes A_{kj} \xrightarrow[(*)]{\Sigma\,\mathrm{ev.}} \mathbb{C} \\ 0 & \ell \neq j \end{cases}$

counit

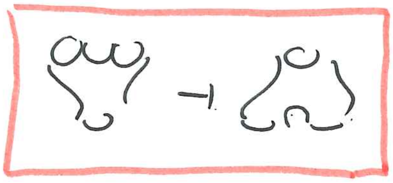
$c_i \to RF(c_i)$

$c_i \to \bigoplus_{j\ell} A_{ij} \otimes A^*_{\ell j} \otimes c_\ell$

$\begin{cases} \mathbb{C} \xrightarrow[(*)]{\Sigma\,\mathrm{coev}} \bigoplus_j A_{ij} \otimes A^*_{ij} \\ 0 & i \neq \ell \end{cases}$

unit

Recall that we had adjunctions



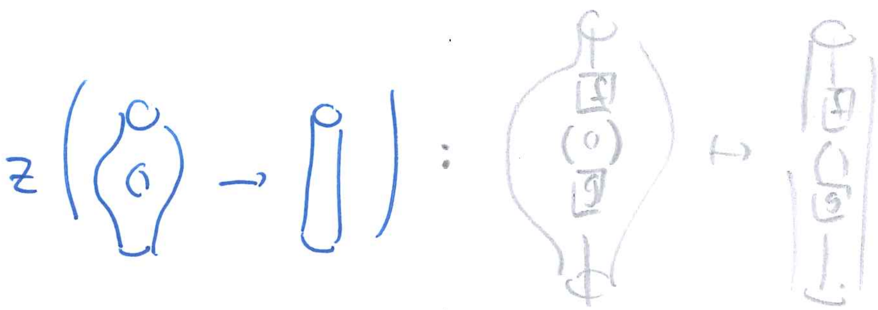
and

and that those adjunctions were our way to figure out $\Delta = Z($$)$ and $\varepsilon = Z($$)$.

But an adjunction is more than just the information of what the adjoint is: it also contains the data of the unit & counit

$\Rightarrow$ Once we know Given $Z($$)$, we get

- $Z($$)$

- 

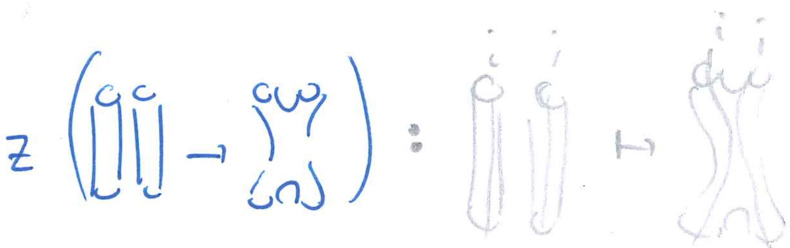
Indeed, if we work out (*) in our example, we get

$\mathrm{Hom}(c_i, c_j \otimes c_k) \otimes$

$\mathrm{Hom}(c_j \otimes c_k, c_i) \xrightarrow{ev} \mathbb{C}$

$\qquad = \mathrm{Hom}(c_i, c_i)$

also known as "composition".

- 

Indeed, if we work out (*), we get

$\mathrm{coev} \in \bigoplus_k \mathrm{Hom}(c_i \otimes c_j, c_k) \otimes$

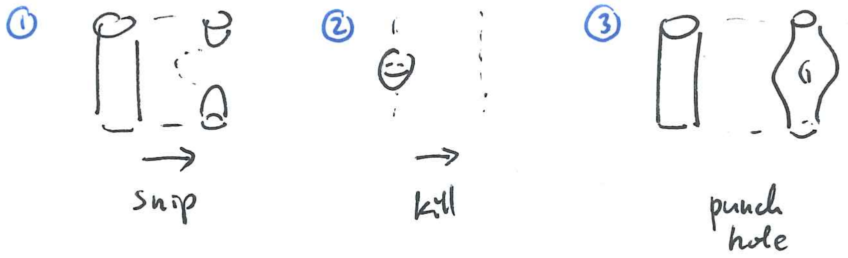
$\mathrm{Hom}(c_k, c_i \otimes c_j)$

which composes to $\mathrm{id}_{c_i \otimes c_j}$.

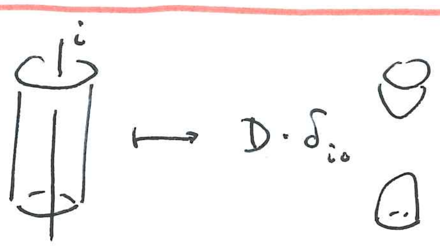
(Similarly for  and )

see handout [left column]

Our next goal is to compute Z on:

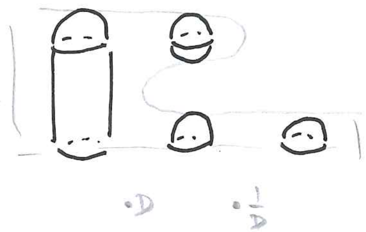
① 

At this point, the balanced category no longer contains all the data to determine the TQFT: there's some freedom in the way to choose ①. Let's pick it to be

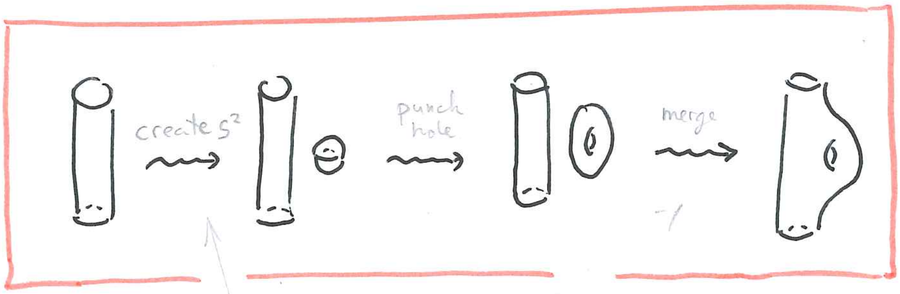


at the level of string diagrams, for some yet-to-be determined constant D [~~later~~ later, we'll see that $\boxed{D^2 = \sum_i \dim(c_i)^2}$ so the freedom is $D \leftrightarrow -D$ but for the moment we don't know]

It follows that  acts like $1 \mapsto \frac{1}{D}$ because this cobordism is trivial:



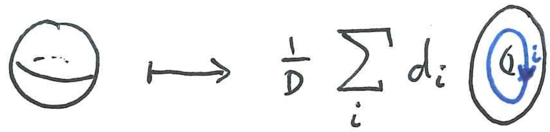
To compute ③ let us rewrite it as the composite



those we already know

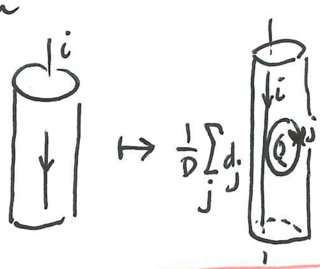
so we only need to figure out

Most general form:

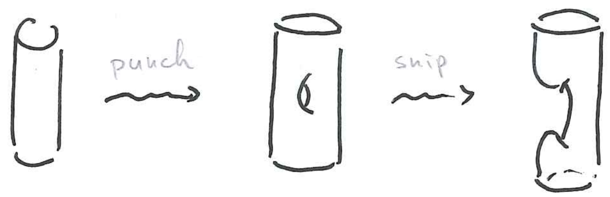


[here, I'm using a suggestive name, and we'll later see that $d_i = \bigcirc^i = \dim(c_i)$ but for the moment $d_i \in \mathbb{C}$ are unknown]

then

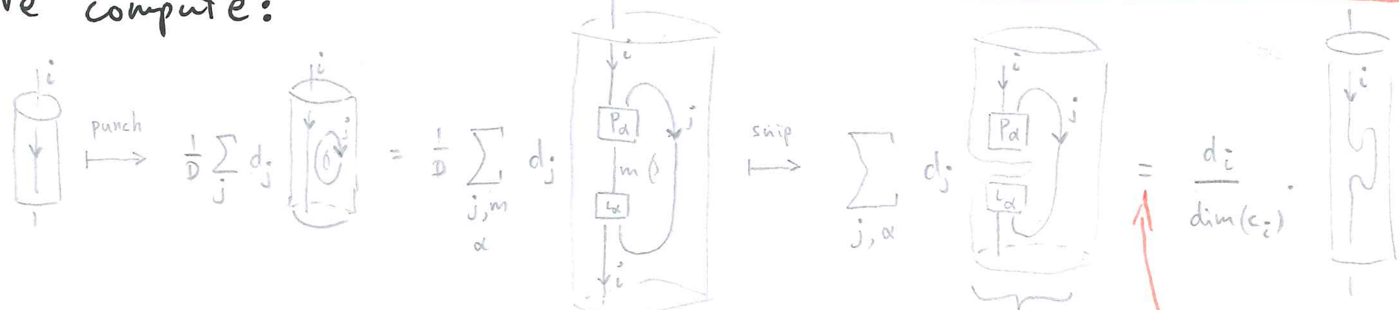


Now



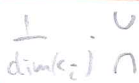
is a trivial cobordism (two Morse singularities in cancelling position)

We compute:



∴ $d_i = \dim(c_i)$

this is only non-zero when $i=j$, in which case there's no α-summation, as the relevant hom-space is 1-dim.



is the projection

Now let's compute some 3-manifold invariants:

- S^3:

$$\emptyset \longrightarrow \emptyset \longrightarrow \emptyset$$

$$1 \longmapsto 1 \longmapsto \tfrac{1}{D}$$

$$Z(S^3) = \tfrac{1}{D}$$

- $S^1 \times S^2$:

$$\emptyset \mapsto \quad \mapsto \frac{1}{D}\sum_i d_i \quad \longmapsto \frac{1}{D}\sum_i d_i^2 \quad \longmapsto \frac{1}{D^2}\sum_i d_i^2 = 1$$

$S^1 \times D^2$ $S^1 \times D^2$

[glued along the identity]

- T^3: [see handout]

Note: The 1-dim TQFT $\int_{S^2} Z$ given by $\left(\int_{S^2} Z\right)(M) := Z(M \times S^2)$ assigns $Z(S^2) = \mathbb{C}$ to a pt and $Z(S^1 \times S^2)$ [computed above] to S^1.

But we know that in a 1d TQFT, the value on S^1 is always the dimension of the value on a pt. $\Rightarrow Z(S^1 \times S^2) = 1$, ✓

Really: that's a confirmation of the eqn $\boxed{D^2 = \sum_i d_i^2}$.

- S^3 (v2):

$$\emptyset \mapsto \quad \mapsto \frac{1}{D}\sum_i d_i \quad \xrightarrow{\theta} \frac{1}{D}\sum_i d_i$$

$$= \frac{1}{D}\sum_i d_i\theta_i \quad \mapsto \frac{1}{D}\sum_i d_i^2\theta_i \quad \longmapsto \frac{1}{D^2}\sum_i d_i^2\theta_i \overset{?}{=} \frac{1}{D}$$

NO!

So it's not an invariant!

Actually, modular tensor categories don't produce functors $\mathrm{Bord}_{1,2,3} \to \mathrm{LinCat}$ (sometimes they do, but rarely)

They produce functors $\mathrm{Bord}^{\partial}_{1,2,3} \to \mathrm{LinCat}$

everything is equipped with a bounding manifold

obj:

arrows: 3d bundle

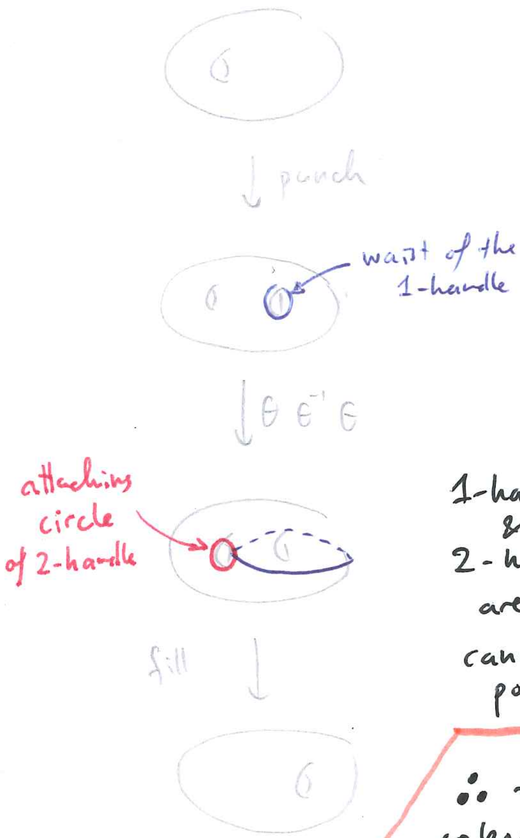
2-mph: $M^3 + W^4$/cobordism

Thm (Barlett – Douglas – SP – Vicary):

$$\left[\begin{array}{l}\mathrm{Fun}^{\otimes}\left(\mathrm{Bord}^{\partial}_{1,2,3},\ \mathrm{LinCat}\right)\\ \text{s.t. } 1\in Z(S^1) \\ \text{is simple}\end{array}\right] \xleftrightarrow{1:1} \left[\begin{array}{l}\text{Ribbon categories} \quad + \quad \text{square root of } \textstyle\sum d_i^2\\ \text{s.t. } 1\in\mathcal{C} \text{ is simple}\\ \text{\& modular [see handout]}\\ \text{“Modular tensor category”}\end{array}\right]$$

Modularity: Now let me tell you why modularity is needed:

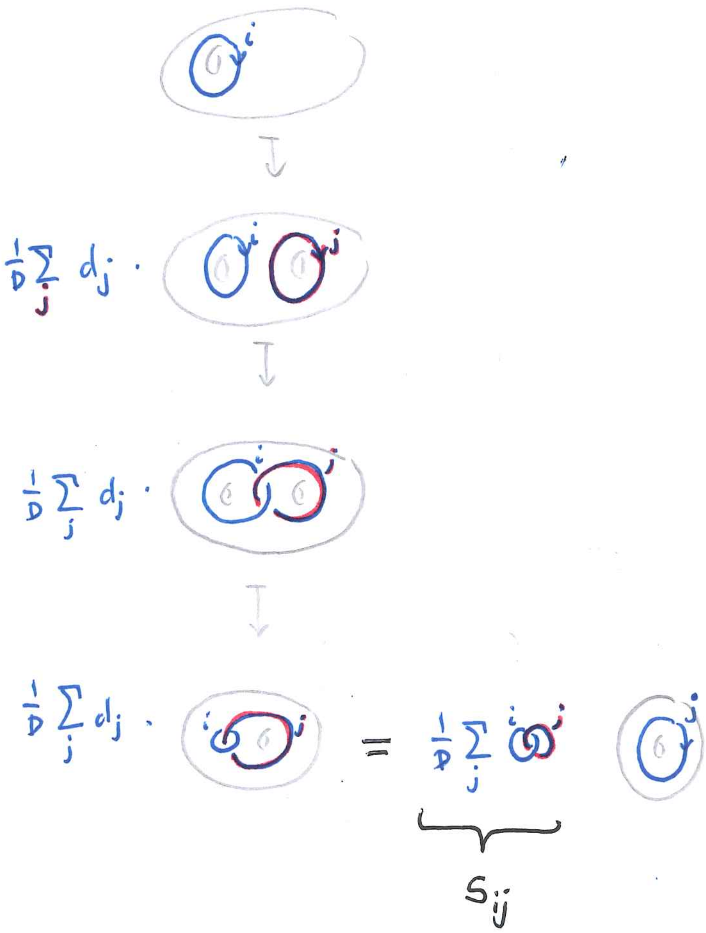
Consider the following cobordism:



1-handle & 2-handle are in cancelling position

$\therefore$ this cobordism is the mapping cylinder of a diffeomorphism. In particular it's invertible.

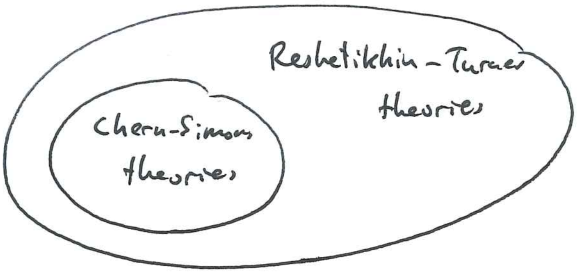
Now let's compute:



the map is invertible, so the matrix should be invertible. $\boxtimes$

All this being said, there is something important that I should confess to you:

what I've been talking about is not Chern–Simons theory: what I've been talking about is Reshetikhin–Turaev theory.



Fix a group G and a level k.

By definition, $CS_{G,k}$ is the RT-theory associated to the modular tensor category $\mathrm{Rep}_k(LG)$.

- $LG = \mathrm{Map}(S^1, G)$
- $k \rightsquigarrow$ central extension
- "positive energy" representations

Actually, this way of saying things only works when G is a connected Lie group. If one wants to treat the most general case, when G is allowed to be disconnected (e.g. finite group), then

$$\cancel{\mathrm{Rep}_k(LG)} \rightsquigarrow \bigoplus_{P:\ G\text{-bundles over } S^1} \mathrm{Rep}_k(\underbrace{\mathrm{Aut}\,P}_{\text{twisted loop group}}) = \text{"(k-twisted) vector bundles" over } \mathrm{Bun}_G(S^1)$$

again: positive energy reps

Next time: I'll tell you a bunch of things that you can do for CS theories and that you can't do for general RT theories

Let me remind you that there is one Chern-Simons theory for every compact Lie group G and level k.

If G is an arbitrary compact Lie group, then k is an element of $H^4(BG, \mathbb{Z})$, but if we restrict to the case when G is simple and simply connected, then this cohomology group is canonically isomorphic to $\mathbb{Z}$, and so k is just a number.

Now the claim is that

$$\boxed{CS_{G,k} = RT \text{ for } Rep_k(LG)}$$

the category of positive energy representations of the loop group of G, at level k (and I will tell you later what that means)

But this claim of course needs justification, because the official definition of CS is not that. The official definition is that Chern-Simons is the quantum field theory given by the action functional

$$\boxed{S[A] = \frac{k}{4\pi} \int_{M^3} tr(A\, dA + \tfrac{2}{3} A \wedge A \wedge A)}$$

where A is a connection 1-form on a principal G-bundle.

Strictly speaking, this expression only makes sense for $G = SU(n)$ in which case A is a matrix valued 1-form, and I know how to multiply matrices, and how to take traces. For other groups, this is definitely a sloppy way of writing things down.

So what does one do when confronted with such a Lagrangian? There's a couple of things one can do:

① Path integral quantization

Given a 3-manifold M, you write down the path integral $\boxed{\int_{\mathrm{Bun}_G^\nabla(M)} e^{iS[A]}\, \mathcal{D}A}$ and pretend that it makes sense.

This is supposed to give you the Chern-Simons invariant of M.

One of the things one can do in order to try to give a meaning to the above expression is a so-called "stationary phase approximation" where you say: this is an oscillatory integral, so unless you're at a critical point of the action, the integrand is going to mostly cancel itself

This leads one to try to compute the critical points of the action:

Say ε is very small. Then

$$S[A+\varepsilon] \cong S[A] + \frac{k}{4\pi}\int_M \mathrm{tr}\,(\varepsilon\, dA + A\, d\varepsilon + 2\, \varepsilon\wedge A\wedge A)$$

those two contributions are equal by Stokes' thm

and so we get: $\quad = 2\int_M \mathrm{tr}(\varepsilon\cdot(dA + A\wedge A))$

For the connection A to be a critical point, this should be zero $\forall \varepsilon$, which is equivalent to saying $\boxed{\overset{\text{critical point}}{dA + A \wedge A = 0}}$. In other words, the connection should be flat.

This is a very standard procedure in quantum field theory, and the solutions of that equation are called the "classical solutions".

② Canonical quantization (Hamiltonian formalism):

This is a method for computing the values of the quantum field theory on manifolds of codimension one (in our case: surfaces).

Given a surface Σ, we're now going to start by looking at the space of classical solutions on Σ:
$$\boxed{\mathrm{Loc}_G(\Sigma)}$$

One realizes that this is canonically a

$\boxed{\text{is a symplectic manifold} \quad \omega_A(\alpha,\beta) = \frac{k}{4\pi}\int_\Sigma \mathrm{tr}(\alpha \wedge \beta)}$ ← (for α, β in the tangent space at A of $\mathrm{Bun}_G^{D=0}(\Sigma)$)

and performs geometric quantization.

to get a vector space.

If one now takes Σ to be a surface with boundary and applies canonical quantization to it, the moduli space that ends up carrying a symplectic structure is the moduli space of flat G-bundles on Σ, equipped with a trivialization over $\partial\Sigma$. Let's call it $\mathrm{Loc}_G(\Sigma;\partial\Sigma)$.

That moduli space carries an action of LG for every boundary component of Σ, which induces an action of LG (this time a projective action) on the corresponding Hilbert space.

The conjecture is then that if we let $Z = RT(\mathrm{Rep}_k LG)$ then

$$\Gamma_{hol}\left(\mathrm{Loc}_G(\Sigma;\partial\Sigma), \mathcal{L}_k\right) \cong \bigoplus_{\lambda_1 \ldots \lambda_n} \underbrace{Z(\Sigma)_{\lambda_1\ldots\lambda_n}} \otimes \underbrace{H_{\lambda_1}} \otimes \ldots \otimes \underbrace{H_{\lambda_n}}$$

up to some completion

unclear what this means

as representations of $(LG)^n$.

these are the spaces I've been talking about on monday & wednesday

irreducible representations of the loop group

Of course this is still just a conjecture, but at least the two sides of the equation live in the same category, which gives a fair amount of evidence that indeed $CS_{G,k} = RT(\mathrm{Rep}_k LG)$.

I should point out that there is also, since a couple of months ago, a way of taking a Lagrangian and directly assigning categories to codimension two manifolds. I will call it

(3) Freed - Teleman quantization

What they did is they considered the category of {twisted curved Fredholm complexes on $Loc_G(S^1)$} and proved that this category is on the nose $Rep_k(LG)$.

Here, instead of a prequantum line bundle we have a prequantum gerbe over $Loc_G(S^1)$: that's the "twisted". A taking "curved Fred. complexes" is a particular way of making sense of what it means to take sections of this gerbe.

By now, I hope that I've convinced you that

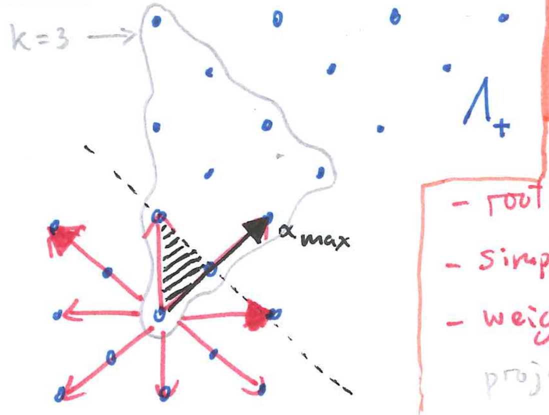
$$\boxed{CS_{G,k}(S^1) = Rep_k(LG).}$$

So let me tell you what $Rep_k(LG)$ looks like. (and later I'll give you the precise mathematical definition)

First I'll tell you what the set of irreducible objects of $Rep_k(LG)$ looks like. For that I'll need to introduce a little bit of Lie theory, and I'll do it by example.

My example will be rank two, because then I can draw on the board, and so this leaves A_2 B_2 & G_2

... let me do B_2 (i.e. $Spin(5)$)



- root system
- simple roots
- weight lattice (= things whose projection on every simple root is a $\frac{1}{2}$-integer times that root)
- dominant weights (in bijective correspondence with irreps of G)
- highest root α_{max}
- Weyl alcove

Simple objects $H_\lambda \in Rep_k(LG)$ parametrized by $\lambda \in \Lambda_+$ s.t. $\langle \alpha_{max}, \lambda \rangle \overset{(*)}{\leq} k$. [where $\langle \cdot, \cdot \rangle$ is normalized so that $\|\alpha_{max}\|^2 = 2$]

Now let me tell you a bit what it means to be an object of $Rep_k(LG)$ and where that condition (*) comes from.

It's easier to do things at the Lie algebra level, so let me do that. We have:

$\mathfrak{g}$: the complexified Lie algebra of G

$L\mathfrak{g} := C^\infty(S^1, \mathfrak{g})$ with bracket pointwise

$\widetilde{L\mathfrak{g}}_k := L\mathfrak{g} \oplus \mathbb{C}$ with bracket

$$[(f,a),(g,a')]_k := \left([f,g], \frac{-k}{2\pi i} \int_{S^1} \langle f, dg \rangle \right)$$

dual inner product to the above inner product on $\mathfrak{g}^*$: the so-called "basic inner product".

The "affine Lie algebra"

Given a dominant weight λ, consider the corresponding finite dimensional representation of G (equivalently $\mathfrak{g}$).

One can then induce it up to a representation of the affine Lie algebra:

$$\boxed{W_\lambda := \operatorname{Ind}_{L\mathfrak{g}_+}^{\widetilde{L\mathfrak{g}}_k} V_\lambda}$$

where $L\mathfrak{g}_+ \subset L\mathfrak{g}$ denotes the subalgebra of functions $\left\{ f : S^1 \xrightarrow{f} \mathfrak{g} \right\}$ that extend to holomorphic functions

$$\begin{tikzcd} S^1 \arrow[r, "f"] \arrow[d, phantom, "\cap" description] & \mathfrak{g} \\ D^2 \arrow[ur, dashed] & \end{tikzcd}$$

. The action of $L\mathfrak{g}_+$ on V_λ is then given by evaluating at $0 \in D^2$ and then acting on V_λ.

The representation W_λ is not unitary, and does not exponentiate to a representation of $\widetilde{LG}$, but when $\langle \lambda, \alpha_{max} \rangle \leq k$, there is a way of solving both problems at once:

If we let $E_0 \in \widetilde{L\mathfrak{g}}_k$ denote the loop $z \mapsto z^{-1} \cdot E_{max}$ where $E_{max} \in \mathfrak{g}$ is the raising operator corresponding to α_{max}, and if we let $v_\lambda \in V_\lambda$ be the highest weight vector, then it turns out that $\boxed{E_0^{\,k - \langle \lambda, \alpha_{max} \rangle + 1}(v_\lambda)}$ generates a proper submodule. The quotient module H_λ is unitary and carries an action of $\widetilde{LG}_k$.

Now let me tell you about von Neumann algebras

If H is a Hilbert space and $B(H)$ denotes its algebra of bounded operators, then

Def A **von Neumann algebra** on H is a $*$-algebra $A \subset B(H)$ such that $A'' = A$, where $A' := \{ b \in B(H) \mid ab = ba \ \forall a \in A \}$ is the commutant of A.

Let now H_0 be **vacuum representation** of LG (it's always projective representations, so I'll stop mentioning it).

Given an interval $I \subset S^1$, we can consider the **local loop group**

$$L_I G := \{ \gamma : S^1 \to G \mid \mathrm{supp}(\gamma) \subset I \}$$

By definition, the Local von Neuman algebras are the von Neumann algebras

$$\boxed{\mathcal{A}(I) := L_I G''}$$

generated by the local loop groups on the vacuum Hilbert space H_0.

I should say that those algebras are **type III factors**, which means that they have only one $\neq 0$ rep. The rep is not irreducible it can be written as $\oplus$ of two subreps but the subreps are isomorphic to the big rep.

Then there's a general theorem, called **Haag duality**, which tells us that $\mathcal{A}(S^1 \setminus I) = \mathcal{A}(I)'$

Now there's a natural question that arises: what would happen if we had taken some other positive energy representation (not the vacuum) Would the corresponding von Neumann algebras be the same as these ones?

$$\begin{tikzcd} L_I G \arrow[r] & B(H_\lambda) \\ \mathcal{A}(I) \arrow[u, phantom, "\subseteq" rotate=90] \arrow[ur, dashed, "?"'] & \end{tikzcd}$$

And the answer is: For $SU(n)$ yes, and for the other Lie groups it's an open question.

So from now on, everything that I say will have the status of a theorem for $SU(n)$, and a conjecture for other Lie groups.

So far we just have $Rep_k(LG)$ as a category, but I haven't said anything about the fusion product. That's where von Neumann algebras are going to be really useful:

$$H \boxtimes K = Hom_{I_2}(H_0, H) \otimes_{I_1} K$$

$$= Hom_{I_2}(H_0, H) \otimes_{I_1} H_0 \otimes_{I_2} Hom_{I_1}(H_0, K)$$

("Connes fusion")

$$= H \otimes_{I_2} Hom_{I_1}(H_0, K)$$

⚠ the fact that these 3 formulas agree depends essentially on Haag duality

[algebraic Hom, and algebraic tensor product]

Now I can associate algebras $\mathcal{A}(I)$ not just to submanifolds of the circle, but to any 1-manifold (compact) whatsoever:

- I — 1-manifold
- $C_0^\infty(I, \mathfrak{g})$ — Lie algebra
- $\frac{-k}{2\pi i} \int_I \langle f, dg \rangle$ — 2-cocycles
- $\widetilde{Map_0(I, G)}$ — integrate
- $\mathcal{A}(I)$ — take group algebra and complete.

This functor from 1-manifolds to von Neumann algebras is a so-called $\boxed{\text{conformal net}}$.

Now I'd like to make the point that these $\mathcal{A}(I)$ are what Chern-Simons theory assigns to 1-manifolds with boundary.

(small shift of perspective: until now we've said that a 3d TQFT should assign categories to 1-manifolds, and now I'm saying that it should assign an algebra to a 1-manifold. But that's ok because I can always trade an algebra for its representation category)

Taking that change of perspective into account,

$Z(\Sigma)$ is no longer a functor $Z(\partial_{in}\Sigma) \to Z(\partial_{out}\Sigma)$

before: $\boxed{Z(\Sigma): Z(\partial_{in}\Sigma) \to Z(\partial_{out}\Sigma)}$

but rather a bimodule for the algebras

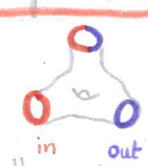
$\mathcal{A}(\partial_{in}\Sigma)$ and $\mathcal{A}(\partial_{out}\Sigma)$, and to avoid any possible

confusion we'll change our notation and call this

bimodule now: $\boxed{{}_{\mathcal{A}(\partial_{in}\Sigma)}H(\Sigma)_{\mathcal{A}(\partial_{out}\Sigma)}}$

[except that now

now:



is allowed.]

And note that this space $H(\Sigma)$ should be exactly

the space $\Gamma_{hol}(\mathrm{Loc}_G(\Sigma;\partial\Sigma), \mathcal{L}_k)$ that I had talked

about earlier.

The beauty of this is that the simplest example

$$\boxed{H(D^2) = H_0}$$

disc

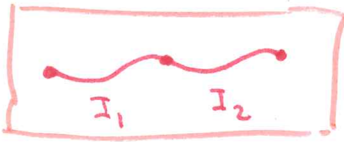
(*)

is enough to determine $H(\Sigma)$ for arbitrary

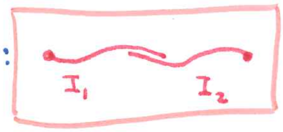
surface Σ, because we are now gluing

2-manifolds along 1-manifolds with boundary.

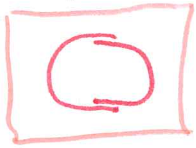
(*) Caveat: Recall that the construction of H_0 involved a choice of complex structure on D^2. In the absence of such a complex structure, one only gets a projective Hilbert space.

Now what remains to tell you is what Chern-Simons theory assigns to a point. But I will delay the pleasure and instead first tell you how 

assuming you know $\mathcal{A}(I_1)$ and $\mathcal{A}(I_2)$ you can compute $\mathcal{A}(I_1 \cup I_2)$

Actually, in order to do that it's convenient to let the 1-manifolds overlap a bit: 

.

<u>Also</u>: intervals are boring. Let's do the circle: 

We know $\subset$ and $\supset$, and we want to reconstruct the algebra that corresponds to the whole circle.

More generally, suppose that we know the algebra associated to "small" intervals $I \subset S^1$, and we want to recover $\mathcal{A}(S^1)$.

<u>Theorem</u> Let $\{I_k\}_{k=1..n}$ be a 2-cover of S^1.

~~If $I \mapsto \mathcal{A}(I)$ was a cosheaf, we would have~~

then $\mathcal{A}(S^1) = \operatorname{colim} \left(\begin{tikzcd} \mathcal{A}(I_1 \cap I_2) \arrow[r] \arrow[dr] & \mathcal{A}(I_1) \\ \mathcal{A}(I_1 \cap I_3) \arrow[ur] \arrow[dr] & \mathcal{A}(I_2) \\ & \mathcal{A}(I_3) \end{tikzcd} \right)$

but that's not the case, and instead we have...

Now I just need to tell you what a 2-cover is:

$\boxed{\text{Def } \{I_k\} \text{ is a 2-cover if } \forall x,y \in S^1 \ \exists k \text{ such that } \{x,y\} \subset I_k.}$

(this is a special case of a so-called "Weiss topology").

More generally, given two von Neumann algebras M and N and homomorphisms $\boxed{M \leftarrow \mathcal{A}([0,1]) \rightarrow N}$ (subject to some conditions), there is a way of forming the "glued von Neumann algebra".

$M^c :=$ commutant of $\mathcal{A}[\frac{1}{3},1]$ inside M

$N^c :=$ commutant of $\mathcal{A}[0\ \frac{2}{3}]$ in N

$$\mathrm{colim}\left(\begin{tikzcd} M & & M^c \bar{\otimes} N^c & & N \\ M^c \bar{\otimes} \mathcal{A}[\frac{2}{3},1] \arrow[u] \arrow[urr] & & \mathcal{A}[0,1] \arrow[ull] \arrow[urr] & & \mathcal{A}[0,\frac{1}{3}] \bar{\otimes} N^c \arrow[u] \arrow[ull] \end{tikzcd}\right)$$

Now I want to tell you what it is that we are gluing over.

Recall that one can think of M and N above as being placeholders for a category.

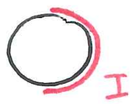
Similarly, $\mathcal{A}([0,1])$ is a placeholder for a $\otimes$-category.

First of all, the underlying category is $\mathcal{A}([0,1])$-mod.

And the $\otimes$ functor comes from the fact that we have two embedings

$$[0,1] \overset{x \mapsto \frac{x}{2}}{\underset{x \mapsto \frac{x+1}{2}}{\rightrightarrows}} [0,1].$$

Now there's a way of rephrasing all that using loop groups.



let's go back to using the more generic notation I instead of $[0,1]$:

it's (a certain) category of $L_I G$-modules

where the monoidal product corresponds to vertical stacking of circles

and we're doing fusion along this little bit.

Now there's one draw-back with that category, which is that it's not unital (no unit object). But that can be fixed, by shrinking this part to a point, and so now I'd like to present you my ultimate proposal of what $CS_{G_k}(pt)$ should be:

Definition: The category of positive energy representations of the based loop group has objects the unitary reps

$$\widetilde{\Omega G} \to U(H)$$

st $\forall\, I \subset S^1$ that does not contain the base point $* \in S^1$ in its interior, $H|_{\widetilde{L_I G}} \cong H_0|_{\widetilde{L_I G}}$.

(or $H = \{0\}$.)

The tensor product is given by Connes fusion, as in the case of LG-reps. (now it's no longer braided)