What Chern–Simons theory assigns to a point

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Chern–Simons theory

Parameters:

- G, a compact Lie group.
- $k \in \mathbb{N}$, the *level*.

Action functional:

$$S[A] = \frac{k}{4\pi} \int_{M^3} tr(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A)$$

Closed 3-manifold

Connection 1-form

Chern–Simons theory

$$S[A] = \frac{k}{4\pi} \int_{M^3} tr(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A)$$

Variation of the action:

$$S[A+\varepsilon] = S[A] + \frac{k}{4\pi} \int_{M^3} tr(\varepsilon \wedge dA + A \wedge d\varepsilon + 2\,\varepsilon \wedge A \wedge A)$$

equal by Stokes' theorem

$$= 2 \int_{M^3} tr\big(\varepsilon \wedge (dA + A \wedge A)\big)$$

Classical solutions:

$dA + A \wedge A = 0$: *flat connections.*

Chern–Simons theory

$$S[A] = \frac{k}{4\pi}\int_{M^3} tr(A \wedge dA + \tfrac{2}{3} A\wedge A\wedge A)$$

Partition function:

$$Z_{CS}(M) = \int_{\left\{\substack{G\text{-bundles with connection } A \\ \text{modulo gauge transform.}}\right\}} e^{iS[A]}\,\mathcal{D}A$$

Hilbert space:

Classical solutions on $\Sigma\times\mathbb{R}$

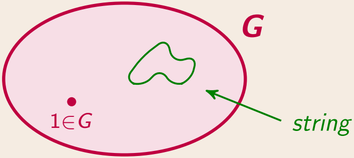
$\mathcal{H}_{CS}(\Sigma)$ = geom. quantiz. of $\underbrace{\text{moduli space of } \textit{flat} \text{ connections}}$

symplectic manifold, $\omega(\alpha,\beta) = \frac{k}{4\pi}\int_\Sigma tr(\alpha\wedge\beta)$

Chern–Simons $=$ Reshetikhin–Turaev for $Rep^k(LG)$

Loop group:

$$LG = Map(S^1, G)$$
$$L\mathfrak{g} = C^\infty(S^1, \mathfrak{g})$$



Central extension: $U(1) \to \widetilde{LG} \to LG$

$$Rep^k(LG) = \text{representations of } LG \text{ at } \underbrace{\text{level } k}.$$

$U(1)$ acts by $\lambda \mapsto \lambda^k$

Central extension:

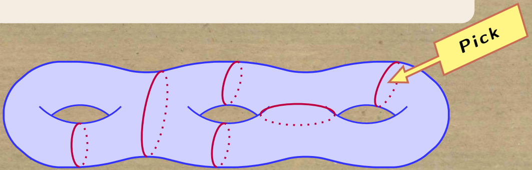
$$L\mathfrak{g} \oplus \mathbb{C} \qquad \big[(f,a),(g,a')\big]_{L\mathfrak{g}\oplus\mathbb{C}} = \Big([f,g]_{L\mathfrak{g}}, \tfrac{k}{2\pi i}\int_{S^1}\langle f, dg\rangle\Big)$$

Chern–Simons = Reshetikhin–Turaev for $Rep^k(LG)$

Partition function: M : 3-manifold

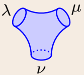
$$Z_{RT}(M) \;=\; \ldots\ldots$$

Σ :



Hilbert space:

$$\mathcal{H}_{RT}(\Sigma) \;=\; \bigoplus_{\substack{\text{labellings}\\ \text{of } S^1\text{'s by}\\ LG\text{-reps}}} \;\bigotimes\; \underbrace{\mathrm{Hom}_{LG}\left(\lambda \otimes \mu,\ \nu\right)}$$



finite dimensional

$\boldsymbol{Rep^k(LG)}$

positive energy representations
of the loop group at level k.

mathematically

physically

$Rep^k(LG) = CS(S^1)$

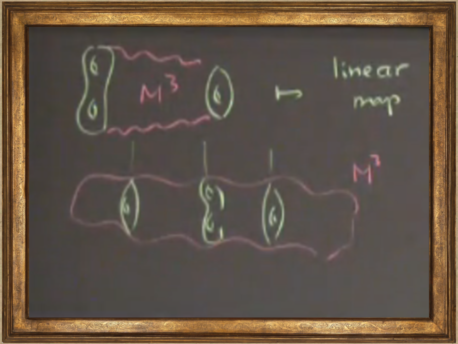
Charges of
Wilson lines

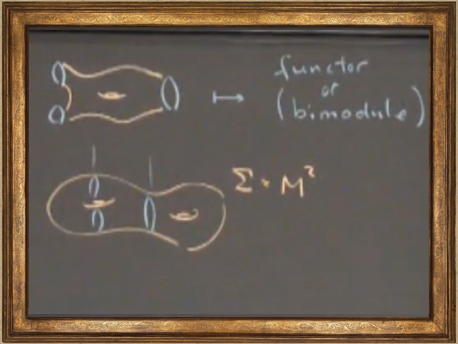
What Chern–Simons theory assigns to a circle

Extended TQFT:

M^3	$\longmapsto$	number	(partition function)
M^2	$\longmapsto$	vector space	(Hilbert space)
M^1	$\longmapsto$	category or algebra	$Rep^k(LG)$
M^0	$\longmapsto$	2-category or $\otimes$-category	**?**

Cobordism hypothesis: a TQFT is *entirely determined* by what it assigns to a point



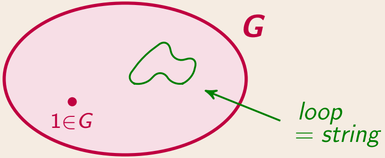


Cobordism hypothesis: a TQFT is *entirely determined* by what it assigns to a point

Loop group:

$LG = Map(S^1, G)$

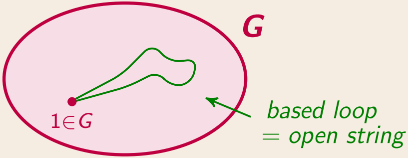
$Rep^k(LG) = CS(S^1)$



Based loop group:

$\Omega G = \{\gamma \in LG \mid \gamma(1) = 1\}$

$Rep^k(\Omega G) = CS(pt)$



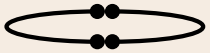
If $\mathcal{Z}$ is a 3D extended TQFT and $\mathcal{C} := \mathcal{Z}(pt)$,
then: ($\mathcal{C}$ is a tensor category)

$\mathcal{Z}($  $) = {}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}}$

$\mathcal{Z}($  $) = {}_{\mathcal{C}\otimes\mathcal{C}^{\mathrm{op}}}\mathcal{C}$

$\mathcal{Z}($  $) = \mathcal{C}_{\mathcal{C}\otimes\mathcal{C}^{\mathrm{op}}}$

$\mathcal{C}$ as a mere category, but where one remembers the left action of $\mathcal{C}$ on itself and the right action of $\mathcal{C}$ on itself.

$\mathcal{Z}(S^1) = \mathcal{Z}($  $) = \mathcal{C} \underset{\mathcal{C}\otimes\mathcal{C}^{\mathrm{op}}}{\boxtimes} \mathcal{C} =: \mathbb{D}(\mathcal{C})$

If $\mathcal{Z}$ is an extended TQFT, then $\mathcal{Z}(S^1) = \mathbb{D}\big(\mathcal{Z}(pt)\big)$

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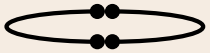
$\mathcal{Z}(S^1) = \mathcal{Z}($  $)$ $= \mathcal{C} \underset{\mathcal{C}\otimes\mathcal{C}^{\mathrm{op}}}{\boxtimes} \mathcal{C} =: \mathbb{D}(\mathcal{C})$

Drinfeld center = quantum double

Theorem (H. 2017): $\mathbb{D}\big(Rep^k(\Omega G)\big) = Rep^k(LG)$.

If $\mathcal{Z}$ is an extended TQFT, then $\mathcal{Z}(S^1) = \mathbb{D}\big(\mathcal{Z}(pt)\big)$

- It is generally accepted that $CS(S^1) = Rep^k(LG)$.

- If we set $CS(pt) = Rep^k(\Omega G)$, we recover what we know:
  $CS(S^1) = \mathbb{D}\big(CS(pt)\big) = \mathbb{D}\big(Rep^k(\Omega G)\big) = Rep^k(LG)$.

$\Longrightarrow$ • We propose: $\boldsymbol{CS(pt) = Rep^k(\Omega G)}$

Theorem (H. 2017): $\mathbb{D}\big(Rep^k(\Omega G)\big) = Rep^k(LG)$.

Physical interpretation

CS: $S[A] = \frac{k}{4\pi}\int_M tr\left(AdA + \frac{2}{3}A^3\right)$

WZW ∂-condition: $A_{\bar{z}}|_{\partial M} = 0$

$CS(M^3)$ = partition function

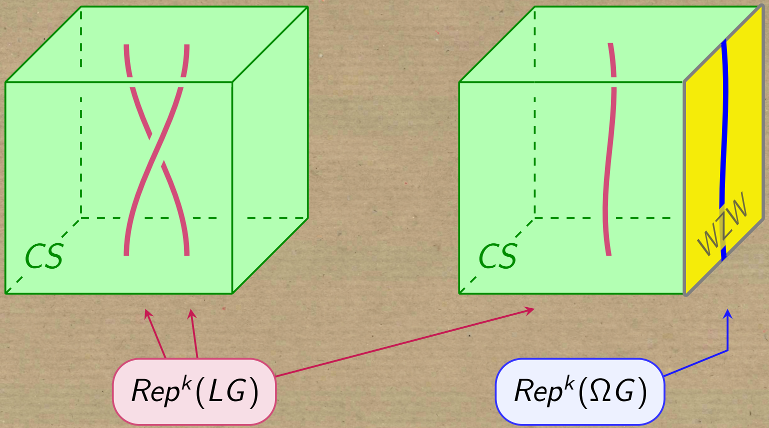
$CS(\Sigma^2)$ = quantum Hilbert space

$CS(S^1) = Rep^k(LG)$ = charges of Wilson lines

$CS(pt) = Rep^k(\Omega G) \overset{(?)}{=}$ line defects in chiral WZW

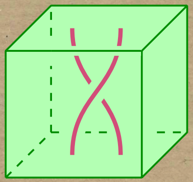
Bulk lines
(= Wilson lines)

Boundary lines
in chiral WZW

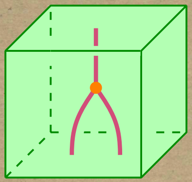


Structure of $Rep^k(LG)$ and of $Rep^k(\Omega G)$.

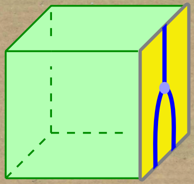
Braiding of bulk lines



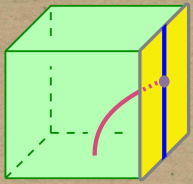
Fusion of bulk lines



Fusion of boundary lines



Fusion of bulk line with a bdry line



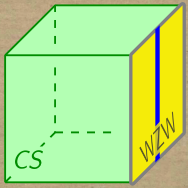
Structure of $Rep^k(LG)$ and of $Rep^k(\Omega G)$.

Theorem (H. 2017):

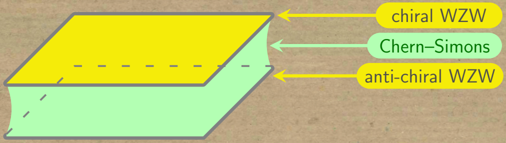
- The categories $Rep^k(LG)$ and $Rep^k(\Omega G)$ carry all the structures mentioned above (braiding, fusion, etc.).
- $Rep^k(LG)$ is the *maximal* category that carries that structure with $Rep^k(\Omega G)$.

$$\Big(\Rightarrow \ Rep^k(LG) \text{ can be reconstructed from } Rep^k(\Omega G). \Big)$$

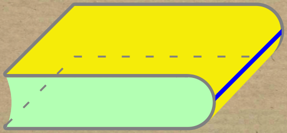
$Rep^k(\Omega G)$ $=$ line defects in chiral WZW



Full WZW



$Rep^k(\Omega G) \overset{(?)}{=}$ ∂-conditions for full WZW



Question: *Can one use the description via* $Rep^k(\Omega G)$ *to classify line defects in chiral WZW?*

Conjecture:

Line defects [1,2] in chiral WZW $\longleftrightarrow$

- a fusion category $\mathcal{C}$
- an object $x \in \mathcal{C}$
- a central functor $Rep^k(LG) \to \mathcal{C}$

1 : only those defects whose fusion algebra closes after finitely many steps.

2 : only up to conjugation by invertible defects.

Summary

- Chern–Simons theory appears to fit in the formalism of extended quantum field theory.
- $CS(pt) = Rep^k(\Omega G) =$ line defects in chiral WZW.
- Prospects of classification results for line defects in chiral WZW $=$ boundary conditions for full WZW.

Thank you!